

Research Article

\mathcal{H}_∞ Estimates for Discrete-Time Markovian Jump Linear Systems

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This paper deals with the problem of \mathcal{H}_∞ filtering for discrete-time Markovian jump linear systems. Predicted and filtered estimates are obtained based on the game theory. Both filters are solved through recursive algorithms. The Markovian system considered assumes that the jump parameters are not accessible. Necessary and sufficient conditions are provided to the existence of the filters. A numerical example is provided in order to show the effectiveness of the approach proposed.

1. Introduction

Filtering of Markovian jump linear systems (MJLSs) has been subject of intensive study in the last years. Different and creative approaches to deal with this class of filter have been considered in the literature. One of the main alternatives to solve this kind of problem is based on Kalman filter algorithms, see for instance [1, 2]. Filters proposed in these references are based on linear minimum mean square error estimates of discrete-time MJLS. They present an interesting feature related with recursiveness; however, they are not robust in nature.

When the robustness is relevant to the filtering process and demands an extra performance of the filtering approach, \mathcal{H}_∞ techniques are always considered as one of the best solutions to be adopted. In this way, a closed-loop transfer function from the unknown disturbances to the estimation error is designed in order to satisfy a prescribed \mathcal{H}_∞ -norm constraint. In general, algorithms developed to deduce \mathcal{H}_∞ Markovian filters are based on linear matrix inequalities (LMIs), see for instance [3–12]. In particular, [4, 6] assume that the jump parameter of the Markov chain is not available.

In this paper, we propose \mathcal{H}_∞ filters for discrete-time MJLS which are calculated in terms of recursive algorithms

based on Riccati equations. Following the approach considered in [13], we develop predicted and filtered estimates based on the two-players game theory. The idea of game theory was also adopted in stochastic $\mathcal{H}_2/\mathcal{H}_\infty$ control, see for instance [14].

We define the status of the players in the filtering problems considered in this paper in order to reach an equilibrium between two contradictory objectives. The first player can be interpreted as the maximizer of the estimation cost, whereas the second player tries to find an estimate that brings the quadratic cost to a minimum. A solution exists for a specified γ -level if the resulting cost is positive. We assume in these Markovian filtering problems that the jump parameter is not accessible. The recursiveness of the approach we are proposing is the main advantage of these Markovian filters if compared with the filters aforementioned. As a by-product of this approach, we provide necessary conditions for the existence of them based on only known parameters of the Markovian system.

This paper is organized as follows: in Section 2, we present the problem statement; in Section 3, \mathcal{H}_∞ filters for discrete-time MJLS are presented; and in Section 4, a comparative study, based on numerical example, between the approach we are proposing and the filter developed in [4] is performed.

2. Problem Definition

The \mathcal{H}_∞ recursive filters developed in this paper are based on the following discrete-time MJLS:

$$\begin{aligned} x_{i+1} &= F_{i,\Theta_i} x_i + G_{i,\Theta_i} u_i, \quad i = 0, 1, \dots, \\ y_i &= H_{i,\Theta_i} x_i + D_{i,\Theta_i} w_i, \\ s_i &= L_{i,\Theta_i} x_i + R_{i,\Theta_i} v_i, \end{aligned} \quad (1)$$

where $x_i \in \mathfrak{R}^n$ is the valued state, $y_i \in \mathfrak{R}^m$ is the valued output sequence, $s_i \in \mathfrak{R}^p$ is the valued signal to be estimated, $u_i \in \mathfrak{R}^p$, $w_i \in \mathfrak{R}^q$, and $v_i \in \mathfrak{R}^q$ are random disturbances; Θ_i is a discrete-time Markov chain with finite state space $\{1, \dots, N\}$ and transition probability matrix $P = [p_{jk}]$. We set $\pi_{i,j} := P(\Theta_i = j)$, $F_{i,k} \in \mathfrak{R}^{n \times n}$, $G_{i,k} \in \mathfrak{R}^{n \times p}$, $H_{i,k} \in \mathfrak{R}^{m \times n}$, $D_{i,k} \in \mathfrak{R}^{m \times q}$, $L_{i,k} \in \mathfrak{R}^{p \times n}$, and $R_{i,k} \in \mathfrak{R}^{p \times q}$, $k \in \{1, \dots, N\}$ for $i \geq 0$. The random disturbances $\{u_i\}$, $\{w_i\}$, and $\{v_i\}$ are assumed to be null mean with finite second order moments, independent wide sense stationary sequences mutually independent with covariance matrices equal to the U_i , W_i , and V_i , respectively. $x_0 1_{\{\Theta_0=k\}}$ are random vectors with $\mathbb{E}\{x_0 1_{\{\Theta_0=k\}}\} = \mu_k$ (where $1_{\{\cdot\}}$ denotes Dirac measure) and $\mathbb{E}\{x_0 x_0^T 1_{\{\Theta_0=k\}}\} = V_k$; x_0 and $\{\Theta_i\}$ are independent of $\{u_i\}$, $\{w_i\}$ and $\{v_i\}$. A scalar $\gamma > 0$ and a sequence of sets of observations

$$\{y_0\}, \{y_0, y_1\}, \dots, \{y_0, \dots, y_l\}, \dots \quad (2)$$

are defined to find at each instant l a filtered estimate $s_{l|l}$ of s_l such that

$$\sup_{x_0} \frac{\|s_{0|0} - L_{0,\Theta_0} x_0\|_{V_0^{-1}}^2}{\|x_0 - F_{-1} \bar{x}_0\|_{P_0^{-1}}^2 + \|y_0 - H_{0,\Theta_0} x_0\|_{W_0^{-1}}^2} < \gamma^2, \quad (3)$$

