Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2014, Article ID 162472, 8 pages
http://dx.doi.org/10.1155/2014/162472

*Research Article*

# Several Dynamical Properties for a Nonlinear Shallow Water Equation

**Ls Yong and Haibo Yan**

*Department of Mathematics, Southwestern University of Finance and Economics, Chengdu 611130, China*

Correspondence should be addressed to Haibo Yan; wengxiang00@163.com

Received 18 April 2014; Accepted 8 May 2014; Published 22 May 2014

Academic Editor: Gongnan Xie



A nonlinear third order dispersive shallow water equation including the Degasperis-Procesi model is investigated. The existence of weak solutions for the equation is proved in the space $L^1(R) \cap BV(R)$ under certain assumptions. The Oleinik type estimate and $L^{2N}(R)$ ($N$ is a natural number) estimate for the solution are obtained.

## 1. Introduction

Constantin and Lannes [1] derived the shallow water wave equation

$$\begin{aligned} u_t + u_x + \frac{3}{2}\rho u u_x + \mu\left(\alpha u_{xxx} + \beta u_{xxt}\right) \\ = \rho\mu\left(\gamma u_x u_{xx} + \delta u u_{xxx}\right), \end{aligned} \tag{1}$$

where the constants $\alpha$, $\beta$, $\gamma$, $\delta$, $\rho$, and $\mu$ satisfy certain conditions. As stated in [1], using suitable mathematical transformations, one can turn (1) into the form

$$u_t - u_{xxt} + 2ku_x + muu_x = au_x u_{xx} + buu_{xxx}, \tag{2}$$

where $a$, $b$, $k$, and $m$ are constants. Clearly, (2) contains both the Camassa-Holm and Degasperis-Procesi models.

The aim of this paper is to investigate the existence of weak solutions for the special case of (2). Namely, we study the shallow water equation

$$\begin{aligned} \partial_t u - \partial_{txx}^3 u + mu\partial_x u = 3\partial_x u \partial_{xx}^2 u + u\partial_{xxx}^3 u, \\ (t,x) \in R_+ \times R, \end{aligned} \tag{3}$$

where $m > 0$ is a constant. Letting $y = u - \partial_{xx}^2 u$, $v = \left(m - \partial_{xx}^2\right)^{-1} u$ and using (3), we derive the conservation law

$$\begin{aligned} \int_R yv\,dx &= \int_R \frac{1+\xi^2}{m+\xi^2}|\widehat{u}(\xi)|^2 d\xi \\ &= \int_R \frac{1+\xi^2}{m+\xi^2}|\widehat{u}_0(\xi)|^2 d\xi \sim \|u_0\|_{L^2(R)}, \end{aligned} \tag{4}$$

where $u_0 = u(0,x)$. In fact, the conservation law (4) takes an important role in our further investigations of (3).

For $m = 4$, (3) reduces to the Degasperis-Procesi equation [2]

$$\begin{aligned} \partial_t u - \partial_{txx}^3 u + 4u\partial_x u = 3\partial_x u \partial_{xx}^2 u + u\partial_{xxx}^3 u, \\ (t,x) \in R_+ \times R, \end{aligned} \tag{5}$$

which has been studied by many scholars (see [3–5]). Lundmark and Szmigielski [6] developed an inverse scattering approach for computing $n$-peakon solutions to (5). The traveling wave solutions of (5) were investigated in Vakhnenko and Parkes [5]. Holm and Staley [7] studied stability of solitons and peakons numerically. Lin and Liu [8] proved the stability of peakons for the Degasperis-Procesi equation (5) under certain assumptions. The precise blowup scenario result, a blowup result, and the global existence of strong solutions and global weak solutions to (5) can be found in [9]. Matsuno [10] studied multisoliton solutions and their peakon limits.

Analogous to the case of the Camassa-Holm equation, Henry [11] and Mustafa [12] showed that smooth solutions to (5) have infinite speed of propagation. For other methods to handle the problems relating to various dynamic properties of the Degasperis-Procesi equation and other shallow water equations, the reader is referred to [13–15] and the references therein.

Coclite and Karlsen [16] established the existence, uniqueness, and $L^1(R)$ stability of entropy weak solutions belonging to the class $L^1(R) \cap BV(R)$ for (5). They obtained existence of at least one weak solution satisfying a restricted set of entropy inequalities in the space $L^2(R) \cap L^4(R)$ and extended these results to a class of generalized Degasperis-Procesi equations in [16].

Motivated by the desire to extend parts of the results presented in Coclite and Karlsen [16], we consider (3) with its Cauchy problem in the form

$$\begin{aligned} \partial_t u - \partial^3_{txx} u &= -\partial_x \left(\frac{m}{2}u^2\right) + 3\partial_x u \partial^2_{xx} u + u\partial^3_{xxx} u \\ &= -\left(\frac{m}{2}u^2\right)_x + \frac{1}{2}\partial^3_{xxx} u^2, \\ u(0,x) &= u_0(x), \end{aligned} \tag{6}$$

which is equivalent to

$$\begin{aligned} u_t + uu_x &= -\frac{m-1}{2}\Lambda^{-2}\left(u^2\right)_x, \\ u(0,x) &= u_0(x), \end{aligned} \tag{7}$$

where $m > 0$ is a constant and $\Lambda = (1 - \partial_x^2)^{1/2}$.

The objective of this paper is to study (3). We investigate the existence of weak solutions in the space $L^1(R) \cap BV(R)$ under certain conditions. Several dynamical properties such as Oleinik type estimate and $L^{2N}(R)$ ($N$ is a natural number) are obtained. As (3) includes the Degasperis-Procesi equation (5), parts of results presented in [16] are extended. Here we should mention that the generalized Degasperis-Procesi equation discussed in [16] does not include the model (3). We state that the ideas and approaches to prove our main results come from those in [16].

The rest of this paper is organized as follows. Section 2 establishes the $L^2$, $BV$, and $L^\infty$ estimates for the viscous approximations of problem (6). The main result is given in Section 3.

