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Research Article

Monotone Iterative Methods of Positive Solutions for Fractional Differential Equations Involving Derivatives

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This paper studies the existence and computing method of positive solutions for a class of nonlinear fractional differential equations involving derivatives with two-point boundary conditions. By applying monotone iterative methods, the existence results of positive solutions and two iterative schemes approximating the solutions are established. The interesting point of our method is that the iterative scheme starts off with a known simple function or the zero function and the nonlinear term in the fractional differential equation is allowed to depend on the unknown function together with derivative terms. Two explicit numerical examples are given to illustrate the results.

1. Introduction

In this paper, we discuss the existence and computing method of positive solutions of the αth-order fractional boundary value problem consisting of the nonlinear fractional differential equation

$$D_{0^+}^{\alpha} u(t) + q(t) f\left(t, u(t), u'(t), \ldots, u^{([\beta])}(t)\right) = 0, \quad 0 < t < 1, \tag{1}$$

and the two-point boundary conditions

$$\begin{gathered} u^{(i)}(0) = 0, \quad i = 0, 1, \ldots, n-2, \\ \left[D_{0^+}^{\beta} u(t)\right]_{t=1} = 0, \quad 1 \leqslant \beta \leqslant n-2 \text{ is fixed}, \end{gathered} \tag{2}$$

where $n \geqslant 4$ is an integer, $n-1 < \alpha \leqslant n$ is a real number, $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative of order α, $[\beta]$ denotes the integer part of real number β, and $u^{(i)}$ in boundary conditions (2) represents the ith (ordinary) derivative of u.

Fractional differential equations arise in many fields such as physics, mechanics, chemistry, economics, engineering, and biological sciences. Recently, there have been many papers dealing with the solutions or positive solutions of boundary value problems for nonlinear fractional differential equations. We refer the reader to the papers of Agarwal et al. [1], Ahmad and Sivasundaram [2], Ahmad and Nieto [3], Babakhani and Daftardar-Gejji [4], Bai and Sun [5], Bai et al. [6], Bai and Qiu [7], Caballero et al. [8], Delbosco and Rodino [9], Graef et al. [10], Jiang and Yuan [11], Lakshmikantham and Vatsala [12], Liang and Zhang [13], Qiu and Bai [14], Tian and Liu [15], Wang et al. [16], Yang and Chen [17], Yuan et al. [18], Zhang [19], Zhang and Han [20], and Zhang et al. [21, 22] and the references therein. Nonlinear fractional differential equations with two-point boundary conditions (2) have been studied by several authors. For example, in [23], Goodrich studied a fractional differential equation of the form

$$D_{0^+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1 \tag{3}$$

with boundary conditions (2), where $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous. The author obtained Green's function of the problem and proved that Green's function satisfied a Harnack-like inequality. By using a fixed point theorem due to Krasnosel'skii, the author established the existence results for at least one positive solution. Graef et al. in [24] found sufficient conditions to guarantee that the following fractional differential equation:

$$D_{0^+}^{\nu} u(t) + \lambda f(t, u(t)) + e(t) = 0, \quad 0 < t < 1, \tag{4}$$

with boundary conditions (2) has at least one or two positive solutions when λ is small and large, where λ is a parameter, $n \geqslant 3$, $n-1 < \nu \leqslant n$, $f : [0,1] \times [0,\infty) \to \mathbb{R}$, and $e : [0,1] \to \mathbb{R}$ are continuous functions. Zhai and Hao [25] discussed the existence and uniqueness of positive solutions for the following fractional differential equation:

$$D_{0^+}^{\alpha} u(t) + f(t, u(t), u(t)) + g(t, u(t)) = 0, \quad 0 < t < 1 \tag{5}$$

with boundary conditions (2), where $f : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ and $g : [0,1] \times [0,\infty) \to [0,\infty)$ are continuous functions and satisfy some monotonicity conditions. The analysis relies on two new fixed point theorems for mixed monotone operators with perturbation. In [26], Su and Feng studied a fractional differential equation with deviating argument of the form

$$D_{0^+}^{\alpha} u(t) + h(t) f(u(\theta(t))) = 0, \quad 0 < t < 1 \tag{6}$$

with boundary conditions (2), where $f : [0,\infty) \to [0,\infty)$, $h : [0,1] \to (0,\infty)$, and $\theta : (0,1) \to (0,1]$ are continuous functions. The author obtained novel sufficient conditions for the existence of at least one or two positive solutions by using Krasnosel'skii's fixed point theorem, and some other new sufficient conditions for the existence of at least triple positive solutions by using the fixed point theorems developed by Leggett and Williams, and so forth. Yuan [27] gave sufficient conditions for the existence of multiple positive solution for the semipositone (n,p)-type boundary value problems of nonlinear fractional differential equations