is satisfied for $l = 0$, and

$$\sup_{\{x_i\}_{i=0}^l} \frac{\sum_{i=0}^l \|s_{i|l} - L_{i,\Theta_i} x_i\|_{V_i^{-1}}^2}{\|x_0 - F_{-1} \bar{x}_0\|_{P_0^{-1}}^2 + \sum_{i=0}^l \|y_i - H_{i,\Theta_i} x_i\|_{W_i^{-1}}^2 + \sum_{i=0}^{l-1} \|x_{i+1} - F_{i,\Theta_i} x_i\|_{U_i^{-1}}^2} < \gamma^2 \quad (4)$$

is satisfied for $l > 0$. For the predicted estimate, given $\gamma > 0$ and a sequence of observations (2), the problem is to find at each instant l a prediction $s_{l+1|l}$ of s_{l+1} such that

$$\sup_{x_0} \frac{\|s_{0|-1} - L_{0,\Theta_0} x_0\|_{V_0^{-1}}^2}{\|x_0 - F_{-1} \bar{x}_0\|_{P_0^{-1}}^2} < \gamma^2 \quad (5)$$

is satisfied for $l = -1$ and

$$\sup_{\{x_i\}_{i=0}^{l+1}} \frac{\sum_{i=0}^{l+1} \|s_{i|l} - L_{i,\Theta_i} x_i\|_{V_i^{-1}}^2}{\|x_0 - F_{-1} \bar{x}_0\|_{P_0^{-1}}^2 + \sum_{i=0}^l \|y_i - H_{i,\Theta_i} x_i\|_{W_i^{-1}}^2 + \sum_{i=0}^l \|x_{i+1} - F_{i,\Theta_i} x_i\|_{U_i^{-1}}^2} < \gamma^2 \quad (6)$$

is satisfied for $l \geq 0$. The sequence of solutions $s_{l|l}$ and $s_{l+1|l}$ are outputs of the respective \mathcal{H}_∞ filters. Due to the hybrid nature of this class of system, at each instant of time a new

model of a countable set of models is used to calculate the functionals (4) and (6). In general, to synthesize recursive filtering algorithms for this class of problems is not an easy task. Thanks to the augmented model of (1) proposed in [1] we can redefine these functionals in order to develop \mathcal{H}_∞ recursive filters similar to those we find in the literature for standard state-space systems, without jumps. The augmented model of (1) can be written as follows:

$$\begin{aligned} z_{i+1} &= \mathcal{F}_i z_i + \psi_i, \quad i = 0, 1, \dots \\ y_i &= \mathcal{H}_i z_i + \varphi_i, \\ s_i &= \mathcal{L}_i z_i + \sigma_i, \end{aligned} \quad (7)$$

where the parameter matrices are given by

$$\begin{aligned} \mathcal{F}_i &:= \begin{bmatrix} p_{11} F_{i,1} & \cdots & p_{N1} F_{i,N} \\ \vdots & \ddots & \vdots \\ p_{1N} F_{i,1} & \cdots & p_{NN} F_{i,N} \end{bmatrix}, \\ \mathcal{H}_i &:= [H_{i,1} \quad \cdots \quad H_{i,N}], \\ \mathcal{L}_i &:= [L_{i,1} \quad \cdots \quad L_{i,N}], \end{aligned} \quad (8)$$

whose dimensions are defined by $\mathcal{F}_i \in \mathfrak{R}^{Nn \times Nn}$, $\mathcal{H}_i \in \mathfrak{R}^{m \times Nn}$, and $\mathcal{L}_i \in \mathfrak{R}^{p \times Nn}$; and the state variable is defined as

$$\begin{aligned} z_i &:= [z_{i,1}^T \quad \cdots \quad z_{i,N}^T]^T \in \mathfrak{R}^{Nn}, \\ z_{i,k} &:= x_i 1_{\{\Theta_i=k\}} \in \mathfrak{R}^n. \end{aligned} \quad (9)$$

We define for $i \geq 0$ and $k \in \{1, \dots, N\}$,

$$\begin{aligned} Z_{i,k} &:= \mathbb{E} \{z_{i,k} z_{i,k}^T\} \in \mathfrak{R}^{n \times n}, \\ Z_i &:= \mathbb{E} \{z_i z_i^T\} = \text{diag} [Z_{i,k}] \in \mathfrak{R}^{Nn \times Nn}, \end{aligned} \quad (10)$$

where $Z_{i,k}$ is given by the following recursive equation:

$$\begin{aligned} Z_{i+1,k} &= \sum_{j=1}^N p_{jk} F_{i,j} Z_{i,j} F_{i,j}^T + \sum_{j=1}^N p_{jk} \pi_{i,j} G_{i,j} U_i G_{i,j}^T, \\ Z_{0,k} &= V_k. \end{aligned} \quad (11)$$

We define also in the next lemma, weighting matrices Λ_i , Π_i , and Γ_i as variances of random disturbances given by ψ_i , φ_i , and σ_i , respectively.