## 2. Viscous Approximations and Estimates

Firstly, we give some notations.

Set $R_+ = [0, +\infty)$. The space of all infinitely differentiable functions $\phi(t,x)$ with compact support in $R_+ \times R$ is denoted by $C_0^\infty$. We let $L^p = L^p(R)$ $(1 \le p < +\infty)$ be the space of all measurable functions $h$ such that $\|h\|^p_{L^p} = \int_R |h(t,x)|^p dx < \infty$. We define $L^\infty = L^\infty(R)$ with the standard norm $\|h\|_{L^\infty} = \inf_{m(e)=0}\sup_{x\in R\setminus e}|h(t,x)|$. For any real number $s$, we let $H^s = H^s(R)$ denote the Sobolev space with the norm defined by

$$\|h\|_{H^s} = \left(\int_R \left(1 + |\xi|^2\right)^s \left|\widehat{h}(t,\xi)\right|^2 d\xi\right)^{1/2} < \infty, \tag{8}$$

where $\widehat{h}(t,\xi) = \int_R e^{-ix\xi} h(t,x)dx$.

For $T > 0$ and nonnegative number $s$, let $C([0,T); H^s(R))$ denote the Frechet space of all continuous $H^s$-valued functions on $[0,T)$.

Defining

$$\phi(x) = \begin{cases} e^{1/(x^2-1)}, & |x| < 1, \\ 0, & |x| \ge 1 \end{cases} \tag{9}$$

and letting $\phi_\varepsilon(x) = \varepsilon^{-1/4}\phi(\varepsilon^{-1/4}x)$ with $0 < \varepsilon < 1/4$ and $u_{0,\varepsilon} = \phi_\varepsilon \star u_0$, we know that $u_{0,\varepsilon} \in C^\infty$ for any $u_0 \in H^s$ with $s \ge 0$.

For simplicity, throughout this paper, we let $c_0$ denote any positive constant, which is independent of parameter $\varepsilon$ and time $t$.

To establish the existence of solutions to the Cauchy problem (6), we will analyze the limiting behavior of a sequence of smooth functions $\{u_\varepsilon\}_{\varepsilon>0}$, where each function $u_\varepsilon$ satisfies the viscous problem

$$\begin{aligned} &\partial_t u_\varepsilon - \partial^3_{txx} u_\varepsilon + m u_\varepsilon \partial_x u_\varepsilon \\ &\quad = 3\partial_x u_\varepsilon \partial^2_{xx} u_\varepsilon + u_\varepsilon \partial^3_{xxx} u_\varepsilon + \varepsilon \partial^2_{xx} u_\varepsilon - \varepsilon \partial^4_{xxxx} u_\varepsilon, \\ &\qquad\qquad (t,x) \in R_+ \times R, \\ &u_\varepsilon(0,x) = u_{0,\varepsilon}(x), \quad x \in R, \end{aligned} \tag{10}$$

which is equivalent to the parabolic-elliptic system

$$\begin{aligned} \partial_t u_\varepsilon + \partial_x\left(\frac{u_\varepsilon^2}{2}\right) + \partial_x P_\varepsilon &= \varepsilon \partial_{xx} u_\varepsilon, \\ P_\varepsilon - \partial^2_{xx} P_\varepsilon &= \frac{m-1}{2} u_\varepsilon^2, \\ u_\varepsilon(0,x) &= u_{0,\varepsilon}(x). \end{aligned} \tag{11}$$

From the second identity of (11), we get

$$P_\varepsilon(t,x) = \frac{m-1}{4}\int_R e^{|x-y|} u_\varepsilon^2(t,y)\, dy. \tag{12}$$

*2.1. $L^2$ Estimates and Several Consequences.* Several properties for the smooth function $u_{0,\varepsilon}$ are given in the following Lemma.

**Lemma 1.** *The following estimates hold for any $\varepsilon$ with $0 < \varepsilon < 1/4$ and $s \geq 0$:*

$$\begin{gathered}\|u_{0,\varepsilon}\|_{L^2(R)} \leq c_0\|u_0\|_{L^2(R)},\\ \|u_{0,\varepsilon}\|_{L^1(R)} \leq c_0\|u_0\|_{L^1(R)},\\ \|u_{0,\varepsilon}\|_{BV(R)} \leq c_0\|u_0\|_{BV(R)},\\ \|u_{0,\varepsilon}\|_{L^p(R)} \leq c_0\|u_0\|_{L^p(R)} \quad \text{for } 1 \leq p \leq \infty,\\ u_{0,\varepsilon} \longrightarrow u_0 \quad (\varepsilon \longrightarrow 0) \text{ in } L^p(R) \text{ for } 1 \leq p \leq \infty,\\ \|u_{0,\varepsilon}\|_{H^q} \leq c_0\|u_0\|_{H^s}, \quad \text{if } q \leq s,\end{gathered} \tag{13}$$

*where $c_0$ is a constant independent of $\varepsilon$.*

The proof of Lemma 1 is similar to that of Lemma 5 presented in [14]. Here we omit it.

