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Research Article

A Bijection between Lattice-Valued Filters and Lattice-Valued Congruences in Residuated Lattices

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The aim of this paper is to study relations between lattice-valued filters and lattice-valued congruences in residuated lattices. We introduce a new definition of congruences which just depends on the meet $\wedge$ and the residuum $\rightarrow$. Then it is shown that each of these congruences is automatically a universal-algebra-congruence. Also, lattice-valued filters and lattice-valued congruences are studied, and it is shown that there is a one-to-one correspondence between the set of all (lattice-valued) filters and the set of all (lattice-valued) congruences.

1. Introduction and Preliminaries

The interest in lattice-valued logic has been rapidly growing recently. Several algebras playing the role of structures of true values have been introduced and axiomatized [1–3]. The most general structure considered in this paper is that of a residuated lattice [4].

In a narrow sense, a residuated lattice is an algebra $L = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ of type (2,2,2,2,0,0) satisfying the following: (i) $(L, \wedge, \vee)$ is a bounded lattice with $0, 1$ as the bottom element, and the top element respectively; (ii) $(L, \otimes, 1)$ is a commutative monoid and monotone at both arguments; (iii) $a \otimes b \leq c$ if and only if $a \leq b \rightarrow c$ (for all $a, b, c \in L$). The operations $\otimes, \rightarrow$ are called the multiplication and residuum, respectively. A residuated lattice in this paper is generally called a bounded, integral, and commutative residuated lattice in [4].

Residuated lattices were first introduced as a generalization of ideal lattices of rings in 1939 by Ward and Dilworth [5]. In their original definition, a residuated lattice was what we would call an integral commutative one.

For a residuated lattice L, the negation operation $\neg : L \rightarrow L$ is defined by $\neg x = x \rightarrow 0$ (for all $x \in L$).

Residuated lattices are very common in mathematical science and a lot of lattices and algebras are residuated lattices firstly. For example, an integral commutative Girard-monoid [2] is a residuated lattice satisfying the law of double negation: $x = \neg\neg x$; a Heyting algebra [6] is a residuated lattice with $\otimes = \wedge$; an MV-algebra [7] is a residuated lattice where $x \vee y = (x \rightarrow y) \rightarrow y$ holds; an MTL-algebra [1] is a residuated lattice satisfying $(x \rightarrow y) \vee (y \rightarrow x) = 1$; a BL-algebra [8] is an MTL-algebra satisfying $x \wedge y = x \otimes (x \rightarrow y)$; a product algebra (or Π-algebra) [8] is a BL-algebra satisfying $\neg\neg z \leq ((x \otimes z) \rightarrow (y \otimes z)) \rightarrow (x \rightarrow y)$ and $x \wedge \neg x = 0$; a G-algebra (Gödel algebra) [2] is both a Heyting algebra and an MTL-algebra; an R_0-algebra [3] is a residuated lattice where $x \otimes y = \neg(x \rightarrow \neg y)$; a lattice implication algebra [9] is a residuated lattice with $a \otimes b = (a \rightarrow b')'$ (where $' : L \rightarrow L$ is an order-reversing involution).

Since the class of all residuated lattices is a variety of algebras (Proposition 2 in [10]), we can study them as universal algebras. Now, consider a residuated lattice $L = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ as a universal algebra; a congruence $\sim$ on L is an equivalence relation which preserves all operators on L; that is, $(a, b), (c, d) \in \sim$ implies that $(a \wedge c, b \wedge d), (a \vee c, b \vee d), (a \otimes c, b \otimes d), (a \rightarrow c, b \rightarrow d) \in \sim$.

The aim of this paper is to study the relation between lattice-valued filters and lattice-valued congruences in residuated lattices. We will introduce a new definition of congruences just depending on the meet $\wedge$ and the residuum $\rightarrow$.

Then it is shown that each of these congruences is automatically a universal-algebra-congruence. Also, lattice-valued filters and lattice-valued congruences are studied, and it is shown that there is a one-to-one correspondence between the set of all (lattice-valued) filters and the set of all (lattice-valued) congruences.

2. Filters and Congruences

In pure mathematics, (lattice-valued) filters (or ideals) and (lattice-valued) congruences are useful tools in investigating the structure of the corresponding algebras.