Lemma 1. Let ψ_i , φ_i , and σ_i defined by

$$\begin{aligned} \psi_i &:= \mathcal{M}_{i+1} z_i + \vartheta_i, \\ \varphi_i &:= D_{i,\Theta_i} w_i, \\ \sigma_i &:= R_{i,\Theta_i} v_i, \end{aligned} \quad (12)$$

for $i \geq 0$ and $j, k \in \{1, \dots, N\}$, where

$$\begin{aligned} \mathcal{M}_{i+1}^T &:= [\mathcal{M}_{i+1,1}^T \cdots \mathcal{M}_{i+1,N}^T], \\ \mathcal{M}_{i+1,k} &:= \left[(1_{\{\Theta_{i+1}=k\}} - p_{1k}) F_{i,1} 1_{\{\Theta_i=1\}} \cdots \right. \\ &\quad \left. (1_{\{\Theta_{i+1}=k\}} - p_{Nk}) F_{i,N} 1_{\{\Theta_i=N\}} \right], \\ \Theta_i &:= \begin{bmatrix} 1_{\{\theta(i+1)=1\}} G_{i,\Theta_i} \mathbf{u}_i \\ \vdots \\ 1_{\{\theta(i+1)=N\}} G_{i,\Theta_i} \mathbf{u}_i \end{bmatrix}. \end{aligned} \quad (13)$$

The variances of ψ_i , φ_i , and σ_i can be obtained by the following equations:

$$\begin{aligned} \Gamma_i &:= \text{diag} \left[\sum_{j=1}^N p_{jk} F_{i,j} Z_{i,j} F_{i,j}^T \right] - \mathcal{F}_i Z_i \mathcal{F}_i^T \\ &\quad + \text{diag} \left[\sum_{j=1}^N p_{jk} \pi_{i,j} G_{i,j} U_i G_{i,j}^T \right], \\ \Pi_i &:= \mathcal{D}_i \mathcal{D}_i^T, \\ \Lambda_i &:= \mathcal{R}_i \mathcal{R}_i^T, \end{aligned} \quad (14)$$

respectively, where

$$\mathcal{D}_i := [D_{i,1} \pi_{i,1}^{1/2} W_i^{1/2} \cdots D_{i,N} \pi_{i,N}^{1/2} W_i^{1/2}], \quad (15)$$

$$\mathcal{R}_i := [R_{i,1} \pi_{i,1}^{1/2} V_i^{1/2} \cdots R_{i,N} \pi_{i,N}^{1/2} V_i^{1/2}]. \quad (16)$$

Proof. The proof follows the arguments proposed in [15], Chapter 3, and in [1]. \square

3. \mathcal{H}_∞ Estimates for DMJLS

In this section, we propose recursive \mathcal{H}_∞ estimates for discrete-time MJLS (DMJLS). It is known that for standard state-space systems, this kind of filtering approach is difficult to be implemented *online* due to the fact that we do not know the minimum γ for each step of the recursion. In general, it depends on the estimate error variance matrix which should be calculated at the same instant of time. In this sense, the existence condition of this filter is not known *a priori*. To ameliorate this limitation we provide, for this Markovian problem, necessary conditions to tune this parameter depending on the variances of the random disturbances ψ_i , φ_i , and σ_i and on the known parameter matrices of the augmented model of (1) given in (7).

3.1. \mathcal{H}_∞ Filter. Consider (3) and (4) for the augmented model (7):

$$\sup_{z_0} \frac{\|s_{0|0} - \mathcal{L}_0 z_0\|_{\Lambda_0^{-1}}^2}{\|z_0 - \mathcal{F}_{-1} \bar{z}_0\|_{\bar{Z}_0^{-1}}^2 + \|y_0 - \mathcal{H}_0 z_0\|_{\Pi_0^{-1}}^2} < \gamma_a^2, \quad (17)$$

for $l = 0$, and

$$\sup_{\{z_i\}_{i=0}^l} \frac{\sum_{i=0}^l \|s_{i|l} - \mathcal{L}_i z_i\|_{\Lambda_i^{-1}}^2}{\|z_0 - \mathcal{F}_{-1} \bar{z}_0\|_{\bar{Z}_0^{-1}}^2 + \sum_{i=0}^l \|y_i - \mathcal{H}_i z_i\|_{\Pi_i^{-1}}^2 + \sum_{i=0}^{l-1} \|z_{i+1} - \mathcal{F}_i z_i\|_{\Gamma_i^{-1}}^2} < \gamma_a^2, \quad (18)$$

for $l > 0$. We can rewrite this \mathcal{H}_∞ filtering problem in terms of the following optimization problem:

$$\min_{\{z_{i|l}\}_{i=0}^l} \max_{\{s_{i|l}\}_{i=0}^l} J_l^f \left(\{y_i\}_{i=0}^l, \{s_{i|l}\}_{i=0}^l, \{z_{i|l}\}_{i=0}^l \right) > 0, \quad (19)$$

where

$$\begin{aligned} J_0^f &:= \|z_{0|0} - \mathcal{F}_{-1} \bar{z}_0\|_{\bar{Z}_0^{-1}}^2 + \|y_0 - \mathcal{H}_0 z_{0|0}\|_{\Pi_0^{-1}}^2 \\ &\quad - \gamma_a^{-2} \|s_{0|0} - \mathcal{L}_0 z_{0|0}\|_{\Lambda_0^{-1}}^2, \quad l = 0, \end{aligned} \quad (20)$$

$$\begin{aligned} J_l^f &:= \|z_{0|l} - \mathcal{F}_{-1} \bar{z}_0\|_{\bar{Z}_0^{-1}}^2 + \sum_{i=0}^l \|y_i - \mathcal{H}_i z_{i|l}\|_{\Pi_i^{-1}}^2 \\ &\quad + \sum_{i=0}^{l-1} \|z_{i+1|l} - \mathcal{F}_i z_{i|l}\|_{\Gamma_i^{-1}}^2 \end{aligned} \quad (21)$$

$$- \gamma_a^{-2} \sum_{i=0}^l \|s_{i|l} - \mathcal{L}_i z_{i|l}\|_{\Lambda_i^{-1}}^2, \quad l > 0.$$