**Lemma 2.** *Provided that $u_0 \in L^2(R)$, for any fixed $\varepsilon > 0$, there exists a unique global smooth solution $u_\varepsilon = u_\varepsilon(t,x)$ to the Cauchy problem (11) belonging to $C([0,\infty); H^s(R))$ with $s \geq 0$.*

*Proof.* We omit the proof since it is similar to the one found in [16] or [17] by using $u_{0,\varepsilon} \in C^\infty(R)$. □

**Lemma 3.** *Assume that $u_0 \in L^2(R)$ holds and $u_\varepsilon$ is a solution of problem (10). Then, the following bounds hold for any $t \geq 0$:*

$$\|u_\varepsilon\|_{L^2} \leq c_0\|u_0\|_{L^2}, \tag{14}$$

$$\sqrt{\varepsilon}\|\partial_x u_\varepsilon\|_{L^2} \leq c_0\|u_0\|_{L^2}, \tag{15}$$

*where $c_0$ is a positive constant independent of $\varepsilon$ and $t$.*

*Proof.* Letting

$$m v_\varepsilon - \partial_{xx}^2 v_\varepsilon = u_\varepsilon \tag{16}$$

derives

$$v_\varepsilon = \left(m - \partial_{xx}^2\right)^{-1} u_\varepsilon. \tag{17}$$

Multiplying the first equation of problem (11) by $v_\varepsilon - \partial_{xx}^2 v_\varepsilon$ and integrating over $R$ yield

$$\begin{aligned}&\int_R \partial_t u_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon\right) dx - \varepsilon \int_R \partial_{xx}^2 u_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon\right) dx\\ &\quad = -\int_R u_\varepsilon \partial_x u_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon\right) dx\\ &\qquad - \int_R \partial_x P_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon\right) dx.\end{aligned} \tag{18}$$

For the left-hand side of this identity, using (16), we get

$$\begin{aligned}&\int_R \partial_t u_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon\right) dx - \varepsilon \int_R \partial_{xx}^2 u_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon\right) dx\\ &\quad = \int_R \left(m\partial_t v_\varepsilon - \partial_{txx}^3 v_\varepsilon\right)\left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon\right) dx\\ &\qquad - \varepsilon \int_R \left(m\partial_{xx}^2 v_\varepsilon - \partial_{xxxx}^4 v_\varepsilon\right)\left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon\right) dx\\ &\quad = \int_R \Big(m v_\varepsilon \partial_t v_\varepsilon - v_\varepsilon \partial_{txx}^3 v_\varepsilon - m\partial_t v_\varepsilon \partial_{xx}^2 v_\varepsilon\\ &\qquad\qquad + \partial_{xx}^2 v_\varepsilon \partial_{txx}^3 v_\varepsilon\Big) dx\\ &\qquad - \varepsilon \int_R \Big(m v_\varepsilon \partial_{xx}^2 v_\varepsilon - m\left(\partial_{xx}^2 v_\varepsilon\right)^2 - v_\varepsilon \partial_{xxxx}^4 v_\varepsilon\\ &\qquad\qquad + \partial_{xx}^2 v_\varepsilon \partial_{xxxx}^4 v_\varepsilon\Big) dx\\ &\quad = \int_R \left(m v_\varepsilon \partial_t v_\varepsilon - (m+1) v_\varepsilon \partial_{txx}^3 v_\varepsilon + \partial_{xx}^2 v_\varepsilon \partial_{txx}^3 v_\varepsilon\right) dx\\ &\qquad - \varepsilon \int_R \Big(m v_\varepsilon \partial_{xx}^2 v_\varepsilon - (m+1) v_\varepsilon \partial_{xxxx}^4 v_\varepsilon\\ &\qquad\qquad + \partial_{xx}^2 v_\varepsilon \partial_{xxxx}^4 v_\varepsilon\Big) dx\\ &\quad = \int_R \left(m v_\varepsilon \partial_t v_\varepsilon + (m+1)\partial_x v_\varepsilon \partial_{tx}^2 v_\varepsilon + \partial_{xx}^2 v_\varepsilon \partial_{txx}^3 v_\varepsilon\right) dx\\ &\qquad - \varepsilon \int_R \Big(-m\partial_x v_\varepsilon \partial_x v_\varepsilon - (m+1)\partial_{xx}^2 v_\varepsilon \partial_{xx}^2 v_\varepsilon\\ &\qquad\qquad - \partial_{xxx}^3 v_\varepsilon \partial_{xxx}^3 v_\varepsilon\Big) dx\\ &\quad = \frac{1}{2}\frac{d}{dt}\int_R \left(m v_\varepsilon^2 + (m+1)\left(\partial_x v_\varepsilon\right)^2 + \left(\partial_{xx}^2 v_\varepsilon\right)^2\right) dx\\ &\qquad + \varepsilon \int_R \left(m\left(\partial_x v_\varepsilon\right)^2 + (m+1)\left(\partial_{xx}^2 v_\varepsilon\right)^2 + \left(\partial_{xxx}^3 v_\varepsilon\right)^2\right) dx.\end{aligned} \tag{19}$$

For the right-hand side of (18), we conclude

$$\begin{aligned}&-\int_R u_\varepsilon \partial_x u_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon\right) dx - \int_R \partial_x P_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon\right) dx\\ &\quad = -\int_R u_\varepsilon \partial_x u_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon\right) dx\\ &\qquad + \int_R \left(P_\varepsilon - \partial_{xx}^2 P_\varepsilon\right)\partial_x v_\varepsilon dx\\ &\quad = -\int_R u_\varepsilon \partial_x u_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon\right) dx\end{aligned}$$

$$
\begin{aligned}
&+\frac{m-1}{2}\int_R u_\varepsilon^2\partial_x v_\varepsilon dx\\
&=-\int_R u_\varepsilon\partial_x u_\varepsilon\left(v_\varepsilon-\partial_{xx}^2 v_\varepsilon\right)dx\\
&\quad-(m-1)\int_R u_\varepsilon\partial_x u_\varepsilon v_\varepsilon dx\\
&=\int_R\left(-mu_\varepsilon\partial_x u_\varepsilon v_\varepsilon+u_\varepsilon\partial_x u_\varepsilon\partial_{xx}^2 v_\varepsilon\right)dx\\
&=\int_R u_\varepsilon\partial_x u_\varepsilon\left(-mv_\varepsilon+\partial_{xx}^2 v_\varepsilon\right)dx\\
&=-\int_R u_\varepsilon^2\partial_x u_\varepsilon dx\\
&=0,
\end{aligned}
\tag{20}
$$

where we have used (16) and integration by parts.