The definition of a residuated lattice (in a narrow sense) has been given in Section 1. In the following discussion, L always denotes a residuated lattice.

Proposition 1 (see [3, 8, 10, 11]). *Let L be a residuated lattice. Then*

(R1) $a \otimes b \le a \wedge b$;

(R2) $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$;

(R3) $(a \vee b) \to c = (a \to c) \wedge (b \to c)$;

(R4) $a \otimes (b \vee c) = (a \otimes b) \vee (a \otimes c)$;

(R5) $b \to c \le (a \to b) \to (a \to c)$;

(R6) $a = 1 \to a$;

(R7) $a \le b$ *if and only if* $a \to b = 1$;

(R8) $a \le b \to c$ *if and only if* $b \le a \to c$;

(R9) $a \to (b \to c) = b \to (a \to c) = (a \otimes b) \to c$;

(R10) $a \to b \le (a \otimes c) \to (b \otimes c)$;

(R11) $a \to b \ge b$;

(R12) $b \ge a \otimes (a \to b)$;

(R13) $b \le a \to (a \otimes b)$;

(R14) $a \to (a \otimes b) \le a \to b$;

(R15) $a \le (a \to b) \to b$;

(R16) $(a \to b) \otimes (b \to c) \le a \to c$.

Definition 2 (see [8]). A nonempty subset F of L is called a filter if

(F1) F is an upper set; that is, $x \le y$ and $x \in F$ imply $y \in F$ for all $x, y \in L$;

(F2) F is closed under $\otimes$; that is, $x \otimes y \in F$ holds for all $x, y \in F$.

Proposition 3 (see [8]). *Let F be a nonempty subset of L. Then the following three are equivalent:*

(1) *F is a filter;*

(2) *$1 \in F$ and $x, x \to y \in F$ imply $y \in F$ for all $x, y \in F$;*

(3) *F is closed under $\otimes$ and $x \vee y \in F$ for all $x \in F$ and $y \in L$.*

Denote $F(L)$ as the set of all filters of L. Then $F(L)$ is a complete lattice under the partial order of set inclusion with the largest element L and the least element $\{1\}$. Furthermore, the meets in $F(L)$ are the usual intersection of sets.

For simplification of congruence relation in algebraic structures, related attempts have been made in [12–14].

Definition 4. A nonempty subset $\sim$ of $L \times L$ is called a $\{\wedge, \to\}$-congruence on L if the following conditions hold:

(ER) $\sim$ is an equivalence;

for any $x, y, z \in L$,

(C1) if $(x, y) \in \sim$, then $(x \wedge z, y \wedge z) \in \sim$;

(C2) if $(x, y) \in \sim$, then $(x \to z, y \to z) \in \sim$.

Obviously, a congruence is always a $\{\wedge, \to\}$-congruence. Let $\mathrm{Con}(L)$ denote the set of all congruences on L. It is easy to verify that $\mathrm{Con}(L)$ is a complete lattice, where the meets are the usual intersection of sets and $L \times L$, $\{(1,1)\}$ are the largest and the least elements, respectively.

Let $\sim$ be a congruence on L and $L/\sim = \{[x] \mid x \in L\}$, where $[x]$ is the congruence class of x with respect to $\sim$. Define $[x] \wedge [y] = [x \wedge y]$, $[x] \vee [y] = [x \vee y]$, $[x] \otimes [y] = [x \otimes y]$, $[x] \to [y] = [x \to y]$ (for all $x, y \in L$). It is easy to verify that $(L/\sim, \wedge, \vee, \otimes, \to, [0], [1])$ is also a residuated lattice.

Proposition 5 (see [15]). *Let F be a filter of L. Then $\sim_F = \{(x, y) \in L \times L \mid x \to y, y \to x \in F\}$ is a $\{\wedge, \to\}$-congruence on L.*

Proof. (ER) Obviously, $\sim_F$ is reflexive and symmetric. To show the transitivity of $\sim_F$, suppose that $(x, y), (y, z) \in \sim_F$; we have $x \to y$, $y \to x$, $y \to z$, $z \to y \in F$. Then

$$(x \longrightarrow y) \otimes (y \longrightarrow z), \qquad (z \longrightarrow y) \otimes (y \longrightarrow x) \in F. \tag{1}$$

By (R16)

$$\begin{aligned} (x \longrightarrow y) \otimes (y \longrightarrow z) &\le x \longrightarrow z, \\ (z \longrightarrow y) \otimes (y \longrightarrow x) &\le z \longrightarrow x, \end{aligned} \tag{2}$$

and F is an upper set; we have $x \to z$, $z \to x \in F$.

