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Research Article

Backward Stochastic H_2/H_∞ Control: Infinite Horizon Case

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The mixed H_2/H_∞ control problem is studied for systems governed by infinite horizon backward stochastic differential equations (BSDEs) with exogenous disturbance signal. A necessary and sufficient condition for the existence of a unique solution to the H_2/H_∞ control problem is derived. The equivalent feedback solution is also discussed. Contrary to deterministic or stochastic forward case, the feedback solution is no longer feedback of the current state; rather, it is feedback of the entire history of the state.

1. Introduction

H_∞ control is one of the most important robust control approaches in which control law is sought to efficiently eliminate the effect of the exogenous disturbance in the practical system. We refer the reader to [1–3] and the references therein. If the purpose is to select control not only to restrain the exogenous disturbance, but also to minimize a cost function when the worst case disturbance d^* is implemented, this is the so-called mixed H_2/H_∞ control problem. Mixed H_2/H_∞ control problem has attracted much attention and has been widely applied to various fields. Please refer to [4, 5] for more information.

It should be pointed out that the above-mentioned works are concerned only with the forward stochastic systems. The case of systems governed by backward stochastic differential equations with exogenous disturbance signal, to our best knowledge, seems to be open. The objective of this paper is to develop an H_2/H_∞ control theory for infinite horizon backward stochastic systems.

A BSDE is an Itô stochastic differential equation (SDE) for which a random terminal condition on the state has been specified. Since BSDEs are well-defined dynamic systems, it is very natural and appealing to study the control problems involving BSDEs as well as their applications in lots of different fields, especially in finance, economics, insurance, and so forth. Please refer to [6–12] for more details. This paper is concerned with mixed H_2/H_∞ control of backward systems governed by infinite horizon linear BSDEs, namely, an infinite horizon backward stochastic H_2/H_∞ control problem. This means that our purpose is to study mixed H_2/H_∞ backward stochastic control problem in infinite horizon which presents more robust and stable sense in practise. For that, as preliminaries, we first need to review some results on infinite horizon BSDEs in Section 2. Chen and Wang [13] gave an existence and uniqueness result under a kind of Lipschitz condition suitable for one-dimensional infinite horizon BSDEs. Wu [14] generalized the result of [13] into the poisson jump process case in unbounded stopping time duration and obtained the corresponding comparison theorem. In this section, under this frame, we get the existence and uniqueness result for the infinite horizon matrix-valued BSDEs.

In Section 3, similar to the deterministic or stochastic forward case, we formulate the infinite horizon backward stochastic H_2/H_∞ control problem. In Section 4, a necessary and sufficient condition for the existence of a unique solution to the H_2/H_∞ control problem is derived. It is shown that the existence of a unique solution to the control problem is equivalent to the corresponding uncontrolled perturbed system to have a $\mathbb{L}_2$-gain less than or equal to γ and the resulting solution is characterized by the solution of an uncontrolled forward backward stochastic differential equation (FBSDE).

Under some monotone assumptions, Hu and Peng [15] and Peng and Wu [16] obtained the existence and uniqueness results in an arbitrarily prescribed time duration. Wu and Xu [17] gave some comparison theorems for FBSDEs. Riccati equation plays an important role to get the feedback form of the optimal control; please refer to Yong and Zhou [18] for the details. Section 5 gives the equivalent linear feedback solution by virtue of the solution of a Riccati-type equation. As it turns out, the infinite horizon backward stochastic H_2/H_∞ control can no longer be expressed as a linear feedback of the current state like that in deterministic or stochastic forward case. Rather, it depends, in general, on the entire past history of the state pair $(x(\cdot), z(\cdot))$.

2. Notations and Preliminary Results of Infinite Horizon BSDEs

To treat the infinite horizon backward stochastic H_2/H_∞ control problem, we need the following preliminary results of infinite horizon BSDEs.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a completed filtering probability space; let $(W_t)_{t\geq 0}$ be a standard one-dimensional Wiener process (our assumption that $W(\cdot)$ is scalar-valued is for the sake of simplicity; no essential difficulties are encountered when extending our analysis to the case of vector-valued Wiener process). $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration generated by this Wiener process $W(\cdot)$ up to time t, where $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$ and $\mathcal{F}_\infty = \vee_{t\geq 0}\mathcal{F}_t$.