From (18), (19), and (20), we have

$$
\begin{aligned}
&m\|v_\varepsilon\|_{L^2}^2+(m+1)\|\partial_x v_\varepsilon\|_{L^2}^2+\left\|\partial_{xx}^2 v_\varepsilon\right\|_{L^2}^2\\
&\quad+2\varepsilon\int_R\left(m\|\partial_x v_\varepsilon\|_{L^2}^2+(m+1)\left\|\partial_{xx}^2 v_\varepsilon\right\|_{L^2}^2\right.\\
&\qquad\left.+\left\|\partial_{xxx}^3 v_\varepsilon\right\|_{L^2}^2\right)dx\\
&=m\|v_\varepsilon(0,\cdot)\|_{L^2}^2+(m+1)\|\partial_x v_\varepsilon(0,\cdot)\|_{L^2}^2\\
&\quad+\left\|\partial_{xx}^2 v_\varepsilon(0,\cdot)\right\|_{L^2}^2.
\end{aligned}
\tag{21}
$$

From (17), we obtain

$$
\begin{aligned}
&\|v_\varepsilon(0,\cdot)\|_{L^2},\|\partial_x v_\varepsilon(0,\cdot)\|_{L^2},\left\|\partial_{xx}^2 v_\varepsilon(0,\cdot)\right\|_{L^2}\\
&\quad\le c_0\|u_{0,\varepsilon}\|_{L^2}\le c_0\|u_0\|_{L^2}.
\end{aligned}
\tag{22}
$$

It follows from (16) that

$$
\begin{aligned}
&\|u_\varepsilon(t,\cdot)\|_{L^2(R)}^2\\
&\quad=\int_R\left(-\partial_{xx}^2 v_\varepsilon+mv_\varepsilon\right)^2dx\\
&\quad=\int_R\left(\partial_{xx}^2 v_\varepsilon\right)^2dx-2m\int_R v_\varepsilon\partial_{xx}^2 v_\varepsilon dx\\
&\qquad+m^2\int_R v_\varepsilon^2 dx\\
&\quad=\int_R\left(\partial_{xx}^2 v_\varepsilon\right)^2dx+2m\int_R\left(\partial_x v_\varepsilon\right)^2dx\\
&\qquad+m^2\int_R v_\varepsilon^2 dx.
\end{aligned}
\tag{23}
$$

Using (16), (21), and Lemma 1 derives that

$$
\begin{aligned}
&\|u_\varepsilon\|_{L^2}^2\\
&\quad\le 2\left\|\partial_{xx}^2 v_\varepsilon\right\|_{L^2}^2+2m^2\|v_\varepsilon\|_{L^2}^2\\
&\quad\le\max(2,2m)\left(m\|v_\varepsilon\|_{L^2}^2+(m+1)\|\partial_x v_\varepsilon\|_{L^2}^2\right.\\
&\qquad\left.+\left\|\partial_{xx}^2 v_\varepsilon\right\|_{L^2}^2\right)\\
&\quad\le\max(2,2m)\left(m\|v_\varepsilon(0,\cdot)\|_{L^2}^2+(m+1)\|\partial_x v_\varepsilon(0,\cdot)\|_{L^2}^2\right.\\
&\qquad\left.+\left\|\partial_{xx}^2 v_\varepsilon(0,\cdot)\right\|_{L^2}^2\right)\\
&\quad\le c_0\|u_{0,\varepsilon}\|_{L^2}^2\\
&\quad\le c_0\|u_0\|_{L^2}^2,\\
&\varepsilon\|\partial_x u_\varepsilon\|_{L^2}^2\\
&\quad\le 2\varepsilon\left\|\partial_{xxx}^3 v_\varepsilon\right\|_{L^2}^2+2m^2\varepsilon\|\partial_x v_\varepsilon\|_{L^2}^2\\
&\quad\le\varepsilon\max(2,2m)\left(m\|\partial_x v_\varepsilon\|_{L^2}^2+(m+1)\left\|\partial_{xx}^2 v_\varepsilon\right\|_{L^2}^2\right.\\
&\qquad\left.+\left\|\partial_{xxx}^3 v_\varepsilon\right\|_{L^2}^2\right)\\
&\quad\le\max(2,2m)\left(m\|v_\varepsilon(0,\cdot)\|_{L^2}^2+(m+1)\|\partial_x v_\varepsilon(0,\cdot)\|_{L^2}^2\right.\\
&\qquad\left.+\left\|\partial_{xx}^2 v_\varepsilon(0,\cdot)\right\|_{L^2}^2\right)\\
&\quad\le c_0\|u_{0,\varepsilon}\|_{L^2}^2\\
&\quad\le c_0\|u_0\|_{L^2}^2.
\end{aligned}
\tag{24}
$$

The proof of Lemma 3 follows from (24). □

We give some bounds on the nonlocal term $P_\varepsilon$, in which all are consequences of the $L^2$ bound in Lemma 3.

**Lemma 4.** *Assume that $u_0\in L^2(R)$ holds. Then,*

$$P_\varepsilon\ge 0\quad \textit{for } m\ge 1, \tag{25}$$

$$\|P_\varepsilon(t,\cdot)\|_{L^1(R)},\quad\|\partial_x P_\varepsilon(t,\cdot)\|_{L^1(R)}\le c_0\frac{|m-1|}{2}\|u_0\|_{L^2}^2, \tag{26}$$

$$\|P_\varepsilon\|_{L^\infty(R_+\times R)},\quad\|\partial_x P_\varepsilon\|_{L^\infty(R_+\times R)}\le c_0\frac{|m-1|}{2}\|u_0\|_{L^2}^2, \tag{27}$$

$$\left\|\partial_{xx}^2 P_\varepsilon(t,\cdot)\right\|_{L^1(R)}\le c_0\frac{|m-1|}{2}\|u_0\|_{L^2}^2, \tag{28}$$