Suppose that $(x, y) \in \sim_F$ and $z \in L$. Then $x \to y, y \to x \in F$.

(C1) First,

$$\begin{aligned} (x \wedge z) \longrightarrow (y \wedge z) &= ((x \wedge z) \longrightarrow y) \wedge ((x \wedge z) \longrightarrow z) \\ &\ge (x \longrightarrow y) \wedge 1 = x \longrightarrow y. \end{aligned} \tag{3}$$

Then $(x \wedge z) \to (y \wedge z) \in F$ since F is an upper set. Similarly, we have $(y \wedge z) \to (x \wedge z) \in F$. Hence $(x \wedge z, y \wedge z) \in \sim_F$.

(C2) By (R16), we have

$$\begin{aligned} (y \longrightarrow x) \otimes (x \longrightarrow z) &\le (y \longrightarrow z), \\ y \longrightarrow x &\le (x \longrightarrow z) \longrightarrow (y \longrightarrow z). \end{aligned} \tag{4}$$

Thus $(x \to z) \to (y \to z) \in F$ since F is an upper set. Similarly, we have $(y \to z) \to (x \to z) \in F$. Hence $(x \to z, y \to z) \in \sim_F$. □

Proposition 6. *Let F be a filter of L. If $(a,b),(c,d)\in\sim_F$, then $(a\wedge c,b\wedge d),(a\vee c,b\vee d),(a\otimes c,b\otimes d),(a\to c,b\to d)\in\sim_F$.*

Proof. Suppose that $(a,b),(c,d)\in\sim_F$. Then $a\to b$, $b\to a$, $c\to d$, $d\to c\in F$.

(1) $(a\wedge c,b\wedge d)\in\sim_F$. In fact, by (R1) and (R2),

$$\begin{aligned}(a\wedge c)\longrightarrow(b\wedge d)&=((a\wedge c)\longrightarrow b)\wedge((a\wedge c)\longrightarrow d)\\&\geq(a\longrightarrow b)\wedge(c\longrightarrow d)\\&\geq(a\longrightarrow b)\otimes(c\longrightarrow d)\in F.\end{aligned}\tag{5}$$

It follows that $(a\wedge c)\to(b\wedge d)\in F$. Similarly, $(b\wedge d)\to(a\wedge c)\in F$. Hence $(a\wedge c,b\wedge d)\in\sim_F$.

(2) $(a\vee c,b\vee d)\in\sim_F$. In fact, by (R1) and (R3),

$$\begin{aligned}(a\vee c)\longrightarrow(b\vee d)&=(a\longrightarrow(b\vee d))\wedge(c\longrightarrow(b\vee d))\\&\geq(a\longrightarrow b)\wedge(c\longrightarrow d)\\&\geq(a\longrightarrow b)\otimes(c\longrightarrow d)\in F.\end{aligned}\tag{6}$$

It follows that $(a\vee c)\to(b\vee d)\in F$. Similarly, $(b\vee d)\to(a\vee c)\in F$. Hence $(a\vee c,b\vee d)\in\sim_F$.

(3) $(a\otimes c,b\otimes d)\in\sim_F$. In fact, by (R10), $(a\otimes c)\to(b\otimes c)\geq a\to b\in F$, which implies that $(a\otimes c)\to(b\otimes c)\in F$. Similarly, $(b\otimes c)\to(a\otimes c)\in F$. Thus $(a\otimes c,b\otimes c)\in\sim_F$. Similarly, $(c\otimes b,d\otimes b)\in\sim_F$. Hence $(a\otimes c,b\otimes d)\in\sim_F$ by the transitivity of $\sim_F$.