$$\begin{gathered} D_{0^+}^{\alpha} u(t) + \lambda f(t, u(t)) = 0, \quad 0 < t < 1, \\ u^{(i)}(0) = 0, \quad 0 \leqslant i \leqslant n-2, \quad u^{(p)}(1) = 0, \end{gathered} \tag{7}$$

where λ is a parameter, $\alpha \in (n-1, n]$ is a real number and $n \geqslant 3$, $1 \leqslant p \leqslant \alpha - 1$ is fixed and integer, and $f : (0,1) \times [0,\infty) \to \mathbb{R}$ is a sign-changing continuous function. The author derived an interval of λ such that for any λ lying in this interval, the semipositone boundary value problem has multiple positive solutions. The analysis relied on nonlinear alternative of Leray-Schauder type and the Krasnosel'skii fixed point theorem.

We notice that the methods used in the above papers are all fixed point theorems and the derivatives of unknown function are not involved in the nonlinear term explicitly. Different from the works mentioned above, motivated by the works [28–32], we will use monotone iterative techniques to study the existence and iteration of positive solutions for the problem (1)-(2). We not only obtain the existence of positive solutions, but also give two iterative schemes approximating the solutions. Moreover, this method does not demand the existence of upper-lower solutions. To the best of our knowledge, few authors utilize the monotone methods to study the existence of positive solutions for nonlinear fractional boundary value problems. So, it is worthwhile to investigate the problem (1)-(2) by using monotone iterative techniques.

This paper is organized as follows. In Section 2, we recall some definitions and notations from the theory of fractional calculus and give expression and properties of Green's function. The main results will be given in Section 3. Finally, in Section 4, some examples are included to demonstrate the applicability of our results.

2. Preliminaries

Here we present some necessary basic knowledge and definitions for fractional calculus theory that can be found in the literature [33, 34].

Definition 1. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $h : [0,\infty) \to \mathbb{R}$ is defined to be

$$D_{0^+}^{\alpha} h(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} h(s)\, ds, \qquad n = [\alpha] + 1, \tag{8}$$

where Γ denotes the Euler gamma function and $[\alpha]$ denotes the integer part of number α provided that the right side is pointwise defined on $(0,\infty)$.

Definition 2. The Riemann-Liouville fractional integral of order α is defined as

$$I_{0^+}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds, \quad t > 0, \ \alpha > 0. \tag{9}$$

Provided that the integral exists.

In [23], the author obtain Green's function associated with the problem (1)-(2). More precisely, the author proved the following lemma.

Lemma 3 (see [23]). *Let $w \in C[0,1]$, then the differential equation*

$$D_{0^+}^{\alpha} u(t) + w(t) = 0, \quad 0 \leqslant t \leqslant 1, \tag{10}$$

with boundary conditions (2) *has a unique solution*

$$\begin{aligned} u(t) &= \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} w(s)\, ds \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} w(s)\, ds \\ &= \int_0^1 G(t,s)\, w(s)\, ds, \end{aligned} \tag{11}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{\alpha-1}(1-s)^{\alpha-\beta-1}, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases} \tag{12}$$

Obviously, for $j = 0, 1, 2, \ldots, [\beta]$,

$$\frac{\partial^j G(t,s)}{\partial t^j} = \frac{1}{\Gamma(\alpha - j)} \times \begin{cases} t^{\alpha-j-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-j-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{\alpha-j-1}(1-s)^{\alpha-\beta-1}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{13}$$

is continuous on $[0,1] \times [0,1]$.

The following properties of Green's function $G(t,s)$ defined by (12) will be used later.

Lemma 4. *Green's function $G(t,s)$ defined by* (12) *has the following properties:*

(1) $0 \leqslant \partial^j G(t,s)/\partial t^j \leqslant (t^{\alpha-j-1}/\Gamma(\alpha-j))(1-s)^{\alpha-\beta-1}$ *on* $[0,1] \times [0,1]$, *for* $j = 0, 1, 2, \ldots, [\beta]$,

(2) $t^{\alpha-1}G(1,s) \leqslant G(t,s) \leqslant G(1,s)$ *on* $[0,1] \times [0,1]$.

