# Multiconsensus of Second-Order Multiagent Systems with Input Delays 

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#### Abstract

The multiconsensus problem of double-integrator dynamic multiagent systems has been investigated. Firstly, the dynamic multiconsensus, the static multiconsensus, and the periodic multiconsensus are considered as three cases of multiconsensus, respectively, in which the final multiconsensus convergence states are established by using matrix analysis. Secondly, as for the multiagent system with input delays, the maximal allowable upper bound of the delays is obtained by employing Hopf bifurcation of delayed networks theory. Finally, simulation results are presented to verify the theoretical analysis.


## 1. Introduction

Recently, distributed coordination control of multiagent systems has drawn an increasing attention of researchers. This is mainly due to the fact that its broad applications have ranged from cooperative control of unmanned air vehicles, formation control of mobile robots, and design of sensor network to swarm-based computing. The main objective of the consensus problem, which is one of the most fundamental issues in coordination control, is to design an appropriate control protocol to make a group of agents reach an agreement on certain quantities of interest by negotiating with their neighbors.

As we all know, the earlier study on consensus problem is primarily about single-integrator dynamic multiagent systems [1-6]. In this case, the task of the consensus algorithm is to guarantee that positions of all the agents converge to a constant value. Moreover, the consensus problem of doubleintegrator dynamic multiagent systems has also aroused growing concern [7-10], which is more challenging than the first case. It is worth to point out that the control goal of all the aforementioned studies is to drive the states of all the agents in a network to a consistent value.

However, in reality, sometimes the agreements are different because of the changes of environment, situation, or cooperative tasks. For example, in nature, a flock of foraging
birds may incorporate or evolve into different subgroups for the sake of resisting foreign intrusion. In the study of formation control problem, the formation is split into several subformations in order to fulfill total task or avoid obstacles, which results in different agreements. Hence, it is of vital significant to study multiconsensus and to design algorithms so that the agents in each subnetwork achieve consensus while there is no consensus among different subnetworks. To illustrate, by using pinning control technology, the multiconsensus problem of multiagent systems was investigated in [11, 12], where the pinned agents were chosen in accordance with the topological structure of the underlying graphs. The multiconsensus of first-order multiagent systems was discussed under fixed topology and switching topology, in which the interaction between the two subnetworks was assumed to be balanced [13, 14]. Under the same assumptions, two different kinds of multiconsensus protocols of second-order multiagent systems for networks with fixed communication topology were presented [15]. Under more mild assumptions, the authors proposed necessary and sufficient conditions of group consensus of first-order multiagent systems with directed and fixed topology [16]. Inspired by the progress in the field, this paper tries to further investigate multiconsensus problem and propose a more general control protocol for second-order multiagent systems.

In addition, we found that time delay is inevitable in consensus convergence of multiagent systems. This is because both the movements of the agents and the congestion of the communication and connected controllers by networks may cause time delay. Hence, it is necessary to consider the effect of the time delay. Based on the frequency domain analysis, the consensus of first-order discrete-time multiagent systems with diverse input and communication delays was discussed in [17], which showed that the consensus condition was dependent on input delays but independent of communication delays. The leader following consensus problem with diverse input delays and symmetric coupling weights was explored [18]. Besides, the consensus of heterogeneous multiagent systems with symmetric coupling weights under identical input delays and different input delays was analyzed in [19], respectively. The finite-time consensus problem of multiagent systems with delays was studied, in which the nonsmooth protocol was proposed to make the system reach agreement in finite time [20]. So far, few works have been performed on multiconsensus of second-order multiagent systems with input delays.

Motivated by all the above results, we focus on the multiconsensus of second-order multiagent systems without delay and with delays, respectively. The network is divided into multiple subnetworks, and all the agents in it are divided into multiple groups consequently. We assume that information exchange exists between not only two agents in a group but also in different groups. Based on stability theory, three sorts of multiconsensus, namely, the dynamic multiconsensus, the static multiconsensus, and the periodic multiconsensus, are considered, in which the final multiconsensus convergence states are obtained for the case without delay. As for the case with input delays, we establish a sufficient condition by employing Hopf bifurcation of delayed networks theory, in which multiconsensus can be achieved if the time delay is less than a certain critical value.