(4) $(a\to c,b\to d)\in\sim_F$. In fact, by (R16), we have

$$(b\longrightarrow a)\otimes(a\longrightarrow c)\otimes(c\longrightarrow d)\leq b\longrightarrow d,\tag{7}$$

which implies that

$$(b\longrightarrow a)\otimes(c\longrightarrow d)\leq(a\longrightarrow c)\longrightarrow(b\longrightarrow d).\tag{8}$$

Thus, $(a\to c)\to(b\to d)\in F$. Similarly, $(b\to d)\to(a\to c)\in F$. Hence $(a\to c,b\to d)\sim_F$. □

Proposition 7. *Let $\sim$ be a $\{\wedge,\to\}$-congruence on L. Then $F_\sim=\{x\in L\mid(x,1)\in\sim\}$ is a filter of L.*

Proof. Obviously, $1\in F_\sim$. Suppose that $x,x\to y\in F_\sim$; that is, $(x,1),(x\to y,1)\in\sim$. By (R6) and (C2), we have $(x\to y,y)=(x\to y,1\to y)\in\sim$ and by the transitivity of $\sim$, we have $(y,1)\in\sim$. Thus $y\in F_\sim$. Hence $F_\sim$ is a filter of L. □

Lemma 8. *Let $\sim$ be a $\{\wedge,\to\}$-congruence on L. Then $(x,y)\in\sim$ if and only if $(x\to y,1)\in\sim$ and $(y\to x,1)\in\sim$.*

Proof. Suppose that $(x,y)\in\sim$. Then $(x\to y,1)=(x\to y,y\to y)\in\sim$ and similarly $(y\to x,1)\in\sim$. Conversely, suppose that $(x\to y,1)\in\sim$ and $(y\to x,1)\in\sim$. Then

$$((x\longrightarrow y)\longrightarrow y,y)=((x\longrightarrow y)\longrightarrow y,1\longrightarrow y)\in\sim.\tag{9}$$

By (C1) and (R15),

$$(x,x\wedge y)=(((x\longrightarrow y)\longrightarrow y)\wedge x,y\wedge x)\in\sim.\tag{10}$$

Similarly, we have $(y,x\wedge y)\in\sim$. Hence $(x,y)\in\sim$ by the transitivity of $\sim$. □

Theorem 9. *Let $F,\sim$ be a filter of L and a $\{\wedge,\to\}$-congruence on L, respectively. Then $\sim_{F_\sim}=\sim$ and $F_{\sim_F}=F$. Thus there is a bijection between $F(L)$ and $Con(L)$.*

Proof. (1) By Lemma 8, $(x,y)\in\sim_{F_\sim}$ if and only if $x\to y\in F_\sim$ and $y\to x\in F_\sim$ if and only if $(x\to y,1)\in\sim$ and $(y\to x,1)\in\sim$ if and only if $(x,y)\in\sim$. Hence $\sim_{F_\sim}=\sim$.

(2) $x\in F_{\sim_F}$ if and only if $(x,1)\in\sim_F$ if and only if $x\to 1\in F$ and $1\to x\in F$ if and only if $x\in F$. Hence $F_{\sim_F}=F$. □

Remark 10. (1) By Proposition 6 and Theorem 9, if $\sim$ is a $\{\wedge,\to\}$-congruence on L and $(a,b),(c,d)\in\sim$, then $(a\wedge c,b\wedge d),(a\vee c,b\vee d),(a\otimes c,b\otimes d),(a\to c,b\to d)\in\sim$. That is to say, a $\{\wedge,\to\}$-congruence and a (universal) congruence are equivalent to each other, and so are the symbols $Con(L)$.

(2) In [16], Pavelka firstly showed that there is a one-to-one correspondence between all filters and all congruences in a residuated lattice. And a binary relation is a universal-algebra-congruence if and only if it is an equivalence relation that preserves both $\otimes$ and $\to$ (that is, it just depends on the operations $\otimes,\to$; the other two operations $\vee,\wedge$ are automatically preserved).

3. M-Filters

In the following part of this paper, unless otherwise stated, M always denotes a lattice with a greatest element 1. In a lattice M, an element a is called prime (resp., coprime) if $b\wedge c\leq a$ (resp., $a\leq b\vee c$) always implies $b\leq a$ or $c\leq a$ (resp., $a\leq b$ or $a\leq c$) for all $b,c\in M$. The set of all prime (resp., coprime) elements of M is denoted by $J(M)$ (resp., $P(M)$). A complete lattice M is called a spatial frame [6] if $a=\wedge\{r\in P(M)\mid a\leq r\}$ and M is called a closed set lattice [17] if $a=\vee\{r\in J(M)\mid r\leq a\}$.