*where $c_0$ is a constant independent of $\varepsilon$ and $t$.*

*Proof.* Using (11), we get

$$P_\varepsilon(t,x) = \frac{m-1}{4}\int_R e^{-|x-y|}(u_\varepsilon(t,y))^2 dy, \tag{29}$$

$$\partial_x P_\varepsilon(t,x) = \frac{m-1}{4}\int_R e^{-|x-y|}\operatorname{sign}(y-x)(u_\varepsilon(t,y))^2 dy. \tag{30}$$

From (29), we obtain (25). Using (14) and the Tonelli theorem, we have

$$\begin{aligned}
&\int_R |P_\varepsilon(t,x)|\,dx \\
&\quad\le \frac{|m-1|}{4}\int_R\left(\int_R e^{-|x-y|}dx\right)(u_\varepsilon(t,y))^2 dy \\
&\quad\le c_0\frac{|m-1|}{2}\int_R (u_\varepsilon(t,y))^2 dy \\
&\quad\le c_0\frac{|m-1|}{2}\|u_0\|_{L^2}^2, \\
&\int_R |\partial_x P_\varepsilon(t,x)|\,dx \\
&\quad\le \frac{|m-1|}{4}\int_R\left(\int_R e^{-|x-y|}dx\right)(u_\varepsilon(t,y))^2 dy \\
&\quad\le c_0\frac{|m-1|}{2}\int_R (u_\varepsilon(t,y))^2 dy \\
&\quad\le c_0\frac{|m-1|}{2}\|u_0\|_{L^2}^2, \\
&|P_\varepsilon(t,x)| \le \frac{|m-1|}{4}\int_R (u_\varepsilon(t,y))^2 dy \\
&\qquad\le c_0\frac{|m-1|}{4}\|u_0\|_{L^2}^2, \\
&|\partial_x P_\varepsilon(t,x)| \le \frac{|m-1|}{4}\int_R (u_\varepsilon(t,y))^2 dy \\
&\qquad\le c_0\frac{|m-1|}{4}\|u_0\|_{L^2}^2.
\end{aligned} \tag{31}$$

It follows from (31) that (26) and (27) hold. Using the second identity of problem (11), Lemma 3, and (26), we obtain (28). □

**Lemma 5.** *If $u_0 \in L^1(R)\cap L^2(R)$, it holds that*

$$\|u_\varepsilon(t,\cdot)\|_{L^1} \le \|u_0\|_{L^1} + c_0 t\|u_0\|_{L^2}^2. \tag{32}$$

*Proof.* Let functions $\eta$ and $q: R \to R$ be such that $q'(u) = u\eta'(u)$. Multiplying the first equation in (11) with $\eta'(u_\varepsilon)$ gives rise to

$$\begin{aligned}
&\partial_t\eta(u_\varepsilon) + \partial_x q(u_\varepsilon) + \eta'(u_\varepsilon)\partial_x P_\varepsilon \\
&\quad= \varepsilon\eta'(u_\varepsilon)\partial_{xx}^2 u_\varepsilon \\
&\quad= \varepsilon\partial_{xx}^2\eta(u_\varepsilon) - \varepsilon\eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2.
\end{aligned} \tag{33}$$

Choosing $\eta(u) = |u|$ (modulo an approximation argument, see [16]) and then integrating the resulting equation over $R$ yield

$$\frac{d}{dt}\int_R |u_\varepsilon|\,dx \le \int_R |\operatorname{sign}(u_\varepsilon)\partial_x P_\varepsilon|\,dx. \tag{34}$$

Using (26), we get

$$\frac{d}{dt}\|u_\varepsilon(t,\cdot)\|_{L^1} \le c_0\frac{|m-1|}{2}\|u_0\|_{L^2}^2, \tag{35}$$

from which we have (32). □

*2.2. BV and $L^\infty$ Estimates.* In this subsection we establish several supplementary estimates for the viscous approximations, which also are consequences of the $L^2$ bound in Lemma 3. In particular, we prove that the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded in $BV$, which yields strong compactness of this sequence. To this end, we need to assume that $u_0 \in L^2(R)$ and $u_0 \in BV(R)$.

**Lemma 6.** *Assume that $u_0 \in L^2(R)$ and $u_0 \in BV(R)$ hold. Then,*

$$\|\partial_x u_\varepsilon(t,\cdot)\|_{L^1(R)} \le c_0\left(|u_0|_{BV} + t\|u_0\|_{L^2}^2\right), \quad t \ge 0. \tag{36}$$

*Proof.* Setting $q_\varepsilon := \partial_x u_\varepsilon$, we know that $q_\varepsilon$ satisfies the equation

$$\partial_t q_\varepsilon + u_\varepsilon\partial_x q_\varepsilon + q_\varepsilon^2 + \partial_{xx}^2 P_\varepsilon = \varepsilon\partial_{xx}^2 q_\varepsilon. \tag{37}$$

If $\eta = \eta(u)$ and $q: R \to R$ satisfies $q'(u) = u\eta'(u)$, using the chain rule yields

$$\begin{aligned}
&\partial_t\eta(q_\varepsilon) + \partial_x(u_\varepsilon q(u_\varepsilon)) - q_\varepsilon\eta(q_\varepsilon) \\
&\quad+ \eta'(q_\varepsilon)q_\varepsilon^2 + \eta'(u_\varepsilon)\partial_{xx}^2 P_\varepsilon \\
&\quad= \varepsilon\partial_{xx}^2\eta(q_\varepsilon) - \varepsilon\eta''(q_\varepsilon)(\partial_x q_\varepsilon)^2.
\end{aligned} \tag{38}$$

Choosing $\eta(u) = |u|$ (modulo an approximation argument) and then integrating the resulting equation over $R$ give rise to

$$\frac{d}{dt}\int_R |\partial_x u_\varepsilon|\,dx \le \int_R |\partial_{xx}^2 P_\varepsilon|\,dx. \tag{39}$$