Notice that it is not necessary to maximize J_l^f over $\{s_{i|l}\}_{i=0}^l$ in the optimization problem (19). The solution of the \mathcal{H}_∞ filter is guaranteed if, and only if, there exists a $\{\hat{s}_{i|l}\}_{i=0}^l$ for which $J_l^f(\{y_i\}_{i=0}^l, \{\hat{s}_{i|l}\}_{i=0}^l, \{z_{i|l}\}_{i=0}^l) > 0$ has a minimum $\{\hat{z}_{i|l}\}_{i=0}^l$. Therefore, in order to deal with only the minimization problem (19), for each $l \geq 0$ it is easy to show that (21) can be rewritten as

$$J_l^f := (\mathbf{U}_l \mathbf{x}_{i|l} - \mathbf{B}_l)^T \mathbf{R}_l (\mathbf{U}_l \mathbf{x}_{i|l} - \mathbf{B}_l), \quad (22)$$

where

$$\mathbf{x}_{i|l} := \begin{bmatrix} z_{i|l} \\ \vdots \\ z_{1|l} \\ z_{0|l} \end{bmatrix}, \quad \mathbf{B}_l := \begin{bmatrix} \mathcal{L}_l \\ \vdots \\ \mathcal{L}_1 \\ \mathcal{L}_0 \end{bmatrix},$$

$$\mathbf{R}_l := \begin{bmatrix} \mathcal{R}_l & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathcal{R}_0 \end{bmatrix},$$

$$\mathbf{U}_l := \begin{bmatrix} \mathcal{E}_l & \mathcal{A}_{l-1} & 0 & 0 & 0 \\ 0 & \mathcal{E}_{l-1} & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \mathcal{A}_1 & 0 \\ 0 & 0 & 0 & \mathcal{E}_1 & \mathcal{A}_0 \\ 0 & 0 & 0 & 0 & \mathcal{E}_0 \end{bmatrix},$$

$$\begin{aligned}\mathcal{E}_i &:= \begin{bmatrix} I \\ \mathcal{H}_i \\ \mathcal{L}_i \end{bmatrix}, \quad \mathcal{R}_i := \begin{bmatrix} \Gamma_{i-1}^{-1} & 0 & 0 \\ 0 & \Pi_i^{-1} & 0 \\ 0 & 0 & -\gamma_a^{-2} \Lambda_i^{-1} \end{bmatrix}, \\ \mathcal{A}_{i-1} &:= \begin{bmatrix} -\mathcal{F}_{i-1} \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{L}_i := \begin{bmatrix} 0 \\ y_i \\ s_{i|l} \end{bmatrix}, \quad 1 \leq i \leq l, \\ \mathbb{Z}_0 &:= \begin{bmatrix} \mathcal{F}_{-1} \bar{z}_0 \\ y_0 \\ s_{0|l} \end{bmatrix}, \\ \Gamma_{-1} &:= \bar{Z}_0.\end{aligned}\quad (23)$$

For all $l \geq 0$, we can conclude that the following recurrent relations are valid:

$$\begin{aligned}\mathcal{R}_l &:= \begin{bmatrix} \mathcal{R}_l & 0 \\ 0 & \mathcal{R}_{l-1} \end{bmatrix}, \quad \mathcal{B}_l := \begin{bmatrix} \mathcal{L}_l \\ \mathcal{B}_{l-1} \end{bmatrix}, \\ \mathcal{X}_{l|l} &:= \begin{bmatrix} z_{l|l} \\ \mathcal{X}_{l-1|l-1} \end{bmatrix}, \quad \mathcal{U}_l := \begin{bmatrix} \mathcal{E}_l & \alpha_{l-1} \\ 0 & \mathcal{U}_{l-1} \end{bmatrix}, \\ \alpha_{l-1} &:= [\mathcal{A}_{l-1} \quad 0 \quad \cdots \quad 0].\end{aligned}\quad (24)$$

According to [16], there exists a minimum for (22) if and only if the positiveness of $\mathcal{U}_l^T \mathcal{R}_l \mathcal{U}_l$ is guaranteed.

Lemma 2. Consider matrices \mathcal{U}_l and \mathcal{R}_l and column vectors \mathcal{B}_l and $\mathcal{X}_{l|l}$ of appropriate dimensions with \mathcal{R}_l symmetric. For any \mathcal{B}_l we have

$$\inf_x J_l^f > -\infty, \quad (25)$$

if and only if $\mathcal{U}_l^T \mathcal{R}_l \mathcal{U}_l \geq 0$ and $\text{Ker}(\mathcal{U}_l^T \mathcal{R}_l \mathcal{U}_l) \subset \text{Ker}(\mathcal{R}_l \mathcal{U}_l)$. If the minimum is attained, it is unique if and only if $\mathcal{U}_l^T \mathcal{R}_l \mathcal{U}_l > 0$ and the optimal solution is given by $\hat{\mathcal{X}}_{l|l} = (\mathcal{U}_l^T \mathcal{R}_l \mathcal{U}_l)^{-1} \mathcal{U}_l^T \mathcal{R}_l \mathcal{B}_l$.