The remainder of this paper is shown as follows. In Section 2, we present some concepts in graph theory and formulate the model to be studied. In Section 3, the multiconsensus problem of second-order multiagent system without delay and with input delays is discussed for the directed network, respectively. Meanwhile, the simulation results are presented to illustrate the effectiveness of the theoretical results in Section 4. Finally, we draw the conclusion in Section 5.

Notation. Throughout this paper, we let $R$ be the set of real number. $1_{n}$ is denoted as an $n$-dimensional column with all the elements being one and $0_{n}$ is denoted as an $n$-dimensional column with all the elements being zero. $0_{m \times n}$ denotes the $m \times n$ matrix with all zero entries and $I_{n}$ denotes $n \times n$ identity matrix. $\operatorname{Re}(\mu)$ and $\operatorname{Im}(\mu)$ denote the real part and imaginary part of a complex number $\mu$, respectively. $\operatorname{det}(A)$ denotes determinant of matrix $A$.

## 2. Preliminaries and Problem Formulation

Let $\mathscr{G}=(\mathscr{V}, \varepsilon, \mathscr{A})$ be a weighted directed graph with the vertex set $\mathscr{V}=\left\{\nu_{1}, \nu_{2}, \ldots, \nu_{n}\right\}$, the edge set $\varepsilon \subseteq \mathscr{V} \times \mathscr{V}$, and a nonsymmetric matrix $\mathscr{A}=\left(a_{i j}\right)_{n \times n}$. An edge $\varepsilon_{i j}=\left(v_{j}, v_{i}\right)$
means that agent $i$ can receive information from agent $j . \mathscr{A}=$ $\left(a_{i j}\right)_{n \times n}$ is defined as $a_{i j} \neq 0$ if $\varepsilon_{i j} \in \varepsilon$ and $a_{i j}=0$, otherwise. Moreover, we assume that $a_{i i}=0$ for all $i$. The set of neighbors of agent $i$ is denoted by $\mathscr{N}_{i}=\left\{\nu_{j} \mid \varepsilon_{i j} \in \varepsilon\right\}$.

In this paper, we consider a complex network $(\mathscr{G}, \boldsymbol{x})$ consisting of $n+m$ agents. All the agents are divided into two parts and the agents in each part build up a subnetwork. Therefore, it consists of two subnetworks $\left(\mathscr{G}_{1}, x^{1}\right)$ and $\left(\mathscr{G}_{2}, x^{2}\right)$. We denote $l_{1}=\{1,2, \ldots, n\}, l_{2}=\{n+1, \ldots, n+m\}$, and $l=l_{1} \cup l_{2}$. Furthermore, we denote $\mathscr{V}_{1}=\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$ and $\mathscr{V}_{2}=$ $\left\{v_{n+1}, \ldots, v_{n+m}\right\}$. Finally, we let $\mathcal{N}_{1 i}=\left\{\nu_{j} \in \mathscr{V}_{1}: \varepsilon_{i j} \in \varepsilon\right\}$, $\mathcal{N}_{2 i}=\left\{\nu_{j} \in \mathscr{V}_{2}: \varepsilon_{i j} \in \varepsilon\right\}$, and $\mathcal{N}_{i}=\mathcal{N}_{1 i} \cup \mathcal{N}_{2 i}$.

Consider vehicles with double-integrator dynamics given by

$$
\begin{align*}
& \dot{x}_{i}=v_{i}, \quad i=1,2, \ldots, n+m  \tag{1}\\
& \dot{v}_{i}=u_{i},
\end{align*}
$$

where $x_{i} \in R, v_{i} \in R$ is the state and the velocity of the agent $i$, respectively, and $u_{i} \in R$ is the control input.