Using (28), we have

$$\frac{d}{dt}\int_R |\partial_x u_\varepsilon|\,dx \le c_0\|u_0\|_{L^2(R)}^2, \tag{40}$$

from which we obtain (36). □

**Lemma 7.** *Assume that $u_0 \in BV(R)\cap L^2(R)$ holds. Then,*

$$\|u_\varepsilon(t,\cdot)\|_{L^\infty(R)} \le c_0\left(|u_0|_{BV(R)} + t\|u_0\|_{L^2(R)}^2\right), \quad t \ge 0, \tag{41}$$

$$\begin{aligned}
&\|\partial_{xx}^2 P_\varepsilon(t,\cdot)\|_{L^\infty} \\
&\quad\le c_0\left[\|u_0\|_{L^2(R)}^2 + \left(|u_0|_{BV(R)} + t\|u_0\|_{L^2(R)}\right)^2\right],
\end{aligned} \tag{42}$$

*where $c_0$ is independent of $\varepsilon$ and $t$.*

*Proof.* Using

$$\left|u_\varepsilon(t,\cdot)\right| \le \int_R \left|\partial_x u_\varepsilon(t,y)\right| dy = \left\|u_\varepsilon(t,\cdot)\right\|_{BV(R)} \tag{43}$$

and Lemma 6 derives (41). Using (27), (41), and the second equation of problem (11), we obtain inequality (42). □

*2.3. $L^{2N}$ Estimate for Nature Number $N \ge 2$.* Next we prove that the viscous approximations are bounded in $L^{2N}(R)$ for any nature number $N \ge 2$. From Lemmas 6 and 7, if $u_0 \in BV(R) \cap L^2(R)$, we have the inequality $u_{0,\varepsilon} \in L^\infty(R)$ from which we derive

$$u_{0,\varepsilon} \in L^2(R) \bigcap L^{2N}(R). \tag{44}$$

**Lemma 8.** *Assume that $u_0 \in BV(R) \cap L^2(R)$ holds. For any $0 < \varepsilon < 1/4$, it has*

$$\begin{aligned}\left\|u_\varepsilon(t,\cdot)\right\|_{L^{2N}(R)} &\le c_0\left(1+\left\|u_0\right\|_{L^2}^2\right)^2 t \\ &\quad + \left\|u_\varepsilon(0,\cdot)\right\|_{L^{2N}(R)}^{2N} e^{c_0\left(1+\left\|u_0\right\|_{L^2}^2\right)^2 t}.\end{aligned} \tag{45}$$

*Proof.* Choosing $\eta(u) = (1/2N)u^{2N}$ in (33), writing

$$\begin{aligned}&\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) + \eta'(u_\varepsilon)\partial_x P_\varepsilon \\ &\quad = \varepsilon\partial_{xx}^2 \eta(u_\varepsilon) - \varepsilon\eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2 \\ &\quad = \varepsilon\eta'(u_\varepsilon)\partial_{xx}^2 u_\varepsilon \\ &\quad = \varepsilon u_\varepsilon^{2N-1}\partial_{xx}^2 u_\varepsilon,\end{aligned} \tag{46}$$

and integrating (46) over $R$ gives rise to

$$\begin{aligned}\frac{1}{2N}\frac{d}{dt}\left\|u_\varepsilon(t,\cdot)\right\|_{L^{2N}}^{2N} &= -\int_R u_\varepsilon^{2N-1}\partial_x P_\varepsilon dx \\ &\quad + \varepsilon\int_R u_\varepsilon^{2N-1}\partial_{xx}^2 u_\varepsilon dx.\end{aligned} \tag{47}$$

Integration by parts shows

$$\begin{aligned}&\varepsilon\int_R u_\varepsilon^{2N-1}\partial_{xx}^2 u_\varepsilon dx \\ &\quad = -(2N-1)\varepsilon\int_R u_\varepsilon^{(2N-2)}(\partial_x u_\varepsilon)^2 dx \le 0.\end{aligned} \tag{48}$$

Letting $p_1 = 2N/(2N-1)$, $q_1 = 2N$, we have $1/p_1 + 1/q_1 = 1$. Using Hölder's inequality, (26), and (27), we obtain

$$\begin{aligned}&\left|\int_R u_\varepsilon^{2N-1}\partial_x P_\varepsilon dx\right| \\ &\quad \le \left(\int_R u_\varepsilon^{2N} dx\right)^{(2N-1)/2N}\left(\int_R (\partial_x P_\varepsilon)^{2N}\right)^{1/2N} \\ &\quad \le \left(1+\int_R u_\varepsilon^{2N} dx\right)^{(2N-1)/2N} \\ &\qquad \times \left(\left\|\partial_x P_\varepsilon\right\|_{L^\infty(R_+\times R)}^{2N-1}\right)^{1/2N}\left\|\partial_x P_\varepsilon\right\|_{L^1(R)}^{1/2N} \\ &\quad \le \left(1+\int_R u_\varepsilon^{2N} dx\right)\left(1+\left\|\partial_x P_\varepsilon\right\|_{L^\infty(R_+\times R)}\right)\left\|\partial_x P_\varepsilon\right\|_{L^1(R)}^{1/2N} \\ &\quad \le c_0\left(1+\left\|u_0\right\|_{L^2}^2\right)\left(1+\left\|u_0\right\|_{L^2}^2\right)^{1/2N}\left(1+\int_R u_\varepsilon^{2N} dx\right) \\ &\quad \le c_0\left(1+\left\|u_0\right\|_{L^2}^2\right)^2\left(1+\int_R u_\varepsilon^{2N} dx\right).\end{aligned} \tag{49}$$