Throughout this paper, we adopt the following conventional notations. S^n: the set of symmetric $n \times n$ matrices with real elements; A^T: the transpose of the matrix A; $A \geq 0$ $(A > 0)$: A is positive semidefinite (positive definite) real matrix; I: identity matrix; $\|x\| := x^T x = (\sum_{i=1}^n |x_i|^2)^{1/2}$ for n-dimensional vector $x = (x_1, \ldots, x_n)^T$; $\|A\| := \max_{x\in R^n, \|x\|=1}\|Ax\|$ for $A \in R^{n\times n}$; $N(\cdot) >$ (≥ 0): $N(t) >$ $(\geq) 0$ for a.s. $t \in R^+$; $M(\cdot) > (\geq) N(\cdot)$: $M(\cdot) - N(\cdot) > (\geq) 0$; X: a given Hilbert space;

$$
\begin{aligned}
L^2_{\mathcal{F}}(R^+; X) &=: \Big\{ f : R^+ \times \Omega \longrightarrow X \text{ is an } \mathcal{F}_t \\
&\quad - \text{adapted process such that } \mathbb{E}\int_0^\infty \|f(t)\|^2 dt \\
&\quad < \infty \Big\}; \\
\mathcal{S}^2 &=: \Big\{ v_t,\ 0 \leq t \leq \infty, \text{ is an } \mathcal{F}_t \\
&\quad - \text{adapted process such that } \mathbb{E}\Big[\sup_{0\leq t\leq\infty}\|v_t\|^2\Big] \\
&\quad < \infty \Big\};
\end{aligned}
$$

$$
\begin{aligned}
L^2 &=: \Big\{\xi,\ \xi \text{ is a vector-valued } \mathcal{F}_\infty \\
&\quad - \text{measurable random variable such that } \mathbb{E}\|\xi\|^2 \\
&\quad < \infty\Big\}.
\end{aligned} \tag{1}
$$

We consider the infinite horizon BSDE:

$$
x_t = \xi + \int_t^\infty f(s, x_s, z_s)\, ds - \int_t^\infty z_s dW_s, \quad t \in [0, \infty]; \tag{2}
$$

(x, z) take value in $R^n \times R^n$, $\xi \in L^2$, and f is a map from $\Omega \times [0, \infty] \times R^n \times R^n$ onto R^n which satisfies the following.

(H2.1) For all $(x, z) \in R^n \times R^n$, $f(\cdot, x, z)$ is progressively measurable and

$$
\mathbb{E}\left(\int_0^\infty \|f(s, 0, 0)\|\, ds\right)^2 < \infty. \tag{3}
$$

(H2.2) There exist two positive deterministic functions $u_1(t)$ and $u_2(t)$ such that, for all $(x_i, z_i) \in R^n \times R^n$, $i = 1, 2$,

$$
\begin{aligned}
&\|f(t, x_1, z_1) - f(t, x_2, z_2)\| \\
&\quad \leq u_1(t)\|x_1 - x_2\| + u_2(t)\|z_1 - z_2\|, \quad t \in [0, \infty), \\
&\int_0^\infty u_1(t)\, dt < \infty, \qquad \int_0^\infty u_2^2(t)\, dt < \infty.
\end{aligned} \tag{4}
$$

Then we have the following.

Theorem 1 (see Wu [14]). *There exists a unique solution* $(x, z) \in \mathcal{S}^2 \times L^2_{\mathcal{F}}(R^+; R^n)$ *satisfying the BSDE* (2).

Let us again consider a function F, which will be in the sequel the generator of the BSDE, defined on $\Omega\times[0,\infty]\times S^n\times S^n$, *with values in* S^n, *such that the process* $(F(t, y, z))_{t\in[0,\infty]}$ *is a progressively measurable process for each* $(y, z) \in S^n \times S^n$.

Along the line of Chen and Wang [13] or Wu [14] combined with that in Peng [19] for matrixed-valued BSDEs result in finite horizon, we get the following existence and uniqueness theorem for infinite horizon matrix-valued BSDEs.

Theorem 2. *Suppose that F satisfies the following.*

(H2.1′) *For all* $(y, z) \in S^n \times S^n$, $F(\cdot, y, z)$ *is progressively measurable and*

$$
\mathbb{E}\left(\int_0^\infty \|F(s, 0, 0)\|\, ds\right)^2 < \infty. \tag{5}
$$

(H2.2′) *There exist two positive deterministic functions* $u_1(t)$ *and* $u_2(t)$ *such that, for all* $(y_i, z_i) \in R^n \times R^n$, $i = 1, 2$,

$$
\begin{aligned}
&\|F(t, y_1, z_1) - F(t, y_2, z_2)\| \\
&\quad \leq u_1(t)\|y_1 - y_2\| + u_2(t)\|z_1 - z_2\|, \quad t \in [0, \infty),
\end{aligned} \tag{6}
$$

and $\int_0^\infty u_1(t)dt < \infty$, $\int_0^\infty u_2^2(t)dt < \infty$, ξ *is a given* S^n*-valued random variable, and* $\xi \in L^2$. *Then, the following matrix-valued infinite horizon BSDE*

$$Y_t = \xi + \int_t^\infty F(s, Y_s, Z_s)\, ds - \int_t^\infty Z_s dW_s \tag{7}$$

admits a unique solution $(Y, Z) \in \mathcal{S}^2 \times L^2_{\mathcal{F}}(R^+; R^n)$.