With this fundamental lemma in mind, we can obtain an existence condition for the filter we are proposing in the following, based on the known parameter matrices of the Markovian model. For $l > 0$, the term $\mathcal{U}_l^T \mathcal{R}_l \mathcal{U}_l$ of $\hat{\mathcal{X}}_{l|l}$ can be written as

$$\begin{bmatrix} \mathcal{E}_l^T \mathcal{R}_l \mathcal{E}_l & \mathcal{E}_l^T \mathcal{R}_l \alpha_{l-1} \\ \alpha_{l-1}^T \mathcal{R}_l \mathcal{E}_l & \mathcal{U}_{l-1}^T \mathcal{R}_{l-1} \mathcal{U}_{l-1} + \alpha_{l-1}^T \mathcal{R}_l \alpha_{l-1} \end{bmatrix}, \quad (26)$$

and a necessary condition for $\mathcal{U}_l^T \mathcal{R}_l \mathcal{U}_l > 0$ is the positiveness of $\mathcal{E}_l^T \mathcal{R}_l \mathcal{E}_l$, which in terms of the augmented Markovian model of (1) can be established as follows:

$$\Gamma_{l-1}^{-1} + \mathcal{H}_l^T \Pi_l^{-1} \mathcal{H}_l - \gamma_a^{-2} \mathcal{L}_l^T \Lambda_l^{-1} \mathcal{L}_l > 0. \quad (27)$$

Remark 3. A lower bound for γ_a can be calculated through (27) based on the augmented model (7) and Lemma 1. Notice that (27) is a natural and interesting extension of the filtering

of standard state-space systems [13], without jumps, to the DMJLS we are dealing with in this paper.

Now we are in a position to deduce the \mathcal{H}_∞ recursive filtered estimate for DMJLS. The next theorem provides the solution for this problem based on the augmented model (7) and on the sequence of measurements (2) of the original Markovian system (1).

Theorem 4. Consider the augmented Markovian model (7). There exists a recursive \mathcal{H}_∞ filter for this system, defined by the following recursive equations:

$$\begin{aligned}\bar{Z}_{0|0}^{-1} &:= \bar{Z}_0^{-1} + \mathcal{H}_0^T \Pi_0^{-1} \mathcal{H}_0 - \gamma_a^{-2} \mathcal{L}_0^T \Lambda_0^{-1} \mathcal{L}_0, \\ \bar{Z}_{l|l}^{-1} &:= \left(\Gamma_{l-1} + \mathcal{F}_{l-1} \bar{Z}_{l-1|l-1} \mathcal{F}_{l-1}^T \right)^{-1} \\ &\quad + \mathcal{H}_l^T \Pi_l^{-1} \mathcal{H}_l - \gamma_a^{-2} \mathcal{L}_l^T \Lambda_l^{-1} \mathcal{L}_l,\end{aligned}\quad (28)$$

$$\begin{aligned}\bar{z}_{0|0} &:= \left(P_{0|0}^{-1} + \gamma_a^{-2} \mathcal{L}_0^T \Lambda_0^{-1} \mathcal{L}_0 \right)^{-1} \\ &\quad \times \left(\bar{Z}_0^{-1} \mathcal{F}_{-1}^T \bar{z}_0 + \mathcal{H}_0^T \Pi_0^{-1} y_0 \right), \\ \bar{z}_{l|l} &:= \left(\bar{Z}_{l|l}^{-1} + \gamma_a^{-2} \mathcal{L}_l^T \Lambda_l^{-1} \mathcal{L}_l \right)^{-1} \\ &\quad \times \left(X_{l-1}^{-1} \mathcal{F}_{l-1} \bar{z}_{l-1|l-1} + \mathcal{H}_l^T \Pi_l^{-1} y_l \right),\end{aligned}\quad (29)$$

$$\begin{aligned}X_{l-1} &:= \Gamma_{l-1} + \mathcal{F}_{l-1} \bar{Z}_{l-1|l-1} \mathcal{F}_{l-1}^T, \\ \hat{s}_{l|l} &:= \mathcal{L}_l \bar{z}_{l|l},\end{aligned}\quad (30)$$

if and only if $\bar{Z}_{l|l} > 0$, $l = 0, 1, \dots$, and the attenuation level γ_a is positive.

Proof. This proof follows standard arguments used to deduce Kalman filters and basically consists in checking the positiveness of (26) and in finding the equation of the filter through the Lemma 2. In order to have $\mathcal{U}_l^T \mathcal{R}_l \mathcal{U}_l > 0$, we can rewrite this term as (26). The (2, 2) sub block of (26) must be positive definite. Assuming that the positiveness of $\mathcal{U}_{l-1}^T \mathcal{R}_{l-1} \mathcal{U}_{l-1}$ is guaranteed and

$$\alpha_{l-1}^T \mathcal{R}_l \alpha_{l-1} = \begin{bmatrix} \mathcal{F}_{l-1}^T \Gamma_{l-1}^{-1} \mathcal{F}_{l-1} & 0 \\ 0 & 0 \end{bmatrix} \geq 0, \quad (31)$$

the (2, 2) term of (26) is positive definite and $\mathcal{U}_l^T \mathcal{R}_l \mathcal{U}_l > 0$ if and only if the Schur complement of the (2, 2) block,

$$M_{l|l}^{-1} := \mathcal{E}_l^T \left(\mathcal{R}_l^{-1} + \alpha_{l-1} (\mathcal{U}_{l-1}^T \mathcal{R}_{l-1} \mathcal{U}_{l-1})^{-1} \alpha_{l-1}^T \right)^{-1} \mathcal{E}_l, \quad (32)$$

is positive definite. For $l - 1$, $M_{l-1|l-1}$ is the (1, 1) block of $(\mathcal{U}_{l-1}^T \mathcal{R}_{l-1} \mathcal{U}_{l-1})^{-1}$. With α_{l-1} defined in (24), we obtain

$$M_{l|l}^{-1} := \mathcal{E}_l^T \left(\mathcal{R}_l^{-1} + \mathcal{A}_{l-1} M_{l-1|l-1} \mathcal{A}_{l-1}^T \right)^{-1} \mathcal{E}_l. \quad (33)$$