In this section, we will study M-filters and their properties in the residuated lattice L.

Definition 11. We call a mapping $A:L\to M$ a lattice-valued filter of L if

(FF1) $A(1)=1$;

(FF2) $A(y)\geq A(x)\ \&\ A(x\to y)$ for all $x,y\in L$.

Remark 12. The definition of a lattice-valued filter [13] is a lattice-valued set A of L satisfying (FF2) and

(FF1′) for all $x\in L$, $A(1)\geq A(x)$,

which is different from Definition 11. It is easy to see that a lattice-valued filter in this paper is always a lattice-valued filter in [13]. In a common sense, a lattice-valued filter should be equivalent to a crisp one if we replaced M by $\{0,1\}$. Thus, the lattice-valued filter in [13] is not a direct generalization of a crisp one since 0_L (the constant map valued at 0) is a lattice-valued filter of L while $\emptyset$ (the crisp counterpart) is not a crisp one.

Denote $FF(L)$ as the set of all lattice-valued filters of L.

Proposition 13. *Let $A : L \to M$ be a mapping with $A(1) = 1$. The following two are equivalent:*

(1) $A \in FF(L)$;

(2) *A is monotone with respect to the order on L and $A(x) \wedge A(y) \geq A(x \otimes y) \geq A(x)\ \&\ A(y)$ for all $x, y \in L$.*

Proof. (1) $\Rightarrow$ (2): for any $x, y \in L$ with $x \leq y$, we have $x \to y = 1$ and

$$\begin{aligned} A(y) &\geq A(x \longrightarrow y)\ \&\ A(x) \\ &= A(1)\ \&\ A(x) = 1\ \&\ A(x) = A(x). \end{aligned} \tag{11}$$

Thus A is monotone. By (FF2) and (R13),

$$A(x \otimes y) \geq A(x)\ \&\ A(x \longrightarrow (x \otimes y)) \geq A(x)\ \&\ A(y) \tag{12}$$

since A is monotone. Also, $A(x \otimes y) \leq A(x) \wedge A(y)$ since A is monotone. Therefore, $A(x) \wedge A(y) \geq A(x \otimes y) \geq A(x)\ \&\ A(y)$.

(2) $\Rightarrow$ (1): by (R12), for all $x, y \in L, A(y) \geq A(x \otimes (x \to y)) = A(x)\ \&\ A(x \to y)$. □

Corollary 14. *If $\& = \wedge$ in M, then for each $A \in FF(L)$, $A(x \wedge y) = A(x \otimes y) = A(x) \wedge A(y)$.*

Proof. By Proposition 13 and (R1), $A(x \wedge y) \leq A(x) \wedge A(y) = A(x \otimes y) \leq A(x \wedge y)$. Then $A(x \wedge y) = A(x \otimes y) = A(x) \wedge A(y)$. □

Let $A : L \to M$ be a mapping. For any $r \in M$, define

$$\begin{aligned} A_{[r]} &= \{x \in L \mid A(x) \geq r\}, \\ A_{(r)} &= \{x \in L \mid A(x) \nleq r\}. \end{aligned} \tag{13}$$

Proposition 15. *$A \in FF(L)$ if and only if $A_{[r]} \in F(L)$ for any $r \in M$.*

Proof. (1) $\Rightarrow$ (2): clearly, for any $r \in M, 1 \in A_{[r]}$. If $x, x \to y \in A_{[r]}$, then $A(x), A(x \to y) \geq r$. Then $A(y) \geq A(x) \wedge A(x \to y) \geq r$. Thus, $y \in A_{[r]}$. Hence $A_{[r]} \in F(L)$.

(2) $\Rightarrow$ (1): clearly, $A(1) = 1$ since $A_{[1]} \in F(L)$. For any $x, y \in L$, suppose that $A(x) \wedge A(x \to y) = r$. Then $x, x \to y \in A_{[r]}$. Thus $y \in A_{[r]}$ and $A(y) \geq r = A(x) \wedge A(x \to y)$. Hence $A \in FF(L)$. □

Proposition 16. *(1) If M is a closed set lattice, then $A \in FF(L)$ if and only if $A_{[r]} \in F(L)$ for any $r \in J(M)$.*

(2) If M is a spatial frame, then $A \in FF(L)$ if and only if $A_{(r)} \in F(L)$ for any $r \in P(M)$.