3. Problem Statement

Now, we consider the following stochastic control system governed by an infinite horizon linear BSDE:

$$\begin{aligned} x(t) = \xi - \int_t^\infty & [A(s)x(s) + B(s)u(s) \\ & + C(s)d(s) + D(s)z(s)]\, ds \\ & - \int_t^\infty z(s)\, dW(s). \end{aligned} \tag{8}$$

$Z \in \mathcal{R}^{n_Z}$ is the penalty output, and the energy of the output signal Z is given by

$$\|Z\|_2^2 = x_0^T H x_0 + \mathbb{E}\int_0^\infty \left[x_t^T Q_t x_t + z_t^T S_t z_t + u_t^T u_t\right] dt, \tag{9}$$

where H is a nonnegative symmetric constant matrix and $Q_t(\omega)$ and $S_t(\omega)$ are nonnegative symmetric bounded progressively measurable matrix-valued processes. u and d stand for the control input and exogenous disturbance signal, respectively. The energy of the disturbances is

$$\|d\|_2^2 = \mathbb{E}\int_0^\infty d_t^T d_t dt. \tag{10}$$

Later, we will state assumptions on the coefficients $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$, $Q(\cdot)$, $S(\cdot)$ so as to guarantee the existence of a unique solution pair $(x(\cdot), z(\cdot)) \in \mathcal{S}^2 \times L^2_{\mathcal{F}}(R^+; R^n)$ of BSDE (8) for any $u \in L^2_{\mathcal{F}}(R^+; R^{n_u})$, $d \in L^2_{\mathcal{F}}(R^+; R^{n_d})$, and $\xi \in L^2$. We refer to such a four-tuple $(x(\cdot), z(\cdot); u(\cdot), d(\cdot))$ as an admissible triple.

Now, we first define the infinite horizon backward stochastic H_2/H_∞ control as follows.

Definition 3 (backward stochastic H_2/H_∞ control). For given $\gamma > 0$ and $d \in L^2_{\mathcal{F}}(R^+; R^{n_d})$, find, if possible, a control $u = u^* \in L^2_{\mathcal{F}}(R^+; R^{n_u})$, such that

(i) the trajectory of the closed-loop system (8) with $\xi = 0$ satisfies

$$\|Z\|_2^2 \le \gamma^2 \|d\|_2^2, \quad \forall d \ne 0 \in L^2_{\mathcal{F}}(R^+; R^{n_d}) \text{ and} \tag{11}$$

(ii) when the worst case disturbance ([4]) $d^* \in L^2_{\mathcal{F}}(R^+; R^{n_d})$, if existing, is implemented in (8), u^* minimizes the quadratic performance $\|Z\|_2^2$ simultaneously.

If we define

$$\begin{aligned} J_1(u, d) &= \|Z\|_2^2 - \gamma^2 \|d\|_2^2, \\ J_2(u, d) &= \|Z\|_2^2 \end{aligned} \tag{12}$$

then the mixed H_2/H_∞ control problem is equivalent to find the Nash equilibria (u^*, d^*) defined as

$$J_1(u^*, d^*) \ge J_1(u^*, d), \quad \forall d \in L^2_{\mathcal{F}}(R^+; R^{n_d}), \tag{13}$$

$$J_2(u, d^*) \ge J_2(u^*, d^*), \quad \forall u \in L^2_{\mathcal{F}}(R^+; R^{n_u}), \tag{14}$$

$$J_1(u^*, d) \le 0, \quad \forall d \ne 0 \in L^2_{\mathcal{F}}(R^+; R^{n_d}), \quad \xi = 0. \tag{15}$$

Obviously, inequality (15) is associated with the H_∞ performance. The first Nash inequality (13) is to keep that d^* is the worst case disturbance, while the second one (14) is related with the H_2 performance. Clearly, if the Nash equilibria (u^*, d^*) exist and satisfy inequality (15), then u^* is our desired H_2/H_∞ controller and d^* is the worst case disturbance. In this case, we also say that the infinite horizon backward stochastic H_2/H_∞ control admits a solution (u^*, d^*).

Throughout this paper, we assume the following.

(A1) All matrices mentioned in this paper are bounded progressively measurable processes.