Proof. Firstly, we prove that (1) is true. In fact, for all $t, s \in [0,1]$, if $t \leqslant s$, it is obvious that $\partial^j G(t,s)/\partial t^j \geqslant 0$ for $j = 0, 1, 2, \ldots, [\beta]$. If $s \leqslant t$, from (13), we obtain that

$$\begin{aligned} \frac{\partial^j G(t,s)}{\partial t^j} &= \frac{1}{\Gamma(\alpha-j)}\left[t^{\alpha-j-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-j-1}\right] \\ &\geqslant \frac{1}{\Gamma(\alpha-j)}\left[t^{\alpha-j-1}(1-s)^{\alpha-j-1} - (t-s)^{\alpha-j-1}\right] \\ &\geqslant \frac{1}{\Gamma(\alpha-j)}\left[(t-ts)^{\alpha-j-1} - (t-s)^{\alpha-j-1}\right] \\ &\geqslant 0, \quad \text{for } j = 0, 1, 2, \ldots, [\beta]. \end{aligned} \tag{14}$$

On the other hand, by (13), we find

$$\frac{\partial^j G(t,s)}{\partial t^j} \leqslant \frac{t^{\alpha-j-1}}{\Gamma(\alpha-j)}(1-s)^{\alpha-\beta-1}, \quad t, s \in [0,1], \ j = 0, 1, 2, \ldots, n-2. \tag{15}$$

From (14) and (15) we get part (1).

Next, we show part (2). In fact, on the one hand, from (1), we know that $\partial G(t,s)/\partial t \geqslant 0$ for any $t, s \in [0,1]$. Thus, $G(t,s)$ is increasing in t, so

$$G(t,s) \leqslant G(1,s), \quad \text{for } (t,s) \in [0,1] \times [0,1]. \tag{16}$$

On the other hand, if $s \leqslant t$, then from (13), we have

$$\begin{aligned} G(t,s) &= \frac{1}{\Gamma(\alpha)}\left[t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}\right] \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}\left[(1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-1}\right] \\ &\quad + \frac{1}{\Gamma(\alpha)}\left[(t-ts)^{\alpha-1} - (t-s)^{\alpha-1}\right] \\ &\geqslant \frac{t^{\alpha-1}}{\Gamma(\alpha)}\left[(1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-1}\right] \\ &= t^{\alpha-1}G(1,s). \end{aligned} \tag{17}$$

If $s \geqslant t$, then from (13), we have

$$\begin{aligned} G(t,s) &= \frac{1}{\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-\beta-1} \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}\left[(1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-1}\right] \\ &\quad + \frac{1}{\Gamma(\alpha)}(t-ts)^{\alpha-1} \\ &\geqslant \frac{t^{\alpha-1}}{\Gamma(\alpha)}\left[(1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-1}\right] \\ &= t^{\alpha-1}G(1,s). \end{aligned} \tag{18}$$

Thus, (17) and (18) show

$$G(t,s) \geqslant t^{\alpha-1}G(1,s). \tag{19}$$

From (16) and (19), we get part (2). Then the proof is completed. □

3. Main Results

In this section, we discuss the existence and iteration of positive solutions for the problem (1)-(2). In the sequel, the following conditions hold:

($H1$) $f : [0,1] \times [0,\infty)^{[\beta]+1} \to [0,\infty)$ is continuous and $f(t,0,0,\ldots,0) \not\equiv 0$, $t \in [0,1]$.

($H2$) $q \in L^1[0,1]$ is nonnegative and $0 < \int_0^1 (1-s)^{\alpha-\beta-1}q(s)ds < \infty$.

For any $u \in C[0,1]$, we define $\|u\|_\infty = \max_{0\leqslant t\leqslant 1}|u(t)|$. Let the Banach space $\mathscr{E} = C^{[\beta]}[0,1]$ be equipped with the norm

$$\|u\| = \max\left\{\|u\|_\infty, \left\|u'\right\|_\infty, \ldots, \left\|u^{([\beta])}\right\|_\infty\right\}. \tag{20}$$

We define a cone $\mathscr{K}$ by

$$\mathscr{K} = \left\{u \in C^{[\beta]}[0,1] : u(t) \geqslant t^{\alpha-1}\|u\|_\infty, u^{(j)}(t) \geqslant 0, \ t \in [0,1], j = 0, 1, 2, \ldots, [\beta]\right\}, \tag{21}$$

and an integral operator $\mathscr{T}$ by

$$(\mathscr{T}u)(t) = \int_0^1 G(t,s)\,q(s)\,F(s,u(s))\,ds, \quad t \in [0,1], \ u \in C^{[\beta]}[0,1], \tag{22}$$

where

$$F(s,u(s)) = f\left(s, u(s), u'(s), \ldots, u^{([\beta])}(s)\right), \quad s \in [0,1], \ u \in C^{[\beta]}[0,1]. \tag{23}$$

Obviously, the fixed points of $\mathcal{T}$ are solutions of the problem (1)-(2).

