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Research Article

Shock Wave Solution for a Nonlinear Partial Differential Equation Arising in the Study of a Non-Newtonian Fourth Grade Fluid Model

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This study focuses on obtaining a new class of closed-form shock wave solution also known as soliton solution for a nonlinear partial differential equation which governs the unsteady magnetohydrodynamics (MHD) flow of an incompressible fourth grade fluid model. The travelling wave symmetry formulation of the model leads to a shock wave solution of the problem. The restriction on the physical parameters of the flow problem also falls out naturally in the course of derivation of the solution.

1. Introduction

A shock wave (also called shock front or simply "shock") is a type of propagating disturbance through a media. It actually expresses a sharp discontinuity of the parameters that delineate the media. Unlike solitons, the energy is a conserved quantity and thus remains constant during its propagation. Shock wave dissipates energy relatively quickly with distance. One source of a shock wave is when the supersonic jets fly at a speed that is greater than the speed of sound. This results in the drag force on aircraft with shocks. These waves also appear in various interesting phenomena in real-life situations. For example, solitons appear in the propagation of pulses through optical fibers. Another example is where cnoidal waves appear in shallow water waves although an extremely scarce phenomena.

The dynamics and mechanics of non-Newtonain fluid flow problems have been an important area of interest in the recent few years. The flow phenomena of non-Newtonian fluids occur in a variety of industrial and technological applications. Because of the diverse physical structure and behavior of non-Newtonian fluids, there is no single mathematical expression which describes all the characteristics of non-Newtonian fluids. Due to this fact, several models of non-Newtonian fluids have been proposed. Apart from this fact, the model equations of the problem dealing with the flow of non-Newtonian fluids are higher-order nonlinear and complex in nature. Several methods have been developed in the recent years to obtain the solutions of these sort of flow problems. Some of these techniques are variational iteration method, the Adomian decomposition method, homotopy perturbation method, homotopy analysis method, and semi-inverse variational method. But all these techniques fail to develop the exact (closed-form) solutions of the non-Newtonian fluid flow problems.

One of the simplest classes of non-Newtonian fluid models is the second grade fluid [1]. Although the second grade model is found to predict the normal stress differences, it does not take into account the shear thinning and shear thickening phenomena due to its constant apparent shear viscosity. For this reason, some experiments may be well described through fluids of grade three or four. Very little attention has been given to date to the flows of fourth grade fluid [2]. This model is known as the most generalized model amongst the differential-type non-Newtonian fluid models [3]. The fourth grade fluid model describes most of the non-Newtonian flow properties at one time. This model is known to capture the interesting non-Newtonian flow properties such as shear thinning and shear thickening that many other non-Newtonian models do not show. This model is also capable of predicting the normal stress effects that lead to phenomena like "die-swell" and "rod-climbing" [4]. With these facts in mind, we have considered a fourth grade fluid model in this study. In general, the model equations of the problem dealing with the flow of fourth grade fluids are higher-order nonlinear equations. The literature survey witnesses that very limited studies are reported in the literature, up to now, dealing with the flow problems of fourth grade fluid and these investigations that further narrow are down when we discussed the closed-form solutions of these problems. However, some useful and interesting communications in this direction are made in the studies [5–11].

In this study, we have used an interesting method to construct the solution of nonlinear problem arising in the study of non-Newtonian fluid. We have explored the shock wave behavior of the problem which deals with the unsteady flow of fourth grade fluid. We have also taken into account the magnetohydrodynamic nature of the fluid by applying uniform magnetic field as an external body force. This concept is introduced so that our solution can be easily reduced to the problem that deals with the effects of body forces.

2. Mathematical Structure of the Model

The unsteady MHD flow of an incompressible fluid is governed by law of conservation of mass and momentum; namely,

$$\operatorname{div} \mathbf{V} = \mathbf{0}, \tag{1}$$

$$\rho \frac{d\mathbf{V}}{dt} = \operatorname{div} \mathbf{T} - \sigma B_0^2 \mathbf{V}. \tag{2}$$

In the above equations, $\mathbf{V}$ is the velocity vector, ρ the density of the fluid, d/dt the total time derivative, and $\mathbf{T}$ the Cauchy stress tensor. We have considered a uniform magnetic field of strength B_0, which is applied in the transverse direction of the flow as an external body force by assuming that the induced magnetic field and the external field are negligible.