Proof. (1) The necessity is from Proposition 15. Sufficiency: clearly, $A(1) = 1$ since $A_{[r]} \in F(L)$ for any $r \in J(L)$. For any $x, y \in L$, suppose that $r \in J(L)$ and $r \leq A(x) \wedge A(x \to y)$. Then $x, x \to y \in A_{[r]}$. Thus $y \in A_{[r]}$ and $A(y) \geq r$, By the arbitrariness of $r \in J(M)$ and $A \in FF(L)$.

(2) Necessity: clearly, for any $r \in P(M), 1 \in A_{(r)}$ since $1 \notin P(L)$. If $x, x \to y \in A_{(r)}$, then $A(x), A(x \to y) \nleq r$ and $A(y) \geq A(x) \wedge A(x \to y)$. Then $A(y) \nleq r$ and $y \in A_{(r)}$. Hence $A_{(r)} \in F(L)$. Sufficiency: if $A(1) \neq 1$, then there exists $r \in P(M)$ such that $A(1) \leq r$. Then $1 \notin A_{(r)}$, which contradicts $A_{(r)} \in F(L)$. Thus $A(1) = 1$. For any $x, y \in L$, for any $r \in P(M)$ such that $A(x) \wedge A(x \to y) \nleq r$, we have $A(x) \nleq r$ and $A(x \to y) \nleq r$ and then $x, x \to y \in A_{(r)}$, which implies that $y \in A_{(r)}$ and $A(y) \nleq r$. By the arbitrariness of r, we have $A(y) \geq A(x) \wedge A(x \to y)$. Hence $A \in FF(L)$. □

4. Lattice-Valued Congruences

In this section, we will study lattice-valued congruences and the relations among filters, congruences, lattice-valued filters, and lattice-valued congruences in residuated lattices.

Definition 17. A mapping $\theta : L \times L \to M$ is called a lattice-valued congruence on L if it satisfies the following, for any $x, y, z \in L$:

(FC1) $\theta(x, x) = 1$;

(FC2) $\theta(x, y) = \theta(y, x)$;

(FC3) $\theta(x, z) \geq \theta(x, y)\ \&\ \theta(y, z)$;

(FC4) $\theta(x \wedge z, y \wedge z) \geq \theta(x, y)$;

(FC5) $\theta(x \to z, y \to z) \geq \theta(x, y)$.

Denote $F\mathrm{Con}(L)$ as the set of all lattice-valued congruences on L.

Definition 18. Let θ be a lattice-valued congruence on L. Define $\theta^x : L \to M$ by $\theta^x(y) = \theta(x, y)$ (for all $y \in L$). θ^x is called the lattice-valued congruence class of x with respect to θ on L.

Proposition 19. *Let θ be a lattice-valued congruence on L. Then θ^1 is a lattice-valued filter on L, called the lattice-valued filter induced by θ, denoted by A_θ.*

Proof. (FF1) Clearly, $\theta^1(1) = \theta(1, 1) = 1$. (FF2) for all $x, y \in L$, by (FC3),

$$\theta^1(y) = \theta(1, y) \geq \theta(1, x \longrightarrow y)\ \&\ \theta(x \longrightarrow y, y), \tag{14}$$

and by (FC5),

$$\theta(x \longrightarrow y, y) = \theta(x \longrightarrow y, 1 \longrightarrow y) \geq \theta(x, 1) = \theta^1(x). \tag{15}$$

Thus $\theta^1(y) \geq \theta^1(x \to y)\ \&\ \theta^1(x)$. Hence $\theta^1 \in FF(L)$. □

Proposition 20. *Let A be a lattice-valued filter on L and $\theta_A(x, y) = A(x \to y)\ \&\ A(y \to x)$ (for all $x, y \in L$). Then θ_A is a lattice-valued congruence on L, called the lattice-valued congruence induced by A.*