Lemma 5. *$\mathcal{T}$ is completely continuous and $\mathcal{T}(\mathcal{K}) \subseteq \mathcal{K}$.*

Proof. Since $f(t)$, $G(t,s)$ are continuous for $t,s \in [0,1]$ and $q(t)$ is integrable on $[0,1]$, we get that the operator $\mathcal{T}$ is well defined on $\mathcal{K}$. By (13), we get $\mathcal{T} : \mathcal{K} \to \mathcal{E}$. Let $\mathcal{A} \subseteq \mathcal{K}$ be bounded. Then there exists a positive constant $R_1 > 0$ such that $\|u\| \leqslant R_1$, $u \in \mathcal{A}$. Denote

$$\max_{(t,u)\in[0,1]\times\mathcal{A}} \left| f\left(t, u(t), u'(t), \ldots, u^{([\beta])}(t)\right)\right| = M. \tag{24}$$

Then for $u \in \mathcal{A}$, by Lemma 4(1) and (22), we have

$$\begin{aligned} \left|(\mathcal{T}u)^{(j)}(t)\right| &= \int_0^1 \frac{\partial^j G(t,s)}{\partial t^j} q(s) F(s,u(s))\, ds \\ &\leqslant \frac{M t^{\alpha-j-1}}{\Gamma(\alpha-j)} \int_0^1 (1-s)^{\alpha-\beta-1} q(s)\, ds \\ &\leqslant \frac{M}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} q(s)\, ds < \infty, \\ &\qquad j = 0,1,2,\ldots,[\beta]. \end{aligned} \tag{25}$$

Hence, $\mathcal{T}(\mathcal{A})$ is bounded. For $u \in \mathcal{A}$, $t_1, t_2 \in [0,1]$, one has

$$\begin{aligned} &\left|(\mathcal{T}u)^{(j)}(t_2) - (\mathcal{T}u)^{(j)}(t_1)\right| \\ &\quad\leqslant \frac{M\left|t_2^{\alpha-j-1} - t_1^{\alpha-j-1}\right|}{\Gamma(\alpha-j)} \int_0^1 (1-s)^{\alpha-\beta-1} q(s)\, ds \\ &\qquad + \frac{M}{\Gamma(\alpha-j)} \left| \int_0^{t_2} (t_2-s)^{\alpha-j-1} q(s)\, ds \right. \\ &\qquad\qquad \left. - \int_0^{t_1} (t_1-s)^{\alpha-j-1} q(s)\, ds \right| \\ &\quad\leqslant \frac{M\left|t_2^{\alpha-j-1} - t_1^{\alpha-j-1}\right|}{\Gamma(\alpha-j)} \int_0^1 (1-s)^{\alpha-\beta-1} q(s)\, ds \\ &\qquad + \frac{M}{\Gamma(\alpha-j)} \int_0^{t_1} \left|(t_2-s)^{\alpha-j-1} - (t_1-s)^{\alpha-j-1}\right| q(s)\, ds \\ &\qquad + \frac{M}{\Gamma(\alpha-j)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-j-1} q(s)\, ds, \\ &\qquad\qquad j = 0,1,2,\ldots,[\beta]. \end{aligned} \tag{26}$$

Thus,

$$\left\|(\mathcal{T}u)(t_2) - (\mathcal{T}u)(t_1)\right\| \longrightarrow 0, \quad \text{for } t_1 \longrightarrow t_2,\ u \in \mathcal{A}. \tag{27}$$

By means of the Arzela-Ascoli theorem, we claim that $\mathcal{T}$ is completely continuous.

Now, we conclude that $\mathcal{T}(\mathcal{K}) \subseteq \mathcal{K}$. In fact, for any $u \in \mathcal{K}$, it follows from Lemma 4(2) that

$$\begin{aligned} (\mathcal{T}u)(t) &= \int_0^1 G(t,s) q(s) F(s,u(s))\, ds \\ &\leqslant \int_0^1 G(1,s) q(s) F(s,u(s))\, ds, \quad t \in [0,1], \end{aligned} \tag{28}$$

which implies that

$$\|\mathcal{T}u\|_\infty \leqslant \int_0^1 G(1,s) q(s) F(s,u(s))\, ds. \tag{29}$$

On the other hand,

$$\begin{aligned} (\mathcal{T}u)(t) &= \int_0^1 G(t,s) q(s) F(s,u(s))\, ds \\ &\geqslant t^{\alpha-1} \int_0^1 G(1,s) q(s) F(s,u(s))\, ds, \quad t \in [0,1], \end{aligned} \tag{30}$$

which together with (29) implies

$$(\mathcal{T}u)(t) \geqslant t^{\alpha-1} \|\mathcal{T}u\|_\infty, \quad t \in [0,1]. \tag{31}$$

In addition, it follows from Lemma 4(1) that

$$\begin{aligned} (\mathcal{T}u)^{(j)}(t) &= \int_0^1 \frac{\partial^j G(t,s)}{\partial t^j} q(s) F(s,u(s))\, ds \geqslant 0, \\ &\qquad j = 0,1,2,\ldots,[\beta]. \end{aligned} \tag{32}$$

Therefore, (31) and (32) show that $\mathcal{T}u \in \mathcal{K}$; that is, $\mathcal{T}(\mathcal{K}) \subseteq \mathcal{K}$. Then the proof is completed. □

For notational convenience, we denote

$$\Lambda = \left(\int_0^1 (1-s)^{\alpha-\beta-1} q(s) ds\right)^{-1}. \tag{33}$$

By $(H2)$, we know that $\Lambda > 0$ is well defined.