For fourth grade fluid model, the Cauchy stress tensor satisfies the constitutive equations [3]:

$$\mathbf{T} = -p\mathbf{I} + \sum_{j=1}^{n} \mathbf{S}_j \quad \text{with } n = 4, \tag{3}$$

where p is the pressure, $\mathbf{I}$ the identity tensor, and $\mathbf{S}_j$ the extra stress tensor as

$$\begin{aligned}
\mathbf{S}_1 &= \mu \mathbf{A}_1, \\
\mathbf{S}_2 &= \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2, \\
\mathbf{S}_3 &= \beta_1 \mathbf{A}_3 + \beta_2 \left(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1\right) + \beta_3 \left(\operatorname{tr} \mathbf{A}_1^2\right) \mathbf{A}_1, \\
\mathbf{S}_4 &= \gamma_1 \mathbf{A}_4 + \gamma_2 \left(\mathbf{A}_3\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_3\right) + \gamma_3 \mathbf{A}_2^2 \\
&\quad + \gamma_4 \left(\mathbf{A}_2\mathbf{A}_1^2 + \mathbf{A}_1^2\mathbf{A}_2\right) \\
&\quad + \gamma_5 \left(\operatorname{tr} \mathbf{A}_2\right) \mathbf{A}_2 + \gamma_6 \left(\operatorname{tr} \mathbf{A}_2\right) \mathbf{A}_1^2 \\
&\quad + \left[\gamma_7 \operatorname{tr} \mathbf{A}_3 + \gamma_8 \operatorname{tr} \left(\mathbf{A}_2\mathbf{A}_1\right)\right] \mathbf{A}_1.
\end{aligned} \tag{4}$$

Here, μ is the dynamic viscosity; α_i $(i = 1, 2)$, β_i $(i = 1, 2, 3)$, and γ_i $(i = 1, 2, \ldots, 8)$ are material constants. The Rivlin-Ericksen tensors $\mathbf{A}_1$ to $\mathbf{A}_4$ are defined by

$$\mathbf{A}_1 = \left(\operatorname{grad} \mathbf{V}\right) + \left(\operatorname{grad} \mathbf{V}\right)^T, \tag{5}$$

$$\mathbf{A}_n = \frac{d\mathbf{A}_{n-1}}{dt} + \mathbf{A}_{n-1} \left(\operatorname{grad} \mathbf{V}\right) + \left(\operatorname{grad} \mathbf{V}\right)^T \mathbf{A}_{n-1} \quad (n > 1), \tag{6}$$

in which grad is the gradient operator.

Consider the unsteady MHD flow of an incompressible fourth grade fluid which occupies the half-space $y > 0$ over an infinite rigid plate which lies in the xz-plane. The x-axis and y-axis are chosen parallel and perpendicular to the plate. By taking the velocity field $(u(y, t), 0, 0)$, the conservation of mass equation is identically satisfied. To obtain the governing PDE in u, substituting (3)–(6) into (2) and rearranging, we obtain the following model equation in the absence of the modified pressure gradient:

$$\begin{aligned}
\rho \frac{\partial u}{\partial t} &= \mu \frac{\partial^2 u}{\partial y^2} + \alpha_1 \frac{\partial^3 u}{\partial y^2 \partial t} + \beta_1 \frac{\partial^4 u}{\partial y^2 \partial t^2} \\
&\quad + 6\left(\beta_2 + \beta_3\right) \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} + \gamma_1 \frac{\partial^5 u}{\partial y^2 \partial t^3} \\
&\quad + 2\left(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8\right) \frac{\partial}{\partial y} \\
&\quad \times \left[\left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y \partial t}\right] - \sigma B_0^2 u.
\end{aligned} \tag{7}$$

3. Reduction of the Model Equation

We know that from the principal of the Lie symmetry methods that if a differential equation is explicitly independent of any dependent or independent variable, then this particular differential equation remains invariant under the translation symmetry corresponding to that particular variable. We noticed that (7) admits the Lie point symmetry generators, $\partial/\partial t$ (time-translation) and $\partial/\partial y$ (space-translation in y). For a detailed analysis, the readers are referred to [12, 13].