Proof. (FC1) and (FC2) are obvious and omitted. For any $x, y, z \in L$, (FC3) by Proposition 13 and (R16),

$$\begin{aligned}
&\theta_A(x, y) \ \& \ \theta_A(y, z) \\
&\quad = A(x \longrightarrow y) \ \& \ A(y \longrightarrow x) \ \& \ A(z \longrightarrow y) \ \& \ A(y \longrightarrow z) \\
&\quad \le A((x \longrightarrow y) \otimes (y \longrightarrow z)) \ \& \ A((z \longrightarrow y) \otimes (y \longrightarrow x)) \\
&\quad \le A(x \longrightarrow z) \ \& \ A(z \longrightarrow x) \\
&\quad = \theta_A(x, z);
\end{aligned} \tag{16}$$

(FC4) by Proposition 13, (R2), and (R7),

$$\begin{aligned}
&\theta_A(x \wedge z, y \wedge z) \\
&\quad = A((x \wedge z) \longrightarrow (y \wedge z)) \ \& \ A((y \wedge z) \longrightarrow (x \wedge z)) \\
&\quad = A(((x \wedge z) \longrightarrow y) \\
&\qquad \wedge ((x \wedge z) \longrightarrow z)) \ \& \ A(((y \wedge z) \longrightarrow x) \\
&\qquad\qquad \wedge ((y \wedge z) \longrightarrow z)) \\
&\quad = A((x \wedge z) \longrightarrow y) \ \& \ A((y \wedge z) \longrightarrow x) \\
&\quad \ge A(x \longrightarrow y) \ \& \ A(y \longrightarrow x) \\
&\quad = \theta_A(x, y);
\end{aligned} \tag{17}$$

(FC5) by Proposition 13, (R9), and (R15),

$$\begin{aligned}
&\theta_A(x \longrightarrow z, y \longrightarrow z) \\
&\quad = A((x \longrightarrow z) \longrightarrow (y \longrightarrow z)) \ \& \ A((y \longrightarrow z) \longrightarrow (x \longrightarrow z)) \\
&\quad = A(y \longrightarrow ((x \longrightarrow z) \longrightarrow z)) \ \& \ A(x \longrightarrow ((y \longrightarrow z) \longrightarrow z)) \\
&\quad \ge A(y \longrightarrow x) \ \& \ A(x \longrightarrow y) \\
&\quad = \theta_A(x, y).
\end{aligned} \tag{18}$$

□

Theorem 21. *Let θ, A be a lattice-valued congruence and a lattice-valued filter on L, respectively. Then*

(1) $\theta_{A_\theta} = \theta$;

(2) $A_{\theta_A} = A$.

Thus there is a bijection between $FF(L)$ and $FCon(L)$.

Proof. (1) For all $x, y \in L$, by (FC2)–(FC5), (R6), and (R15),

$$\begin{aligned}
&\theta_{A_\theta}(x, y) \\
&\quad = A_\theta(x \longrightarrow y) \ \& \ A_\theta(y \longrightarrow x) \\
&\quad = \theta(1, x \longrightarrow y) \ \& \ \theta(1, y \longrightarrow x) \\
&\quad \le \theta(1 \longrightarrow y, (x \longrightarrow y) \longrightarrow y) \ \& \ \theta(1 \longrightarrow x, (y \longrightarrow x) \longrightarrow x) \\
&\quad = \theta(y, (x \longrightarrow y) \longrightarrow y) \ \& \ \theta(x, (y \longrightarrow x) \longrightarrow x) \\
&\quad \le \theta(y \wedge x, ((x \longrightarrow y) \longrightarrow y) \\
&\qquad \wedge x) \ \& \ \theta(x \wedge y, ((y \longrightarrow x) \longrightarrow x) \wedge y) \\
&\quad = \theta(x \wedge y, x) \ \& \ \theta(x \wedge y, y) \\
&\quad \le \theta(x, y).
\end{aligned} \tag{19}$$

And by (FC5) and (R7),

$$\begin{aligned}
&\theta_{A_\theta}(x, y) \\
&\quad = \theta(1, x \longrightarrow y) \ \& \ \theta(1, y \longrightarrow x) \\
&\quad = \theta(y \longrightarrow y, x \longrightarrow y) \ \& \ \theta(x \longrightarrow x, y \longrightarrow x) \\
&\quad \ge \theta(y, x) \ \& \ \theta(x, y) \\
&\quad = \theta(x, y).
\end{aligned} \tag{20}$$