Theorem 6. *Suppose that $(H1)$ and $(H2)$ hold. In addition, assume that there exists $a > 0$ such that*

- $(H3)$ $f(t, x_0, x_1, \ldots, x_{[\beta]}) \leqslant f(t, y_0, y_1, \ldots, y_{[\beta]})$ *for* $0 \leqslant x_j \leqslant y_j \leqslant a$, $t \in [0,1]$, $j = 0,1,2,\ldots,[\beta]$;
- $(H4)$ $\max_{0\leqslant t\leqslant 1} f(t,a,a,\ldots,a) \leqslant \Lambda a$.

Then, the problem (1)-(2) has two positive solutions v^* and w^* satisfying $0 < \|v^*\| \leqslant \|w^*\| \leqslant a$. Moreover, there exist monotone increasing sequence $\{v_k\}_{k=0}^\infty$ and monotone decreasing sequence $\{w_k\}_{k=0}^\infty$ in $\mathcal{K}$ such that $v_k(t) \leqslant w_k(t)$, $t \in [0,1]$, $k = 0,1,2,\ldots$, and $\lim_{k\to\infty}\|v_k - v^*\| = 0$, and $\lim_{k\to\infty}\|w_k - w^*\| = 0$, where $v_0(t) = 0$, and $w_0(t) = at^{\alpha-1}/\Gamma(\alpha)$.

The iterative schemes in Theorem 6 start off with the zero function and a known simple function, respectively.

Proof. We divide the proof into four steps.

Step 1. Let $\mathscr{K}_a = \{u \in \mathscr{K} : \|u\| \leqslant a\}$. Then $\mathscr{T} : \mathscr{K}_a \to \mathscr{K}_a$.

In fact, if $u \in \mathscr{K}_a$, then $\|u\| \leqslant a$; thus,

$$0 \leqslant u^{(j)}(s) \leqslant \|u\| \leqslant a, \quad s \in [0,1], \ j = 0,1,2,\ldots,[\beta]. \tag{34}$$

By the conditions $(H3)$ and $(H4)$, we have

$$\begin{aligned} 0 &\leqslant f\left(s, u(s), u'(s), \ldots, u^{([\beta])}(s)\right) \\ &\leqslant \max_{0\leqslant s\leqslant 1} f(s,a,a,\ldots,a) \leqslant \Lambda a, \quad s \in [0,1]. \end{aligned} \tag{35}$$

Thus, by the definition of $\mathscr{T}$ and Lemma 4 (2), for $j = 0,1,2,\ldots,[\beta]$, we get

$$\begin{aligned} (\mathscr{T}u)^{(j)}(t) &= \int_0^1 \frac{\partial^j G(t,s)}{\partial t^j} q(s) \\ &\qquad \times f\left(s, u(s), u'(s), \ldots, u^{([\beta])}(s)\right) ds \\ &\leqslant \frac{t^{\alpha-j-1}}{\Gamma(\alpha-j)} \int_0^1 (1-s)^{\alpha-\beta-1} q(s) \\ &\qquad \times f(s,a,a,\ldots,a)\, ds \\ &\leqslant \Lambda a \int_0^1 (1-s)^{\alpha-\beta-1} q(s)\, ds = a, \quad t \in [0,1]. \end{aligned} \tag{36}$$

Then (36) shows that $\|\mathscr{T}u\| \leqslant a$; thus, $\mathscr{T} : \mathscr{K}_a \to \mathscr{K}_a$.

Step 2. Let $v_{k+1} = \mathscr{T}v_k$, $k = 0,1,2,\ldots$. Then $\{v_k\}$ is increasing; there exists $v^* \in \mathscr{K}_a$ such that $\lim_{k\to\infty}\|v_k - v^*\| = 0$, and v^* is a positive solution of the problem (1)-(2).