Definition 1. For second-order multiagent system (1), three sorts of multiconsensus are defined.
(i) The system (1) is said to reach a dynamic multiconsensus asymptotically, if for any initial conditions, we have

$$
\begin{gather*}
\lim _{t \rightarrow \infty}\left\|x_{i}(t)-x_{j}(t)\right\|=0 \\
\lim _{t \rightarrow \infty}\left\|v_{i}(t)-v_{j}(t)\right\|=0, \quad \forall i, \quad j \in l_{k}, \quad k=1,2 \tag{2}
\end{gather*}
$$

(ii) The system (1) is said to reach a static multiconsensus asymptotically if, for any initial conditions, we have

$$
\begin{gather*}
\lim _{t \rightarrow \infty}\left\|x_{i}(t)-x_{j}(t)\right\|=0, \quad \forall i, j \in l_{k}, k=1,2  \tag{3}\\
\lim _{t \rightarrow \infty} v_{i}(t)=0
\end{gather*}
$$

(iii) The system (1) is said to reach a periodic multiconsensus asymptotically if, for any initial conditions, all the agents in the same subnetwork can reach periodic consensus and the agents in different subnetwork can not coincide.

Motivated by consensus protocol in [9, 13], the following multiconsensus protocol is proposed:

$$
u_{i}= \begin{cases}\sum_{v_{j} \in \mathcal{N}_{1 i}} a_{i j}\left[\gamma_{0}\left(x_{j}-x_{i}\right)+\gamma_{1}\left(v_{j}-v_{i}\right)\right] &  \tag{4}\\ \quad+\sum_{v_{j} \in \mathcal{N}_{2 i}} a_{i j}\left[\gamma_{0} x_{j}+\gamma_{1} v_{j}\right]-\beta x_{i}-\alpha v_{i}, & \forall i \in l_{1} \\ \sum_{v_{j} \in \mathcal{N}_{1 i}} a_{i j}\left[\gamma_{0} x_{j}+\gamma_{1} v_{j}\right] & \\ \quad+\sum_{v_{j} \in \mathcal{N}_{2 i}} a_{i j}\left[\gamma_{0}\left(x_{j}-x_{i}\right)+\gamma_{1}\left(v_{j}-v_{i}\right)\right] & \\ -\beta x_{i}-\alpha v_{i}, & \forall i \in l_{2}\end{cases}
$$

where $\gamma_{0}, \gamma_{1}, \alpha$, and $\beta$ are nonnegative constants, $a_{i j} \geq 0$ for all $i, j \in l_{1}, a_{i j} \geq 0$ for all $i, j \in l_{2}$, and $a_{i j} \in R$ for all $i \in l_{1}, j \in l_{2}$ or $i \in l_{2}, j \in l_{1}$.

Remark 2. Obviously, the protocol (4) is a more general case, which contains protocol (3) in [9] as a special case. The consensus is achieved by designing protocol (3) in [9]. But our idea is to establish criteria, which can make the first $n$ agents reach a consistent state while the last $m$ agents reach another consistent state. Therefore, the protocol (4) is proposed.

Let $x=\left[x_{1}, x_{2}, \ldots, x_{n+m}\right]^{T}$ and $v=\left[v_{1}, v_{2}, \ldots, v_{n+m}\right]^{T}$. The systems (1) with protocol (4) can be written as follows:

$$
\left[\begin{array}{c}
\dot{x}  \tag{5}\\
\dot{v}
\end{array}\right]=\underbrace{\left[\begin{array}{cc}
0_{(n+m) \times(n+m)} & I_{(n+m)} \\
-\beta I_{(n+m)}-\gamma_{0} L & -\alpha I_{(n+m)}-\gamma_{1} L
\end{array}\right]}_{\Gamma}\left[\begin{array}{l}
x \\
v
\end{array}\right],
$$

where $L=\left[l_{i j}\right]$ is defined as

$$
l_{i j}= \begin{cases}\sum_{k=1, k \neq i}^{m+n} a_{i k}, & j=i  \tag{6}\\ -a_{i j}, & j \neq i\end{cases}
$$