(2) For all $x \in L, A_{\theta_A}(x) = \theta_A(1, x) = A(1 \to x) \ \& \ A(x \to 1) = A(x) \ \& \ A(1) = A(x)$. □

Lemma 22. *Let θ be a lattice-valued congruence on L. Then*

(1) *for any $r \in M$, one has $\theta_{[r]} \in Con(L)$;*

(2) *if M is a spatial frame, then for any $r \in P(M)$, $\theta_{(r)} \in Con(L)$.*

Proof. This proof is trivial by the definitions of congruences and lattice-valued congruences. □

Proposition 23. *For any $A \in FF(L), \theta \in FCon(L)$, one has for all $r \in M$,*

(1) $(\theta_A)_{[r]} = \sim_{(A_{[r]})}$;

(2) $(A_\theta)_{[r]} = F_{(\theta_{[r]})}$.

Proof. (1) Consider the following:

$$\begin{aligned}
(\theta_A)_{[r]} &= \{(x, y) \in L \times L \mid \theta_A(x, y) \ge r\} \\
&= \{(x, y) \in L \times L \mid A(x \longrightarrow y) \wedge A(y \longrightarrow x) \ge r\} \\
&= \{(x, y) \in L \times L \mid x \longrightarrow y, y \longrightarrow x \in A_{[r]}\} \\
&= \sim_{(A_{[r]})}.
\end{aligned} \tag{21}$$

(2) Consider

$$\begin{aligned}
(A_\theta)_{[r]} &= \{x \in L \mid A_\theta(x) \ge r\} = \{x \in L \mid \theta(1, x) \ge r\} \\
&= \{x \in L \mid (1, x) \in \theta_{[r]}\} = F_{(\theta_{[r]})}.
\end{aligned} \tag{22}$$

□

Replacing " $\ge$ " by " $\nleq$ " in Proposition 23, we have the following.

Theorem 24. *Let M be a spatial frame. Then $A \in FF(L), \theta \in FCon(L)$, and one has for any $r \in P(M)$,*

(1) $(\theta_A)_{(r)} = \sim_{(A_{(r)})}$;

(2) $(A_\theta)_{(r)} = F_{(\theta_{(r)})}$.

By Theorem 9, Proposition 16, Theorem 21, Proposition 23 and Theorem 24, we have the following.

Corollary 25. *(1) $\theta \in FCon(L)$ if and only if for any $r \in M, \theta_{[r]} \in Con(L)$.*

(2) $\theta \in FCon(L)$ if and only if for any $r \in P(M), \theta_{(r)} \in Con(L)$.

By Corollary 25 and Remark 10, we have the following.

Corollary 26. *Let θ be a lattice-valued congruence on L. Then each of $\theta(a \wedge c, b \wedge d), \theta(a \vee c, b \vee d), \theta(a \otimes c, b \otimes d)$, and $\theta(a \rightarrow c, b \rightarrow d)$ is larger than or equal to $\theta(a,b) \wedge \theta(c,d)$.*

At last, we will give some properties of lattice-valued congruence classes of lattice-valued congruences.

Lemma 27. *Let θ be a lattice-valued congruence on L. Then for any $x, y \in L, \theta(1, x \rightarrow y) \wedge \theta(1, y \rightarrow x) = \theta(x, y)$.*

Proof. It is a corollary of Theorem 21(1). □

Proposition 28. *Let θ be a lattice-valued congruence on L and $x, y \in L$. Then the following four are equivalent.*

(1) $\theta^x = \theta^y$.

(2) $\theta(x, y) = 1$.

(3) $\theta(1, x \rightarrow y) = \theta(1, y \rightarrow x) = 1$.

(4) $\theta^{x \rightarrow y} = \theta^{y \rightarrow x} = \theta^1$.

Proof. Clearly, (2) is equivalent to (3) by Lemma 27.

(1) ⇒ (2): $\theta(x, y) = \theta^x(y) = \theta^y(y) = \theta(y, y) = 1$.

(2) ⇒ (1): for all $z \in L, \theta^x(z) = \theta(x, z) \geq \theta(x, y) \wedge \theta(y, z) = 1 \wedge \theta(y, z) = \theta^y(z)$.

Similarly, $\theta^y(z) \geq \theta^x(z)$ and so $\theta^x = \theta^y$.

Similar to (1) ⇔ (2), we can show that (3) ⇔ (4). □

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