Let X_1 and X_2 be time-translation and space-translation symmetry generators, respectively. Then, the solution corresponding to the generator

$$X = X_1 + mX_2 = \frac{\partial}{\partial t} + m\frac{\partial}{\partial y} \quad (m > 0) \tag{8}$$

would represent the travelling wave solution with constant wave speed m. The Langrangian system corresponding to (8) is

$$\frac{dy}{m} = \frac{dt}{1} = \frac{du}{0}. \tag{9}$$

Solving (9), invariant solutions are given by

$$u(y, t) = f(\eta) \quad \text{with } \eta = y - mt, \tag{10}$$

where $f(\eta)$ is an arbitrary function of the characteristic variable $\eta = y - mt$. Making use of (10) into (7) results in a fifth-order ordinary differential for $f(\eta)$

$$\begin{aligned} -m\rho \frac{df}{d\eta} = {} & \mu \frac{d^2 f}{d\eta^2} - \alpha_1 m \frac{d^3 f}{d\eta^3} + \beta_1 m^2 \frac{d^4 f}{d\eta^4} \\ & + 6(\beta_2 + \beta_3)\left(\frac{df}{d\eta}\right)^2 \frac{d^2 f}{d\eta^2} - \gamma_1 m^3 \frac{d^5 f}{d\eta^5} \\ & - 2m(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8)\frac{d}{d\eta} \\ & \times \left[\left(\frac{df}{d\eta}\right)^2 \frac{d^2 f}{d\eta^2}\right] - \sigma B_0^2 f. \end{aligned} \tag{11}$$

Thus, the original fifth-order nonlinear PDE (7) reduced to a fifth-order ODE (11) along certain curves in the y-t plane. These curves are called *characteristic curves* or just the characteristic.

4. Shock Wave Solution

Now, we obtain shock wave solution of the reduced equation (11). The starting hypothesis for shock wave solution is given by [14–17]

$$f(\eta) = A\exp(B\eta), \tag{12}$$

where A and B are the free parameters to be determined. Inserting (12) in (11), we obtain

$$\begin{aligned} 0 = {} & \left[m\rho B + \mu B^2 - m\alpha_1 B^3 + \beta_1 m^2 B^4 - \gamma_1 m^3 B^5 - \sigma B_0^2\right] \\ & + e^{2B\eta}\left[6(\beta_2 + \beta_3)A^2 B^4\right. \\ & \left. -6m(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8)A^2 B^5\right]. \end{aligned} \tag{13}$$

Separating (13) in the powers of e^0 and $e^{2B\eta}$, we find the following:

$$e^0 : m\rho B + \mu B^2 - m\alpha_1 B^3 + \beta_1 m^2 B^4 - \gamma_1 m^3 B^5 - \sigma B_0^2 = 0, \tag{14}$$

$$\begin{aligned} e^{2B\eta} : {} & 6(\beta_2 + \beta_3)A^2 B^4 \\ & - 6m(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8)A^2 B^5 = 0. \end{aligned} \tag{15}$$

From (15), we deduce

$$B = \frac{(\beta_2 + \beta_3)}{m(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8)}. \tag{16}$$

Using the value of B in (14), we obtain

$$\begin{aligned} 0 = {} & \frac{\rho(\beta_2 + \beta_3)}{(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8)} \\ & + \frac{\mu(\beta_2 + \beta_3)^2}{m^2(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8)^2} \\ & - \frac{\alpha_1(\beta_2 + \beta_3)^3}{m^2(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8)^3} \\ & + \frac{\beta_1(\beta_2 + \beta_3)^4}{m^2(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8)^4} \\ & - \frac{\gamma_1(\beta_2 + \beta_3)^5}{m^2(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8)^5} - \sigma B_0^2. \end{aligned} \tag{17}$$

Thus, the solution for $f(\eta)$ (provided the condition (17) holds) can be written as

$$f(\eta) = A\exp\left[\frac{(\beta_2 + \beta_3)}{m(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8)}\eta\right]. \tag{18}$$

So, the solution $u(y,t)$ which satisfies the condition (17) is written as

$$u(y,t) = A\exp\left[\frac{(\beta_2 + \beta_3)(y - mt)}{m(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8)}\right] \quad \text{with } m > 0. \tag{19}$$