Considering $\bar{Z}_{l|l} := M_{l|l}$ in terms of the original data (23), we obtain (28). From Lemma 2 we have the solution for the minimization of (22). Based on the recurrent relations (23)

and (24), considering $\alpha_{l-1} \mathfrak{X}_{l-1|j} = \mathcal{A}_{l-1} x_{l-1|j}$ for $j \geq l-1$, and introducing

$$\begin{bmatrix} \tilde{Z}_{l|l} & P_{12,l} \\ P_{21,l} & P_{22,l} \end{bmatrix} := \begin{bmatrix} \mathcal{E}_l^T \mathcal{R}_l \mathcal{E}_l & \mathcal{E}_l^T \mathcal{R}_l \alpha_{l-1} \\ \alpha_{l-1}^T \mathcal{R}_l \mathcal{E}_l & \mathbf{U}_{l-1}^T \mathfrak{R}_{l-1} \mathbf{U}_{l-1} + \alpha_{l-1}^T \mathcal{R}_l \alpha_{l-1} \end{bmatrix}^{-1} \quad (34)$$

holds (29) taking into account that $s_{l|l} = \hat{s}_{l|l} := \mathcal{L}_l \hat{z}_{l|l}$. \square

3.2. \mathcal{H}_∞ Predictor. In this subsection, we present the recursive \mathcal{H}_∞ predictor filter for DMJLS. The original \mathcal{H}_∞ problem (5)-(6) can be rewritten in terms of the following optimization problem:

$$\min_{\{z_{i|l}\}_{i=0}^{l+1}} \max_{\{s_{i|l}\}_{i=0}^{l+1}} J_l^p \left(\{y_i\}_{i=0}^l, \{s_{i|l}\}_{i=0}^{l+1}, \{z_{i|l}\}_{i=0}^{l+1} \right) > 0, \quad (35)$$

where

$$\begin{aligned} J_{-1}^p &:= \|z_{0|-1} - \mathcal{F}_{-1} \bar{z}_0\|_{\bar{Z}_0}^2 \\ &\quad - \gamma_a^{-2} \|s_{0|-1} - \mathcal{L}_0 z_{0|-1}\|_{\Lambda_0^{-1}}^2, \quad l = -1, \\ J_l^p &:= \|z_{0|l} - \mathcal{F}_{-1} \bar{z}_0\|_{\bar{Z}_0}^2 + \sum_{i=0}^l \|y_i - \mathcal{H}_i z_{i|l}\|_{\Pi_i^{-1}}^2 \\ &\quad + \sum_{i=0}^l \|z_{i+1|l} - \mathcal{F}_i z_{i|l}\|_{\Gamma_i^{-1}}^2 \\ &\quad - \gamma_a^{-2} \sum_{i=0}^{l+1} \|s_{i|l} - \mathcal{L}_i z_{i|l}\|_{\Lambda_i^{-1}}^2, \quad l \geq 0. \end{aligned} \quad (36)$$

The \mathcal{H}_∞ predictor filter exists at the instant l if and only if there exists a sequence $\{\hat{s}_{i|l}\}_{i=0}^{l+1}$ such that $J_l^p(\{y_i\}_{i=0}^l, \{\hat{s}_{i|l}\}_{i=0}^{l+1}, \{z_{i|l}\}_{i=0}^{l+1})$ has a minimum $\{\hat{z}_{i|l}\}_{i=0}^{l+1}$ for which

$$J_l^p \left(\{y_i\}_{i=0}^l, \{\hat{s}_{i|l}\}_{i=0}^{l+1}, \{\hat{z}_{i|l}\}_{i=0}^{l+1} \right) > 0. \quad (37)$$

The prediction optimization problem is equivalent in nature to the filter problem, aforementioned. In the next theorem we present the \mathcal{H}_∞ predictor filter whose proof follows the line of the filtered version.

Theorem 5. *The Markovian \mathcal{H}_∞ prediction problem (35) is solvable if, and only if, $\tilde{Z}_{l+1|l}^{-1} - \gamma_a^{-2} \mathcal{L}_{l+1}^T \Lambda_{l+1}^{-1} \mathcal{L}_{l+1} > 0$, where the sequence $\{\tilde{Z}_{l+1|l}\}$ and the predictor filter are calculated by the recursions*

$$\begin{aligned} \tilde{Z}_{0|-1} &:= \tilde{Z}_0, \\ \tilde{Z}_{l+1|l} &:= \Gamma_l + \mathcal{F}_l \tilde{Z}_{l|l-1} \mathcal{F}_l^T - \mathcal{F}_l \tilde{Z}_{l|l-1} \begin{bmatrix} \mathcal{H}_l \\ \mathcal{L}_l \end{bmatrix}^T \\ &\quad \times W_{\infty,l}^{-1} \begin{bmatrix} \mathcal{H}_l \\ \mathcal{L}_l \end{bmatrix} \tilde{Z}_{l|l-1} \mathcal{F}_l, \end{aligned}$$

$$\begin{aligned} W_{\infty,l} &:= \begin{bmatrix} \Pi_l & 0 \\ 0 & -\gamma_a^2 \Lambda_l \end{bmatrix} \\ &\quad + \begin{bmatrix} \mathcal{H}_l \\ \mathcal{L}_l \end{bmatrix} \tilde{Z}_{l|l-1} \begin{bmatrix} \mathcal{H}_l^T & \mathcal{L}_l^T \end{bmatrix}, \\ \hat{z}_{l+1|l} &:= \mathcal{F}_l \hat{z}_{l|l-1} \\ &\quad + \mathcal{F}_l M_{l|l-1} \mathcal{H}_l^T \left(\Pi_l + \mathcal{H}_l M_{l|l-1} \mathcal{H}_l^T \right)^{-1} \\ &\quad \times (y_l - \mathcal{H}_l \hat{z}_{l|l-1}), \\ M_{l|l-1}^{-1} &:= \tilde{Z}_{l|l-1}^{-1} - \gamma_a^{-2} \mathcal{L}_l^T \Lambda_l^{-1} \mathcal{L}_l, \\ \hat{s}_{l+1|l} &:= \mathcal{L}_{l+1} \hat{z}_{l+1|l}. \end{aligned} \quad (38)$$