(A2)

$$\begin{aligned} &\mathbb{E}\int_0^\infty \|A(t)\|\, dt < \infty, \\ &\mathbb{E}\int_0^\infty \|D(t)\|^2 dt < \infty. \end{aligned} \tag{16}$$

(A3)

$$\begin{aligned} &\gamma > 0, \\ &Q \ge 0, \qquad \mathbb{E}\int_0^\infty \|Q(t)\|\, dt < \infty, \\ &BB^T(\cdot) > \frac{CC^T(\cdot)}{\gamma^2}, \\ &\mathbb{E}\int_0^\infty \left\| B(t)B(t)^T - \frac{C(t)C(t)^T}{\gamma^2} \right\| dt < \infty. \end{aligned} \tag{17}$$

From Theorem 2, we obtain that assumption (A2) is sufficient to guarantee the existence of a unique solution pair $(x(\cdot), z(\cdot)) \in \mathcal{S}^2 \times L^2_{\mathcal{F}}(R^+; R^n)$ of BSDE (8) for any $u \in L^2_{\mathcal{F}}(R^+; R^{n_u})$ and $d \in L^2_{\mathcal{F}}(R^+; R^{n_d})$.

4. The Necessary and Sufficient Condition

In this section, we give a necessary and sufficient condition for the existence of a unique solution to the backward stochastic H_2/H_∞ control problem. We begin our presentation with some preliminaries.

Consider the following uncontrolled stochastic perturbed system:

$$dx_t = [A(t)x_t + C(t)d_t + D(t)z_t]\,dt + z_t dB(t), \quad x(\infty) = \xi, \quad t \in [0,\infty). \tag{18}$$

Let Z be the to-be-controlled output. For any $0 < T < \infty$, define the perturbation operator $\mathbb{L} : L^2_{\mathcal{F}}(R^+; R^{n_d}) \to L^2_{\mathcal{F}}(R^+; R^{n_Z})$ as

$$\mathbb{L}(d) = Z|_{x_\infty=0}, \quad t \geq 0, \ d \in L^2_{\mathcal{F}}(R^+; R^{n_d}), \tag{19}$$

with its norm

$$\|\mathbb{L}\|_2 := \sup_{d\in L^2_{\mathcal{F}}(R^+;R^{n_d}),\, d\neq 0,\, x_\infty=0} \frac{\|\mathbb{L}(d)\|_2}{\|d\|_2} = \sup_{d\in L^2_{\mathcal{F}}(R^+;R^{n_d}),\, d\neq 0,\, x_\infty=0} \frac{\|Z\|_2}{\|d\|_2}, \tag{20}$$

where

$$\|Z\|_2^2 = x_0^T H x_0 + \mathbb{E}\int_0^\infty [x_t^T Q_t x_t + y_t^T S_t y_t]\,dt. \tag{21}$$

Obviously, $\mathbb{L}$ is a nonlinear operator.

Definition 4. Let $\gamma > 0$; system (18) is said to have $\mathbb{L}_2$-gain less than or equal to γ if for any nonzero $d \in L^2_{\mathcal{F}}(R^+; R^{n_d})$, $\|\mathbb{L}\|_2 \leq \gamma$.

Proposition 5. *For system* (18) *and given disturbance attenuation $\gamma > 0$, if there exists a function $\mathbf{P}(\cdot)$, satisfying the following SDE (the variables t and ω are suppressed):*

$$dP = \left[-A^T P - PA - Q - \frac{PCC^T P}{\gamma^2}\right] dt - D^T P dW(t), \quad P + S \leq 0, \quad P(0) = -H, \quad t \in [0,\infty), \tag{22}$$

then $\|\mathbb{L}\|_2 \leq \gamma$.

Proof. It only needs to note the following identity:

$$\begin{aligned}
&\|Z\|_2^2 - \gamma^2\|d\|_2^2 \\
&= \|Z\|_2^2 - \gamma^2\|d\|_2^2 + \mathbb{E}\int_0^\infty d(x^T P x) - x_0^T H x_0 \\
&= -\gamma^2\left\|v - \frac{C^T P x}{\gamma^2}\right\|_2^2 + \mathbb{E}\int_0^\infty y^T (P+S)\, y\, dt \leq 0.
\end{aligned} \tag{23}$$

□

The following theorem is a necessary and sufficient condition for the existence of a unique solution to the infinite horizon backward stochastic H_2/H_∞ control problem.