Obviously, $v_0 \in \mathscr{K}_a$. Since $\mathscr{T} : \mathscr{K}_a \to \mathscr{K}_a$, we have $v_k \in \mathscr{T}(\mathscr{K}_a) \subseteq \mathscr{K}_a$, $k = 1,2,\ldots$. Since $\mathscr{T}$ is completely continuous, we assert that $\{v_k\}_{k=1}^{\infty}$ is a sequentially compact set. Since $v_1 = \mathscr{T}v_0 = \mathscr{T}0 \in \mathscr{K}_a$, we have

$$\begin{aligned} a \geqslant v_1^{(j)}(t) = (\mathscr{T}v_0)^{(j)}(t) = (\mathscr{T}0)^{(j)}(t) \geqslant 0 = v_0^{(j)}(t) \\ t \in [0,1], \ j = 0,1,2,\ldots,[\beta]. \end{aligned} \tag{37}$$

It follows from $(H3)$ that $\mathscr{T}$ is increasing; then

$$\begin{aligned} v_2^{(j)}(t) = (\mathscr{T}v_1)^{(j)}(t) \geqslant (\mathscr{T}v_0)^{(j)}(t) = v_1^{(j)}(t), \\ t \in [0,1], \ j = 0,1,2,\ldots,[\beta]. \end{aligned} \tag{38}$$

Thus, by the induction, we have

$$\begin{aligned} v_{k+1}^{(j)}(t) \geqslant v_k^{(j)}(t), \quad t \in [0,1], \ j = 0,1,2,\ldots,[\beta], \\ k = 0,1,2,\ldots. \end{aligned} \tag{39}$$

Hence, there exists $v^* \in \mathscr{K}_a$ such that $\lim_{k\to\infty}\|v_k - v^*\| = 0$. Applying the continuity of $\mathscr{T}$ and equation $v_{k+1} = \mathscr{T}v_k$, we get $\mathscr{T}v^* = v^*$. Moreover, because the zero function is not a solution of the problem (1)-(2), thus, $\|v^*\|_\infty > 0$. It follows from the definition of the cone $\mathscr{K}$ that we have $v^*(t) \geqslant t^{\alpha-1}\|v^*\|_\infty > 0$, $t \in (0,1)$. That is, $v^*(t)$ is a positive solution of the problem (1)-(2).

Step 3. Let $w_{k+1} = \mathscr{T}w_k$, $k = 0,1,2,\ldots$. Then $\{w_k\}$ is decreasing; there exists $w^* \in \mathscr{K}_a$ such that $\lim_{k\to\infty}\|w_k - w^*\| = 0$, and w^* is a positive solution of the problem (1)-(2).

Obviously, $w_0 \in \mathscr{K}_a$. Since $\mathscr{T} : \mathscr{K}_a \to \mathscr{K}_a$, we have $w_k \in \mathscr{T}(\mathscr{K}_a) \subseteq \mathscr{K}_a$, $k = 0,1,2,\ldots$. Since $\mathscr{T}$ is completely continuous, we assert that $\{w_k\}_{k=1}^{\infty}$ is a sequentially compact set. Since $w_1 = \mathscr{T}w_0 \in \mathscr{K}_a$, by Lemma 4(2), $(H3)$, and $(H4)$, for $j = 0,1,2,\ldots,[\beta]$, we have

$$\begin{aligned} &(\mathscr{T}w_0)^{(j)}(t) \\ &\quad = \int_0^1 \frac{\partial^j G(t,s)}{\partial t^j} q(s) f\left(s, w_0(s), w_0'(s), \ldots, w_0^{([\beta])}(s)\right) ds \\ &\quad \leqslant \frac{t^{\alpha-j-1}}{\Gamma(\alpha-j)} \int_0^1 (1-s)^{\alpha-\beta-1} q(s) f(s,a,a,\ldots,a)\, ds \\ &\quad \leqslant w_0^{(j)}(t)\, \Lambda \int_0^1 (1-s)^{\alpha-\beta-1} q(s)\, ds \\ &\quad = w_0^{(j)}(t), \quad t \in [0,1]. \end{aligned} \tag{40}$$

Thus, we obtain that

$$w_1^{(j)}(t) \leqslant w_0^{(j)}(t), \quad t \in [0,1], \ j = 0,1,2,\ldots,[\beta]. \tag{41}$$

So by $(H3)$, we have

$$\begin{aligned} w_2^{(j)}(t) &= (\mathscr{T}w_1)^{(j)}(t) \\ &= \int_0^1 \frac{\partial^j G(t,s)}{\partial t^j} q(s) F(s, w_1(s))\, ds \\ &\leqslant \int_0^1 \frac{\partial^j G(t,s)}{\partial t^j} q(s) F(s, w_0(s))\, ds \\ &= (\mathscr{T}w_0)^{(j)}(t) \\ &= w_1^{(j)}(t), \quad t \in [0,1], \ j = 0,1,2,\ldots,[\beta]. \end{aligned} \tag{42}$$

By the induction, we have

$$\begin{aligned} w_{k+1}^{(j)}(t) \leqslant w_k^{(j)}(t), \quad t \in [0,1], \ j = 0,1,2,\ldots,[\beta], \\ k = 0,1,2,\ldots. \end{aligned} \tag{43}$$