In order to recover the original predicted and filtered state estimates of System (1), we calculate the following summations:

$$\hat{x}_{i|i-1} = \sum_{j=1}^N \hat{z}_{i,j|i-1}, \quad \hat{x}_{i|i} = \sum_{j=1}^N \hat{z}_{i,j|i}, \quad (39)$$

where $\hat{z}_{i|i-1} := [\hat{z}_{i,1|i-1}^T \cdots \hat{z}_{i,N|i-1}^T]^T \in \mathfrak{R}^{Nn}$ and $\hat{z}_{i,k|i-1} := \hat{x}_{i|i-1} \mathbf{1}_{\{\Theta_i=k\}} \in \mathfrak{R}^n$. $\hat{x}_{i|i}$ is defined in the same way.

Remark 6. To establish the stability of the stationary \mathcal{H}_∞ prediction filter, it is assumed that all matrices of (1) and the transition probabilities p_{jk} are time invariant; System (1) is mean square stable (MSS) and its Markov chain $\{\Theta_i\}$ is ergodic. There exists a unique positive-definite solution \tilde{Z} (with $i \rightarrow \infty$) to the algebraic Riccati equation

$$\tilde{Z} := \Gamma + \mathcal{F} \tilde{Z} \mathcal{F}^T - \mathcal{F} \tilde{Z} \begin{bmatrix} \mathcal{H} \\ \mathcal{L} \end{bmatrix}^T W_{\infty}^{-1} \begin{bmatrix} \mathcal{H} \\ \mathcal{L} \end{bmatrix} \tilde{Z} \mathcal{F}, \quad (40)$$

$$W_{\infty} := \begin{bmatrix} \Pi & 0 \\ 0 & -\gamma_a^2 \Lambda \end{bmatrix} + \begin{bmatrix} \mathcal{H} \\ \mathcal{L} \end{bmatrix} \tilde{Z} \begin{bmatrix} \mathcal{H}^T & \mathcal{L}^T \end{bmatrix},$$

providing that $\tilde{Z}^{-1} - \gamma_a^{-2} \mathcal{L}^T \Lambda^{-1} \mathcal{L} > 0$, for any γ_a fixed, which guarantees that $W_{\infty} > 0$ and thus the above inverse is well defined. Following the guidelines defined in [1], the stability of the predictor filter is assured with

$$r_\sigma \left(\mathcal{F} - \mathcal{F} M \mathcal{H}^T (\Pi + \mathcal{H} M \mathcal{H}^T)^{-1} \mathcal{H} \right) < 1, \quad (41)$$

where $r_\sigma(\cdot)$ denotes spectral radius of the dynamic matrix of the filter with stationary gain. The asymptotic stability of the filtered version can also be assured following this line of argumentation.

Remark 7. Similar to the filtered case, we can also find a necessary existence condition to the Markovian \mathcal{H}_∞ prediction filter given by

$$\begin{aligned} \tilde{Z}_0^{-1} - \gamma_a^{-2} \mathcal{L}_0^T \Lambda_0^{-1} \mathcal{L}_0 &> 0, \quad l = -1, \\ \Gamma_l^{-1} - \gamma_a^{-2} \mathcal{L}_{l+1}^T \Lambda_{l+1}^{-1} \mathcal{L}_{l+1} &> 0, \quad l \geq 0. \end{aligned} \quad (42)$$

Remark 8. If we consider $\gamma_a \rightarrow \infty$, the \mathcal{H}_∞ filter proposed in this paper reduces to the Kalman filter for MJLS proposed in [1]. For the case with no jumps ($N = 1$), the predicted and the filtered \mathcal{H}_∞ estimates reduce to the \mathcal{H}_∞ filters for state-space systems given in [17].

4. Numerical Example

In this section, we compare the filtering approach proposed with the filter developed in [4]. With two Markovian states, the model (1) is defined based on the following parameters:

$$\begin{aligned} P &= \begin{bmatrix} 0.9 & 0.1 \\ 0.9 & 0.1 \end{bmatrix}, & F_1 &= \begin{bmatrix} 0.7 & 0 \\ 0.1 & 0.1 \end{bmatrix}, \\ F_2 &= \begin{bmatrix} 0.6 & 0 \\ 0.1 & 0.2 \end{bmatrix}, \\ G_1 = G_2 &= \begin{bmatrix} 0.8731 & 0 \\ 0 & 0.2089 \end{bmatrix}, \\ H_1 = H_2 &= \begin{bmatrix} 0.1 & 0 \end{bmatrix}, \\ D_1 = D_2 &= \begin{bmatrix} 0.008 & 0 \end{bmatrix}, \\ L_1 = L_2 &= \begin{bmatrix} 0.5 & 0 \end{bmatrix}, \\ R_1 = R_2 &= \begin{bmatrix} 0.1 & 0.3 \end{bmatrix}. \end{aligned} \quad (43)$$