Using (47)–(49) gives rise to

$$\begin{aligned}&\frac{1}{2N}\frac{d}{dt}\left\|u_\varepsilon(t,\cdot)\right\|_{L^{2N}}^{2N} \\ &\quad \le c_0\left(1+\left\|u_0\right\|_{L^2}^2\right)^2\left(1+\left\|u_\varepsilon(t,\cdot)\right\|_{L^{2N}}^{2N}\right),\end{aligned} \tag{50}$$

from which we obtain (45) by Gronwall's inequality. □

*2.4. Oleinik Type Estimate*

**Lemma 9** (Oleinik type estimate). *Assume that $u_0 \in L^2(R) \cap BV(R)$. Then, for each $t \in (0,T]$ with $T$ being fixed,*

$$\partial_x u_\varepsilon(t,x) \le \frac{1}{t} + K_T, \quad x \in R, \tag{51}$$

*where $K_T = c_0\{\|u_0\|_{L^2(R)}^2 + (\|u_0\|_{BV(R)} + T\|u_0\|_{L^2(R)}^2)^2\}^{1/2}$.*

*Proof.* Setting $q_\varepsilon := \partial_x u_\varepsilon$, it follows from (11) and (42) that

$$\partial_t q_\varepsilon + u_\varepsilon\partial_x q_\varepsilon + q_\varepsilon^2 - \varepsilon\partial_{xx}^2 q_\varepsilon = -\partial_{xx}^2 P_\varepsilon \le K_T^2. \tag{52}$$

Considering the ordinary differential equation

$$\frac{df}{dt} + f^2 = K_T^2 \tag{53}$$

and using comparing theorem, we have

$$\partial_x u_\varepsilon(t,x) \le \frac{1}{t} + K_T, \quad x \in R, \tag{54}$$

which completes the proof. □

## 3. Existence in $L^1(R) \cap BV(R)$

Using the estimates established in Section 2, we will show the existence of weak solutions to problem (6) under the assumption $u_0 \in L^1(R) \cap BV(R)$.

We state the concepts of weak solutions.

*Definition 10* (weak solution). We call a function $u : R_+ \times R \to R$ a weak solution of the Cauchy problem (6) provided that

(i) $u \in L^\infty(R_+; L^2(R))$ and

(ii) $\partial_t u + \partial_x(u^2/2) + \partial_x P^u(t,x) = 0$ in $D'([0,\infty)\times R)$; that is, for all $\phi \in C_0^\infty([0,\infty)\times R)$, there holds the identity

$$\int_{R^+}\int_R \left(u\partial_t\phi + \frac{u^2}{2}\partial_x\phi - \partial_x P^u \phi\right) dx\,dt + \int_R u_0(x)\,\phi(0,x)\,dx = 0, \tag{55}$$

where

$$P^u(t,x) = G_1 * \left(\frac{m-1}{2}u^2\right)(t,x) = \frac{m-1}{4}\int_R e^{-|x-y|}\left(u(t,y)\right)^2 dy. \tag{56}$$

*Remark 11.* It follows from part (i) of Definition 10 that $u \in L^1((0,T)\times R)$ for any $T > 0$ and $\partial_x P^u \in L^\infty(R_+ \times R)$ (see Lemma 5). Therefore, (55) makes sense.

We assume that

$$\|u(t_2,\cdot) - u(t_1,\cdot)\|_{L^1(R)} \le C_T\,|t_2 - t_1| \quad \text{for } \forall t_1, t_2 \in [0,T], \tag{57}$$

where $C_T = c_0[\|u_0\|^2_{L^2(R)} + (|u_0|_{BV(R)} + t\|u_0\|_{L^2(R)})^2]$. Therefore, we have

$$\lim_{t\to 0+}\|u(t,\cdot) - u_0\|_{L^1(R)} = 0. \tag{58}$$

Our main results are summarized in the following Theorem.

**Theorem 12.** *Provided that $u_0 \in L^1(R)\cap L^2(R)\cap BV(R)$ and the solution $u(t,x)$ satisfies* (57)*, then there exists a weak solution to the Cauchy problem* (6)*. The weak solution $u$ satisfies the following estimates for any $t \in (0,T)$:*

$$\begin{gathered}\|u(t,\cdot)\|_{L^1(R)} \le \|u_0\|_{L^1} + c_0 t\|u_0\|^2_{L^2},\\ \|u(t,\cdot)\|_{BV},\quad \|u(t,x)\|_{L^\infty} \le c_0\left(|u_0|_{BV(R)} + t\|u_0\|^2_{L^2(R)}\right),\\ \|u_\varepsilon(t,\cdot)\|_{L^{2N}(R)} \le c_0\left(1 + \|u_0\|^2_{L^2}\right)^2 t + \|u(0,\cdot)\|^{2N}_{L^{2N}(R)} e^{c_0(1+\|u_0\|^2_{L^2})^2 t},\end{gathered} \tag{59}$$

*where $c_0$ is a positive constant independent of $t$ and $\varepsilon$.*

*The following Oleinik type estimate holds for a.e. $(t,x) \in (0,T]\times R$:*

$$\partial_x u(t,x) \le \frac{1}{t} + K_T, \tag{60}$$

*where*

$$K_T = c_0\left\{\|u_0\|^2_{L^2(R)} + \left(\|u_0\|_{BV(R)} + T\|u_0\|^2_{L^2(R)}\right)^2\right\}^{1/2}. \tag{61}$$

This theorem is an immediate consequence of Theorem 13 and results are presented in Section 2.

**Theorem 13** (existence). *Assume that $u_0 \in L^1(R)\cap L^2(R)\cap BV(R)$ and the solution $u(t,x)$ satisfies* (57)*. Then, there exists at least one weak solution to problem* (6)*.*

*Proof.* Using the estimates obtained in Section 2, we take a standard argument to see that there exists a sequence of strictly positive numbers $\{\varepsilon_k\}_{k=1}^\infty$ tending to zero such that as $k \to \infty$

$$\begin{gathered}u_{\varepsilon_k} \longrightarrow u \quad \text{a.e. in } R_+\times R,\\ u_{\varepsilon_k} \longrightarrow u \quad \text{in } L^1_{\text{loc}}(R_+\times R).\end{gathered} \tag{62}$$

The previous estimates in Section 2 imply immediately that the limit function $u$ satisfies (59)-(60).