We observe that the solution (19) does satisfy the physically relevant boundary and initial conditions

$$u(0,t) = E(t), \tag{20a}$$

$$u(y,0) = F(y), \tag{20b}$$

$$\frac{\partial u(y,0)}{\partial t} = G(y), \tag{20c}$$

$$\frac{\partial^2 u(y,0)}{\partial t^2} = H(y), \tag{20d}$$

where

$$E(t) = A\exp\left(\frac{-(\beta_2 + \beta_3)mt}{m(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8)}\right), \tag{21a}$$

$$F(y) = A\exp\left(\frac{(\beta_2 + \beta_3)y}{m(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8)}\right), \tag{21b}$$

$$G(y) = \frac{-(\beta_2 + \beta_3)}{(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8)}F(y), \tag{21c}$$

$$H(y) = \left(\frac{(\beta_2 + \beta_3)}{(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8)}\right)^2 F(y), \tag{21d}$$

with

$$A = E(0) = F(0). \tag{22}$$

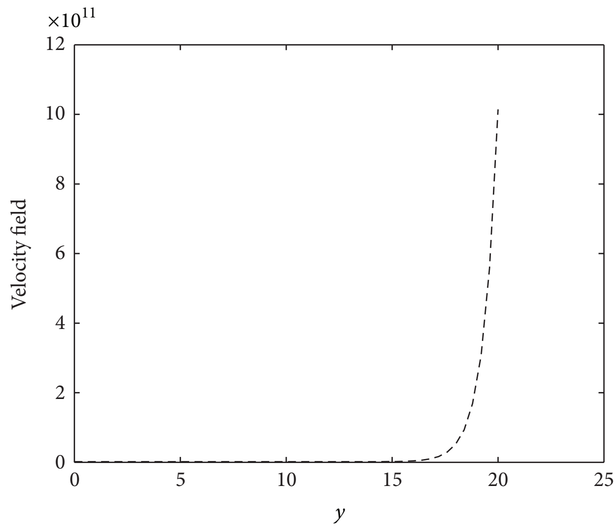

FIGURE 1: Shock wave behaviour of solution (19), where $\beta_2 = 2$, $\beta_3 = 1$, $m = 1$, $t = \pi/2$, and $\gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = \gamma_7 = \gamma_8 = 0.2$.

The functions $E(t)$, $F(y)$, $G(y)$, and $H(y)$ depend on the physical parameters of the flow model. The boundary condition (20a) is the no-slip condition at $y = 0$. The initial velocity $E(0)$ of the rigid plate can be prescribed, but its velocity for $t > 0$, $E(t)$, cannot be arbitrary and is given by (21a). Similarly, the initial velocity profile $u(y, 0)$ cannot be arbitrary and is given by (21b).

Note that the solution (19) is the soliton solution or the shock wave solution to the governing PDE (7). The above solution is valid under the particular condition on the physical parameters of the flow model given in (17). This solution does show the hidden shock wave behavior of the flow problem with slope of the velocity field or the velocity gradient approaches to infinity such that

$$\frac{\partial u}{\partial y} \longrightarrow \infty \quad \text{as } y > 0. \tag{23}$$

If we denote

$$\epsilon = \frac{(\beta_2 + \beta_3)}{(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8)}, \tag{24}$$

thus, the imposing condition (17) can be written as

$$m^2 = \frac{\left[-\gamma_1\epsilon^5 + \beta_1\epsilon^4 - \alpha_1\epsilon^3 + \mu\epsilon^2\right]}{(\sigma B_0^2 - \rho\epsilon)}, \quad \text{with } \sigma B_0^2 \neq \rho\epsilon. \tag{25}$$

We observe that the condition (25) gives the speed m of the travelling shock wave. The range of the values of ϵ for which m is real depends not only on the zeros of the cubic polynomial in ϵ on the numerator of (25) but also on the sign of the denominator. The shock wave behavior of solution (19) is observed from Figure 1.

5. Concluding Remarks

In this study, we have obtained the mathematical structure of the closed-form shock wave solution of higher-order nonlinear PDE arising in the study of a fourth grade non-Newtonian fluid model. Translational symmetry generators in variables t and y have been utilized to perform reduction of governing nonlinear partial differential equation into ordinary differential equation, and, thereafter, the closed-form shock wave solution has been constructed. We have considered a prototype model of the flow problem, but the solution is going to be very helpful in carrying out further analysis of the shock wave behavior associated with the non-Newtonian fluid flow models. The method that we have adopted is also prosperous for tackling wide range of nonlinear problems in non-Newtonian fluid mechanics.

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