Theorem 6. *For system* (8), *the backward stochastic H_2/H_∞ control problem admits a solution if and only if the corresponding uncontrolled system* (18) *has $\mathbb{L}_2$-gain less than or equal to γ. Moreover, if the backward stochastic H_2/H_∞ control problem admits a solution, then the solution is unique with*

$$u^* = \frac{B^T p^*}{2}, \qquad d^* = -\frac{C^T p^*}{2\gamma^2}, \tag{24}$$

where (p^*, x^*, z^*) is the solution of the following FBSDE:

$$\begin{aligned}
dp_t^* &= [2Qx_t^* - A^T p_t^*]\,dt + [2Sz_t^* - D^T p_t^*]\,dB(t), \\
dx_t^* &= \left[Ax_t^* + \frac{BB^T p_t^*}{2} - \frac{CC^T p_t^*}{2\gamma^2} + Dz_t^*\right] dt + z_t^* dB(t), \\
p_0^* &= 2Hx_0^*, \quad x_\infty^* = \xi, \quad t \in [0,\infty).
\end{aligned} \tag{25}$$

Proof. (1) The Sufficient Condition. To show that the backward stochastic H_2/H_∞ control problem admits a unique solution (u^*, d^*) if the corresponding uncontrolled system (18) has $\mathbb{L}_2$-gain less than or equal to γ, we will show that (u^*, d^*) is a solution firstly.

Look at the above FBSDE; from [16], the FBSDE (25) has a unique solution (p_t^*, x_t^*, z_t^*). Now, we try to prove that d^* is the worst case disturbance. For any given $d \in L^2_{\mathcal{F}}(R^+; R^{n_d})$, suppose that x^d is the trajectory corresponding to $(u^*, d) \in L^2_{\mathcal{F}}(R^+; R^{n_u}) \times L^2_{\mathcal{F}}(R^+; R^{n_d})$. It is easy to see the trajectory corresponding to

$$(0, d^* - d) \tag{26}$$

is $x^* - x^d$ with initial state $x_0^* - x_0^d$ and terminal state 0. Hence, $(x^* - x^d, z^* - z^d)$ is the solution corresponding to $d^* - d$ for system (18) with terminal state 0. Since system (18) has $\mathbb{L}_2$-gain less than or equal to γ, then

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\infty \left[-(x^* - x^d)^T Q (x^* - x^d) - (z^* - z^d)^T S (z^* - z^d) \right.\right. \\
&\qquad \left.\left. + \gamma^2 (d^* - d)^T (d^* - d)\right] dt\right] \\
&\quad - (x_0^* - x_0^d)^T H (x_0^* - x_0^d) \geq 0,
\end{aligned} \tag{27}$$

$$\begin{aligned}
&J_2(u^*, d^*) - J_2(u^*, d) \\
&= \mathbb{E}\left[\int_0^\infty \left[x^{*T} Q x^* - x^{dT} Q x^d + z^{*T} S z^* - z^{dT} S z^d \right.\right. \\
&\qquad \left.\left. - \gamma^2 d^{*T} d^* + \gamma^2 d^T d\right] dt\right] \\
&\quad + x_0^{*T} H x_0^* - x_0^{dT} H x_0^d
\end{aligned}$$

$$= \mathbb{E}\left[\int_0^\infty \left[\left(x^* - x^d\right)^T Q\left(x^* - x^d\right) - 2x^{dT} Q\left(x^d - x^*\right) + \left(z^* - z^d\right)^T S\left(z^* - z^d\right) - 2z^{dT} S\left(z^d - z^*\right) - \gamma^2 d^{*T} d^* + \gamma^2 d^T d\right] dt\right] + \left(x_0^* - x_0^d\right)^T H\left(x_0^* - x_0^d\right) - 2x_0^{dT} H\left(x_0^d - x_0^*\right). \quad (28)$$

Applying Itô's formula to $p^{*T}(x^d - x^*)$,

$$\begin{aligned} &-2x_0^{*T} H\left(x_0^d - x_0^*\right) \\ &= \mathbb{E}\left[\int_0^\infty d\left[p^{*T}\left(x^d - x^*\right)\right]\right] \\ &= \mathbb{E}\int_0^\infty \left[2x^{*T} Q\left(x^d - x^*\right) + 2z^{*T} S\left(z^d - z^*\right) + \left(C^T p^*\right)^T (d - d^*)\right] dt \\ &= \mathbb{E}\left[\int_0^\infty \left[2x^{*T} Q\left(x^d - x^*\right) + 2z^{*T} S\left(z^d - z^*\right) - 2\gamma^2 v^{*T} (d - d^*)\right] dt\right]. \end{aligned} \quad (29)$$

Substituting $2x_0^{*T} H(x_0^d - x_0^*)$ into (28), we get

$$\begin{aligned} &J_2(u^*, d^*) - J_2(u^*, d) \\ &= \mathbb{E}\left[\int_0^\infty \left[-\left(x^* - x^d\right)^T Q\left(x^* - x^d\right) - \left(z^* - z^d\right)^T S\left(z^* - z^d\right) + \gamma^2 (d^* - d)^T (d^* - d)\right] dt\right] \\ &\quad - \left(x_0^* - x_0^d\right)^T H\left(x_0^* - x_0^d\right). \end{aligned} \quad (30)$$

From (27), then

$$J_2(u^*, d^*) - J_2(u^*, d) \geq 0. \quad (31)$$

So d^* is the worst case disturbance. Moreover, for $x_\infty = 0$, the FBSDE (25) admits a unique solution $(p^*, x^*, z^*) = (0, 0, 0)$; then

$$J_2(u^*, d) \leq J_2(u^*, d^*) = 0. \quad (32)$$

Hence, u^* restrains the exogenous disturbance. In the following, we will show that u^* also minimizes that cost function when the worst case disturbance d^* is implemented into system (8).