Hence, there exists $w^* \in \mathscr{K}_a$ such that $\lim_{k\to\infty}\|w_k - w^*\| = 0$. Applying the continuity of $\mathscr{T}$ and equation $w_{k+1} = \mathscr{T}w_k$, we get $\mathscr{T}w^* = w^*$. Thus, w^* is a nonnegative solution of the problem (1)-(2). Moreover, the zero function is not a solution of the problem (1)-(2). Thus, $\|w^*\|_\infty > 0$, it follows from the definition of the cone $\mathscr{K}$ that we have

$w^*(t) \geqslant t^{\alpha-1}\|w^*\|_\infty > 0$, $t \in (0,1)$; that is, $w^*(t)$ is a positive solution of the problem (1)-(2).

Step 4. From $v_0^{(j)}(t) \leqslant w_0^{(j)}(t)$, $t \in [0,1]$, $j = 0,1,2,\ldots,[\beta]$, we have

$$\begin{aligned} v_1(t) &= (\mathcal{T}v_0)(t) = \int_0^1 G(t,s)\,q(s)\,F(s,v_0(s))\,ds \\ &\leqslant \int_0^1 G(t,s)\,q(s)\,F(s,w_0(s))\,ds = (\mathcal{T}w_0)(t) = w_1(t). \end{aligned} \tag{44}$$

By the induction, we have

$$v_k(t) \leqslant w_k(t), \quad t \in [0,1],\ k = 0,1,2,\ldots. \tag{45}$$

The proof is complete. □

Remark 7. Of course, $w^* = v^*$ may happen and then the problem (1)-(2) has only one solution in $\mathcal{K}_a$.

Corollary 8. *Assume that* $(H1)$ *and* $(H2)$ *hold. Suppose that* $f(t,x_0,x_1,\ldots,x_{[\beta]})$ *is increasing in* x_j $(j = 0,1,2,\ldots,[\beta])$. *Moreover,*

$$\lim_{x\to+\infty}\max_{0\leqslant t\leqslant 1}\frac{f(t,x,\ldots,x)}{x} < \Lambda. \tag{46}$$

Then the problem (1)-(2) *has at least two monotone positive solutions.*

4. Examples

To illustrate the usefulness of the results, we provide two examples.

Example 1. Consider the fractional boundary value problem

$$\begin{gathered} D_{0^+}^{7/2}u(t) + \frac{1}{4}u^2(t) + \frac{1}{4}\left[u'(t)\right]^2 + \frac{1}{2}t + 1 = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \qquad \left[D_{0^+}^{3/2}u(t)\right]_{t=1} = 0. \end{gathered} \tag{47}$$

Obviously, the problem (47) fits the framework of problem (1)-(2) with $\alpha = 3.5$, $\beta = 1.5$. In addition, we have set $q(t) \equiv 1/4$, $f(t,x,y) = x^2 + y^2 + 2t + 4$. Obviously, $q(t)$ and $f(t,x,y)$ satisfy the conditions $(H1)$ and $(H2)$. Moreover, It is easy to see that $f(t,x,y)$ is increasing in x and y, and

$$\Lambda = \left(\int_0^1 (1-s)^{\alpha-\beta-1}q(s)ds\right)^{-1} = 8. \tag{48}$$

Let $a = 3$; then for any $(t,x,y) \in [0,1]\times[0,a]\times[0,a]$, we have

$$f(t,x,y) \leqslant f(t,3,3) \leqslant f(1,3,3) = 24 = \Lambda a. \tag{49}$$

Then conditions $(H3)$ and $(H4)$ hold. Consequently, applying Theorem 6, the problem (47) has at least two positive solutions $v^*(t)$ and $w^*(t)$ satisfying $0 < \|v^*\| \leqslant \|w^*\| \leqslant 3$.

Moreover, the two iterative schemes are

$$\begin{aligned} v_0(t) &= 0, \quad t \in [0,1], \\ v_{k+1}(t) &= \frac{2t^{5/2}}{15\sqrt{\pi}}\int_0^1 (1-s)\left\{v_k^2(s) + \left[v_k'(s)\right]^2 + 2s + 4\right\}ds \\ &\quad - \frac{2}{15\sqrt{\pi}}\int_0^t (t-s)^{5/2} \\ &\qquad \times \left\{v_k^2(s) + \left[v_k'(s)\right]^2 + 2s + 4\right\}ds, \\ &\qquad t \in [0,1],\ k = 0,1,2,\ldots, \\ w_0(t) &= \frac{8t^{5/2}}{5\sqrt{\pi}}, \quad t \in [0,1], \\ w_{k+1}(t) &= \frac{2t^{5/2}}{15\sqrt{\pi}}\int_0^1 (1-s)\left\{w_k^2(s) + \left[w_k'(s)\right]^2 + 2s + 4\right\}ds \\ &\quad - \frac{2}{15\sqrt{\pi}}\int_0^t (t-s)^{5/2} \\ &\qquad \times \left\{w_k^2(s) + \left[w_k'(s)\right]^2 + 2s + 4\right\}ds, \\ &\qquad t \in [0,1],\ k = 0,1,2,\ldots. \end{aligned} \tag{50}$$