Let us now prove that as $k \to \infty$

$$P_{\varepsilon_k} \longrightarrow P^u,\quad \partial_x P_{\varepsilon_k} \longrightarrow \partial_x P^u \quad \text{in } L^1((0,T)\times R)\ \ \forall T > 0, \tag{63}$$

which follows from the following calculations:

$$\begin{aligned}&\left\|P_{\varepsilon_k} - P^u\right\|_{L^1((0,T)\times R)}\\ &\le \left(\frac{|m-1|}{4}\right)\times\int_0^T\int_R\left(\int_R e^{-|x-y|}\times\left|\left(u_{\varepsilon_k}(t,y)\right)^2 - \left(u(t,y)\right)^2\right| dy\right) dx\,dt\\ &\le \left(\frac{|m-1|}{2}\right)\int_0^T\left(\int_R\left|u_{\varepsilon_k}(t,y) - u(t,y)\right|\times\left|u_{\varepsilon_k}(t,y) + u(t,y)\right| dy\right) dt\\ &\le C_T\int_0^T\int_R\left|u_{\varepsilon_k}(t,y) - \left(u(t,y)\right)\right| dy\,dt \longrightarrow 0 \quad \text{as } k \longrightarrow \infty,\end{aligned}$$

$$\begin{aligned}&\left\|\partial_x P_{\varepsilon_k} - \partial_x P^u\right\|_{L^1((0,T)^1\times R)}\\ &\le \left(\frac{|m-1|}{4}\right)\times\int_0^T\int_R\left(\int_R e^{-|x-y|}\times\left|\left(u_{\varepsilon_k}(t,y)\right)^2 - \left(u(t,y)\right)^2\right| dy\right) dx\,dt\end{aligned}$$

$$\leq \left(\frac{|m-1|}{2}\right)\int_0^T\left(\int_R \left|u_{\varepsilon_k}(t,y)-u(t,y)\right| \times \left|u_{\varepsilon_k}(t,y)+u(t,y)\right| dy\right) dt$$
$$\leq C_T\int_0^T\int_R \left|u_{\varepsilon_k}(t,y)-(u(t,y))\right| dy\, dt \longrightarrow 0 \quad \text{as } k\longrightarrow\infty. \tag{64}$$

From (62) to (64), we complete the proof. □

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgment

This work is supported by the Fundamental Research Funds for the Central Universities (JBK120504).

## References

[1] A. Constantin and D. Lannes, "The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations," *Archive for Rational Mechanics & Analysis*, vol. 192, no. 1, pp. 165–186, 2009.

[2] A. Degasperis and M. Procesi, "Asymptotic integrability," in *Symmetry & Perturbation Theory*, vol. 1, no. 1, pp. 23–37, World Scientific, Singapore, 1999.

[3] J. Escher, Y. Liu, and Z. Y. Yin, "Global weak solutions and blow-up structure for the Degasperis-Procesi equation," *Journal of Functional Analysis*, vol. 241, no. 2, pp. 457–485, 2006.

[4] Y. Fu, Y. Liu, and C. Z. Qu, "On the blow-up structure for the generalized periodic Camassa-Holm and Degasperis-Procesi equations," *Journal of Functional Analysis*, vol. 262, no. 7, pp. 3125–3158, 2012.

[5] V. O. Vakhnenko and E. J. Parkes, "Periodic and solitary-wave solutions of the Degasperis-Procesi equation," *Chaos, Solitons & Fractals*, vol. 20, no. 5, pp. 1059–1073, 2004.

[6] H. Lundmark and J. Szmigielski, "Multi-peakon solutions of the Degasperis-Procesi equation," *Inverse Problems*, vol. 19, no. 6, pp. 1241–1245, 2003.

[7] D. D. Holm and M. F. Staley, "Wave structure and nonlinear balances in a family of evolutionary PDEs," *SIAM Journal on Applied Dynamical Systems*, vol. 2, no. 3, pp. 323–380, 2003.

[8] Z. W. Lin and Y. Liu, "Stability of peakons for the Degasperis-Procesi equation," *Communications on Pure & Applied Mathematics*, vol. 62, no. 1, pp. 125–146, 2009.

[9] Z. Y. Yin, "Global weak solutions for a new periodic integrable equation with peakon solutions," *Journal of Functional Analysis*, vol. 212, no. 1, pp. 182–194, 2004.

[10] Y. Matsuno, "Multisoliton solutions of the Degasperis-Procesi equation and their peakon limit," *Inverse Problems*, vol. 21, no. 5, pp. 1553–1570, 2005.

[11] D. Henry, "Infinite propagation speed for the Degasperis-Procesi equation," *Journal of Mathematical Analysis & Applications*, vol. 311, no. 2, pp. 755–759, 2005.

[12] O. G. Mustafa, "A note on the Degasperis-Procesi equation," *Journal of Nonlinear Mathematical Physics*, vol. 12, no. 1, pp. 10–14, 2005.

[13] Y. Liu and Z. Y. Yin, "Global existence and blow-up phenomena for the Degasperis-Procesi equation," *Communications in Mathematical Physics*, vol. 267, no. 3, pp. 801–820, 2006.

[14] S. Y. Lai and Y. H. Wu, "The local well-posedness and existence of weak solutions for a generalized Camassa-Holm equation," *Journal of Differential Equations*, vol. 248, no. 8, pp. 2038–2063, 2010.

[15] Y. Zhou, "Blow-up of solutions to the DGH equation," *Journal of Functional Analysis*, vol. 250, no. 1, pp. 227–248, 2007.

[16] G. M. Coclite and K. H. Karlsen, "On the well-posedness of the Degasperis-Procesi equation," *Journal of Functional Analysis*, vol. 233, no. 1, pp. 60–91, 2006.

[17] G. M. Coclite, K. H. Karlsen, and H. Holden, "Wellposedness for a parabolic-elliptic system," *Discrete & Continuous Dynamical Systems*, vol. 13, no. 3, pp. 659–682, 2005.