For any $u \in L^2_{\mathcal{F}}(\mathbb{R}^{n_u})$, let x_t^u be the trajectory of the system (8) corresponding to (u, d^*). Let us first consider

$$J_1(u^*, d^*) - J_1(u, d^*) = I_1, \quad (33)$$

where

$$\begin{aligned} I_1 &= -\mathbb{E}\left[\int_0^\infty \left[x^{*T} Q(t) x^* - x^{uT} Q x^u + z^{*T} S(t) z^* - z^{uT} S z^u + u^{*T} u^* - u^T u\right] dt\right] + x_0^{*T} H x_0^* - x_0^{uT} H x_0^u \\ &= \mathbb{E}\left[\int_0^\infty \left[(x^* - x^u)^T Q(x^* - x^u) + (z^* - z^u)^T S(z^* - z^u) + (u^* - u)^T (u^* - u) + 2x^{*T} Q(t)(x^u - x^*) + 2y^{*T} S(t)(z^u - z^*) + 2u^{*T}(u - u^*)\right] dt\right] \\ &\quad + (x_0^* - x_0^u)^T H(x_0^* - x_0^u) + 2x_0^{*T} H(x_0^u - x_0^*). \end{aligned} \quad (34)$$

From $p_0^* = 2Hx_0^*$, we use Itô's formula to $p_t^{*T}(x_t^u - x_t^*)$ and get

$$\begin{aligned} &2x_0^{*T} H(x_0^u - x_0^*) \\ &= -\mathbb{E}\left[\int_0^\infty \left[2x^{*T} Q(t)(x^u - x^*) + 2z^{*T} S(z^u - z^*) + 2u^{*T}(u - u^*)\right] dt\right]. \end{aligned} \quad (35)$$

Then because of Q, S, and H being nonnegative, we have

$$J_1(u^*, v^*) - J_1(u, v^*) = I_1 \geq 0. \quad (36)$$

Therefore, d^* minimizes the cost function when the worst case disturbance d^* is implemented into system (8).

So, $(u^*, d^*) = (B^T p^*/2, -C^T p^*/2\gamma^2)$ is a solution of the backward stochastic H_2/H_∞ control problem.

We are now in a position to prove the uniqueness of the solution. Assume that the backward stochastic H_2/H_∞ control has a solution (u^1, d^1), (x^1, z^1) is the corresponding solution for (8), and p^1 is the solution of the following BSDE:

$$\begin{aligned} dp^1 &= \left[2Qx^1 - A^T p^1\right] dt + \left[2Sz^1 - D^T p^1\right] dB(t), \\ p_0^1 &= 2Hx_0^1. \end{aligned} \quad (37)$$

Implementing d^1, having

$$\inf_{u \in L^2_{\mathcal{F}}(R^{n_u})} J_1\left(u, d^1\right), \quad (38)$$

is a standard LQ optimal control problem. By uniqueness, $u^1 = B^T p^1/2$.

Let x be the trajectory corresponding to $(u^1, d) = (u^1, -C^T p^1/\gamma^2)$; then

$$\begin{aligned}
0 &\geq J_2(u^1, d) - J_2(u^1, d^1) \\
&= \mathbb{E}\left[\int_0^\infty \left[x^T Q x - x^{1T} Q x^1 + z^T S z - z^{1T} S z^1 \right.\right. \\
&\qquad \left.\left. -\gamma^2 d^T d + \gamma^2 d^{1T} d^1 \right] dt\right] \\
&\quad + x_0^T H x_0 - x_0^{1T} H x_0^1 \\
&= \mathbb{E}\left[\int_0^\infty \left[(x^1 - x)^T Q (x^1 - x) \right.\right. \\
&\qquad - 2x^{1T} Q (x^1 - x) + (z^1 - z)^T S (z^1 - z) \\
&\qquad \left.\left. -2z^{1T} S (z^1 - z) + \gamma^2 d^{1T} d^1 - \gamma^2 d^T d \right] dt\right] \\
&\quad + (x_0^1 - x_0)^T H (x_0^1 - x_0) - 2x_0^{1T} H (x_0^1 - x_0).
\end{aligned} \tag{39}$$