After direct calculations, we get

$$\begin{aligned} v_1(t) &= \frac{2t^{5/2}}{15\sqrt{\pi}}\left(\frac{7}{3} - \frac{8}{7}t - \frac{8}{63}t^2\right), \quad t \in [0,1], \\ w_1(t) &= \frac{2t^{5/2}}{15\sqrt{\pi}}\left(\frac{7}{3} - \frac{8}{7}t - \frac{8}{63}t^2\right) \\ &\quad + \frac{8t^{5/2}}{45\pi^{3/2}}\left(\frac{113}{175} - \frac{128}{1001}t^4 - \frac{8192}{1276275}t^6\right), \\ &\qquad t \in [0,1]. \end{aligned} \tag{51}$$

Example 2. Consider the problem

$$\begin{gathered} D_{0^+}^{9/2}u(t) + \frac{1}{2}(1-t)^{-3/2} \\ \times \left\{\sin t + \frac{t}{21}e^{u(t)} + \frac{1}{9}\left[u'(t)\right]^2 + u''(t)\right\} = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = u'''(0) = u''(1) = 0. \end{gathered} \tag{52}$$

In this problem, $\alpha = 4.5$, $\beta = 2$, $q(t) = (1/2)(1-t)^{-3/2}$, and $f(t,x,y,z) = \sin t + (t/21)e^x + (1/9)y^2 + z$. Obviously, $q(t)$ and $f(t,x,y,z)$ satisfy the conditions $(H1)$ and $(H2)$. In addition, $f(t,x,y,z)$ is increasing with regard to x, y, and z and

$$\Lambda = \left(\int_0^1 (1-s)^{\alpha-\beta-1}q(s)ds\right)^{-1} = 2. \tag{53}$$

Let $a = 3$; then for any $(t, x, y, z) \in [0,1]\times[0,a]\times[0,a]\times[0,a]$, by simple computation, we obtain that

$$f(t,x,y,z) \leqslant f(1,3,3,3) < 6 = \Lambda a. \tag{54}$$

Therefore, all assumptions of Theorem 6 are satisfied. Thus, Theorem 6 ensures that the problem (52) has two monotone positive solutions v^* and w^* satisfying $0 < \|v^*\| \leqslant \|w^*\| \leqslant 3$ and $\lim_{k\to\infty}\|v_k - v^*\| = 0$ and $\lim_{k\to\infty}\|w_k - w^*\| = 0$.

Moreover, the two iterative schemes are

$$\begin{aligned}
v_0(t) &= 0, \quad t \in [0,1], \\
v_{k+1}(t) &= \frac{8t^{7/2}}{105\sqrt{\pi}} \\
&\quad \times \int_0^1 \left\{\sin s + \frac{s}{21}e^{v_k(s)} + \frac{1}{9}\left[v_k'(s)\right]^2 + v_k''(s)\right\} ds \\
&\quad - \frac{8}{105\sqrt{\pi}} \\
&\quad \times \int_0^t \frac{(t-s)^{7/2}}{(1-s)^{3/2}} \left\{\sin s + \frac{s}{21}e^{v_k(s)}\right. \\
&\qquad \left. + \frac{1}{9}\left[v_k'(s)\right]^2 + v_k''(s)\right\} ds, \\
&\qquad t \in [0,1], \ k = 0,1,2,\ldots, \\
w_0(t) &= \frac{16t^{7/2}}{35\sqrt{\pi}}, \quad t \in [0,1], \\
w_{k+1}(t) &= \frac{8t^{7/2}}{105\sqrt{\pi}} \\
&\quad \times \int_0^1 \left\{\sin s + \frac{s}{21}e^{w_k(s)} + \frac{1}{9}\left[w_k'(s)\right]^2 + w_k''(s)\right\} ds \\
&\quad - \frac{8}{105\sqrt{\pi}} \\
&\quad \times \int_0^t \frac{(t-s)^{7/2}}{(1-s)^{3/2}} \left\{\sin s + \frac{s}{21}e^{w_k(s)}\right. \\
&\qquad \left. + \frac{1}{9}\left[w_k'(s)\right]^2 + w_k''(s)\right\} ds, \\
&\qquad t \in [0,1], \ k = 0,1,2,\ldots.
\end{aligned} \tag{55}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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