We compare both filters in the predicted form, written in the following way:

$$\begin{aligned} \hat{z}_{l+1|l} &= \mathcal{A}_l \hat{z}_{l|l-1} + \mathcal{B}_l y_l, \\ \hat{s}_{l+1|l} &= \mathcal{L}_{l+1} \hat{z}_{l+1|l}, \end{aligned} \quad (44)$$

where

$$\begin{aligned} \mathcal{A}_l &= \mathcal{F}_l - \mathcal{F}_l M_{l|l-1} \mathcal{H}_l^T (\Pi_l + \mathcal{H}_l M_{l|l-1} \mathcal{H}_l^T)^{-1} \mathcal{H}_l, \\ \mathcal{B}_l &= \mathcal{F}_l M_{l|l-1} \mathcal{H}_l^T (\Pi_l + \mathcal{H}_l M_{l|l-1} \mathcal{H}_l^T)^{-1}. \end{aligned} \quad (45)$$

The filter of [4] is considered in the strictly proper form as

$$\begin{aligned} x_f(k+1) &= A_f x_f(k) + B_f y(k), \\ z_f(k) &= C_f x_f(k), \end{aligned} \quad (46)$$

whose solution is given in terms of linear matrix inequalities. We show in Figure 1 the root-mean-square errors (rms) of both filters. We performed 1000 Monte Carlo simulations from $i = 0, \dots, 6$ with the values of Θ_i generated randomly. The initial condition x_0 is considered Gaussian with mean $[0.196 \ 0.295]^T$ and variance $\begin{bmatrix} 0.0384 & 0.0578 \\ 0.0578 & 0.870 \end{bmatrix}$; $\Theta_i \in \{1, 2\}$, u_i , w_i , and v_i are independent sequences of noises with U_i , W_i , and V_i identity matrices with appropriate dimensions, $\pi_1(0) = 0.05$, and $\pi_2(0) = 0.95$. We obtained $\gamma_a = 1.9523$ for the filter

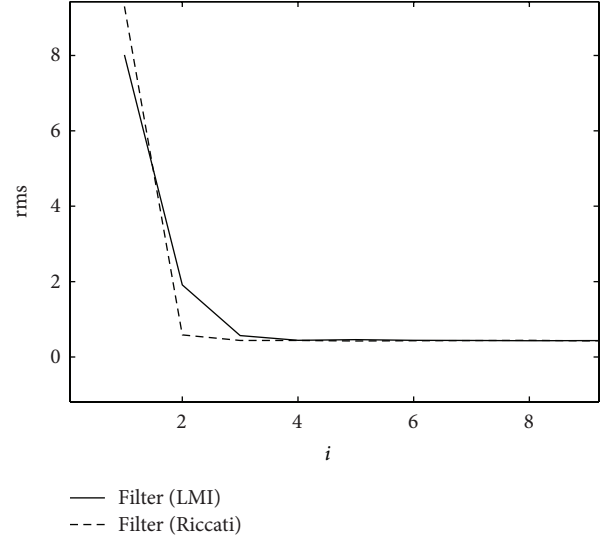


FIGURE 1: Root-mean-square errors for \mathcal{H}_∞ filters based on Riccati recursive equation and linear matrix inequalities.

proposed in this paper and $\gamma = 0.2886$ for the filter proposed in [4]. The parameter matrices of our filter were computed as

$$\mathcal{A}_l = \begin{bmatrix} 0.0116 & 0 & -0.0784 & 0 \\ 0.0003 & 0.09 & 0.0003 & 0.18 \\ 0.0013 & 0 & -0.0087 & 0 \\ 0 & 0.01 & 0 & 0.02 \end{bmatrix}, \quad (47)$$

$$\mathcal{B}_l = \begin{bmatrix} 6.1843 \\ 0.8967 \\ 0.6871 \\ 0.0996 \end{bmatrix}, \quad \mathcal{L}_{l+1} = \begin{bmatrix} 0.5 & 0 & 0.5 & 0 \end{bmatrix},$$

and for the filter of [4] were computed as

$$\begin{aligned} A_f &= \begin{bmatrix} -0.2154 & -0.0122 \\ -0.0342 & -0.3654 \end{bmatrix}, \\ B_f &= \begin{bmatrix} 9.4196 \\ 2.4656 \end{bmatrix}, & C_f &= \begin{bmatrix} 0.4790 & 0.0962 \end{bmatrix}. \end{aligned} \quad (48)$$

In spite of the smaller γ provided by the filter of [4], the rms of both filters are equivalent. This difference in γ is due to the fact that the functionals of both approaches are different in nature. However, if we consider the same numerical value of all parameters but with matrices R_1 and R_2 multiplied by 10^6 , we obtain for the proposed recursive filter $\gamma_a = 0.001$. On the other hand, we obtain an infeasible solution with the LMI proposed in Section IV of [4] (computed through Matlab 7.4.0). The mode-independent filter provided in [4] deals with only sufficient conditions. Considering *offline* computations, our approach provides necessary and sufficient conditions for the existence of the filter.

It is important to recall that the proposed approach enables all parameter matrices of (1) to be time varying. It also provides a practical estimate for the minimum admissible γ through the necessary condition (27), which depends on only known parameters of the Markovian system.

5. Conclusion

This paper developed \mathcal{H}_∞ predicted and filtered estimates for DMJLS. They were deduced based on the assumption that jump parameter is not accessible. The numerical example showed the effectiveness of this approach. Through an augmented model of the standard Markovian system, these filters were deduced based on the game theory. It allows us to deal with recursive algorithms to solve this kind of filtering problem where the parameter matrices can be time varying for each operation mode of the Markovian system. As future works, we intend to solve nonlinear filtering and control problems for DMJLS based on the problems raised by [12, 18].

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