Applying Itô's formula to $p^{1T}(x^1 - x)$,

$$\begin{aligned}
2x_0^{1T} H (x_0^1 - x_0) &= \mathbb{E}\int_0^\infty d\left[p^{1T}(x^1 - x)\right] \\
&= \mathbb{E}\left[\int_0^\infty \left[2x^{1T} Q (x^1 - x) \right.\right. \\
&\qquad \left.\left. +(C^T z^1)^T (d^1 - d) + 2z^{1T} S (z^1 - z)\right] dt\right] \\
&= \mathbb{E}\left[\int_0^\infty \left[2x^{1T} Q (x^1 - x) \right.\right. \\
&\qquad \left.\left. + 2z^{1T} S (z^1 - z) - 2\gamma^2 d^T (d^1 - d)\right] dt\right].
\end{aligned} \tag{40}$$

Substituting $-2x_0^{1T} H(x_0^1 - x_0)$ into (39), then

$$\begin{aligned}
0 &\geq J_2(\gamma, x_0; u^1, d) - J_2(\gamma, x_0; u^1, d^1) \\
&= \mathbb{E}\left[\int_0^\infty \left[(x^1 - x)^T Q (x^1 - x) \right.\right. \\
&\qquad + (z^1 - z)^T S (z^1 - z) \\
&\qquad \left.\left. +\gamma^2 (z - z^1)^T (z - z^1)\right] dt\right].
\end{aligned} \tag{41}$$

Because of Q, S, and M being nonnegative, we get $d^1 = v = C^T z^1/\gamma^2$.

Therefore, $(u^1, d^1) = (u^*, d^*)$.

(2) The Necessary Condition. Here we assume that a solution exists; then from the uniqueness of the solution, we get that (u^*, d^*) is the unique solution and we will show that system (18) has $\mathbb{L}_2$-gain less than or equal to γ.

For $x_T = 0$, the FBSDE (25) has a unique solution $(p^*, x^*, z^*) = (0, 0, 0)$; then $(u^*, d^*) = (0, 0)$ and

$$J_2(u^*, d) \leq J_2(u^*, d^*) = 0, \quad \forall d \in L^2_{\mathcal{F}}(\mathcal{R}^{n_d}). \tag{42}$$

Therefore, system (18) has $\mathbb{L}_2$-gain less than or equal to γ. □

5. The Linear Feedback Solution

The main result of this section gives the equivalent linear feedback solution. For the purpose of this section the coefficients A_t, B_t, C_t, D_t, E_t, Q_t, and S_t are assumed deterministic functions; (18) has $\mathbb{L}_2$-gain less than or equal to γ.

Let (p, x, z) be the solution of (25); we first give the relations between p, x, and z using the undetermined coefficients method. Now, we introduce the following generalized matrix-valued Riccati equation (the variables t are suppressed):

$$\begin{aligned}
&\dot{K} - AK - KA^T + 2KQK - \frac{BB^T}{2} \\
&\quad + \frac{CC^T}{2\gamma^2} + D(I - 2KS)^{-1} K D^T = 0, \qquad K(\infty) = 0.
\end{aligned} \tag{43}$$

Similar to the line developed by Lim and Zhou [6], we can prove that (43) admits a unique solution $K(\cdot)$. Letting $K(\cdot)$ be the solution to (43), we define the following equations:

$$\begin{aligned}
dh &= \left[Ah - 2KQh + D(I - 2KS)^{-1}\eta\right] dt + \eta dB(t), \\
h(\infty) &= \xi.
\end{aligned} \tag{44}$$

Equation (44) is a linear BSDE and admits a unique solution (h, η).

Theorem 7. *Suppose that* $(p(\cdot), x(\cdot), z(\cdot))$, $K(\cdot)$, *and* $(h(\cdot), \eta(\cdot))$ *are the solutions of* (25), (43), *and* (44), *respectively; then the following relations are satisfied:*

$$\begin{aligned}
x(t) &= K(t) p(t) + h(t), \\
z(t) &= (I - 2K(t)S(t))^{-1}\left(\eta(t) - K(t) D(t)^T p(t)\right), \\
x(0) &= (I - 2K(0)H)^{-1} h(0).
\end{aligned} \tag{45}$$

Proof. Let $x(t) = K(t)p(t) + h(t)$. We apply Itô's formula to $x(t)$, $K(t)p(t) + h(t)$, respectively, and it is easy to check that $K(t)$ and $h(t)$ satisfy (43) and (44), respectively. □

From Theorem 7, we know that $x(\cdot)$ can be written to the functions of $K(\cdot)$, $p(\cdot)$, and $h(\cdot)$. Now we would like to derive the feedback solution using the undetermined coefficients method. First, we introduce the generalized matrix-valued Riccati equation and a linear SDE:

$$\begin{aligned}&\dot{\Sigma} + \Sigma A + A^T\Sigma \\ &\quad + \Sigma\left[\frac{BB^T}{2} - \frac{CC^T}{2\gamma^2} - D(I-2KS)^{-1}KD^T\right] \\ &\quad \times \Sigma - 2Q = 0, \\ &\Sigma(0) = 2H,\end{aligned} \tag{46}$$

$$\begin{aligned}&dr \\ &= \left[-A^T r - \frac{\Sigma BB^T r}{2} + \frac{\Sigma CC^T r}{2\gamma^2} + \Sigma D(I-2KS)^{-1}KD^T r \right. \\ &\quad \left. -\Sigma D(I-2KS)^{-1}\eta\right] dt \\ &\quad + \left[(2S-\Sigma)(I-2KS)^{-1}\left[\eta - KD^T(I-\Sigma K)^{-1}(\Sigma h + r)\right]\right. \\ &\quad \left. -D^T(I-\Sigma K)^{-1}(\Sigma h + r)\right] dB(t), \\ &r(0) = 0.\end{aligned} \tag{47}$$

Similar to the line developed by Lim and Zhou [6], we can prove that (46) admits a unique solution $\Sigma(\cdot)$. Equation (47) is a linear SDE and has a unique solution $r(\cdot)$.

Theorem 8. *The backward stochastic H_2/H_∞ control problem has a feedback solution (u^*, v^*),*

$$u^* = \frac{B^T(\Sigma x + r)}{2}, \qquad v^* = -\frac{C^T(\Sigma x + r)}{2\gamma^2}. \tag{48}$$

Proof. Let $p(t) = \Sigma(t)x(t) + r(t)$. We apply Itô's formula to $p(t)$ and $\Sigma(t)x(t) + r(t)$, respectively, and it is easy to check that $\Sigma(t)$ and $r(t)$ satisfy (46) and (47). □

Remark 9. From Theorem 8, we see that the solution involves an additional random nonhomogeneous term $r(\cdot)$. This addition disqualifies (48) from a feedback control of the current state, contrary to the deterministic or stochastic forward H_2/H_∞ (see [4, 5]) cases. The reason is because $r(\cdot)$ depends on $(h(\cdot), \eta(\cdot))$, which in turn depends on ξ, the terminal condition of part of the state variable, $x(\cdot)$. This is one of the major distinctive features of the backward stochastic H_2/H_∞ problem.

Finally, it is important to recognize that the expressions for the backward stochastic H_2/H_∞ control, as presented in Theorems 6 and 8, are equivalent expressions of the same process; that is, this does not contradict the uniqueness of the solution.

We present an example to illustrate the above theoretical results as follows.

Example 10. Consider the backward stochastic H_2/H_∞ control problem of the following one-dimensional system:

$$\begin{aligned}&x(t) \\ &= \xi - \int_t^\infty \left[2e^{-s}x(s) + \sqrt{4e^{-s}+2}\,u(s)\right. \\ &\quad \left. + \sqrt{e^{-s}+1}\,d(s) + 2e^{-s/2}z(s)\right] ds \\ &\quad - \int_t^\infty z(s)\,dW(s),\end{aligned} \tag{49}$$

with controlled output energy

$$\|Z\|_2^2 = \mathbb{E}\int_0^\infty \left[\frac{e^{-t}x_t^2}{2} + u_t^2\right] dt. \tag{50}$$

If we take $\gamma = \sqrt{2}/2$, then the Riccati equation (43) specializes to

$$K(t) = \int_t^\infty \left[e^{-s}\left(K(s)^2 - 1\right)\right] ds. \tag{51}$$

Solving it yields $K(t) = (1 - e^{2e^{-t}})/(1 + e^{2e^{-t}})$. Equation (44) specializes to

$$\begin{aligned}&h(t) \\ &= \xi - \int_t^\infty \left[\left\{2e^{-s} - e^{-s}\frac{1-e^{2e^{-s}}}{1+e^{2e^{-s}}}\right\}h(s) + 2e^{-s/2}\eta(t)\right] dt \\ &\quad - \int_t^\infty \eta(t)\,dW(t).\end{aligned} \tag{52}$$

Then, from Theorem 7, we get a unique solution

$$\begin{aligned}&(u^*, d^*) \\ &= \left(\sqrt{\frac{2e^{-t}+1}{2}} \cdot \frac{1+e^{2e^{-t}}}{1-e^{2e^{-t}}} \cdot [x(t) - h(t)],\right. \\ &\quad \left. -\sqrt{e^{-t}+1} \cdot \frac{1+e^{2e^{-t}}}{1-e^{2e^{-t}}} \cdot [x(t) - h(t)]\right),\end{aligned} \tag{53}$$

of the backward H_2/H_∞ control problem.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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