Before moving on, we make the following assumption as in [12, 13]:

Assumption 3. (i) $\sum_{j=n+1}^{n+m} a_{i j}=0$ for all $i \in l_{1}$ and (ii) $\sum_{j=1}^{n} a_{i j}=$ 0 for all $i \in l_{2}$.

Assumption 3 means that the effect between two subnetworks is balanced. As a result, each row sum of the matrix $L$ is zero. Therefore, 0 is an eigenvalue of $L$.

## 3. Main Results

In this section, we deal with multiconsensus problem of second-order multiagent system (1).
3.1. Multiconsensus of Second-Order Multiagent System. For the linear model (5), eigenvalues of matrix $\Gamma$ are discussed first because they count a lot in stability analysis. Suppose that $\lambda_{i j}(i=1, \ldots, n+m, j=1,2)$ and $\mu_{i}$ are eigenvalues of $\Gamma$ and $-L$, respectively.

Let $\lambda$ be an eigenvalue of matrix $\Gamma$. Then, we have $\operatorname{det}\left(\lambda I_{2(n+m)}-\Gamma\right)=0$.

Note that

$$
\begin{align*}
\operatorname{det} & \left(\lambda I_{2(n+m)}-\Gamma\right) \\
& =\operatorname{det}\left(\left[\begin{array}{cc}
\lambda I_{(n+m)} & -I_{(n+m)} \\
\beta I_{(n+m)}+\gamma_{0} L & \lambda I_{(n+m)}+\alpha I_{(n+m)}+\gamma_{1} L
\end{array}\right]\right) \\
& =\operatorname{det}\left(\lambda^{2} I_{(n+m)}+\left(\alpha I_{(n+m)}+\gamma_{1} L\right) \lambda+\beta I_{(n+m)}+\gamma_{0} L\right) \\
& =\prod_{i=1}^{n+m}\left[\lambda^{2}+\left(\alpha-\gamma_{1} u_{i}\right) \lambda+\left(\beta-\gamma_{0} u_{i}\right)\right]=0 \tag{7}
\end{align*}
$$

Hence,

$$
\begin{align*}
& \lambda_{i 1}=\frac{-\left(\alpha-\gamma_{1} \mu_{i}\right)+\sqrt{\left(\alpha-\gamma_{1} \mu_{i}\right)^{2}-4\left(\beta-\gamma_{0} \mu_{i}\right)}}{2}  \tag{8}\\
& \lambda_{i 2}=\frac{-\left(\alpha-\gamma_{1} \mu_{i}\right)-\sqrt{\left(\alpha-\gamma_{1} \mu_{i}\right)^{2}-4\left(\beta-\gamma_{0} \mu_{i}\right)}}{2}
\end{align*}
$$

Lemma 4 (see [15]). Under Assumption 3, L has a zero eigenvalue whose geometric multiplicity is at least two.

Proof. It is easy to verify that $p_{1}=\left(1_{n}^{T}, 0_{m}^{T}\right)^{T}$ and $p_{2}=$ $\left(0_{n}^{T}, 1_{m}^{T}\right)^{T}$ are two linearly independent right eigenvectors of matrix $L$ associated with zero eigenvalues. This completes the proof.

Theorem 5. Suppose that $-L$ has two simple zero eigenvalues; multiagent system (1) achieves multiconsensus if

$$
\begin{equation*}
\operatorname{Re}\left(\lambda_{i j}\right)<0, \quad(i=3, \ldots, n+m ; j=1,2) \tag{9}
\end{equation*}
$$

where $\lambda_{i j}(i=3, \ldots, n+m, j=1,2)$ are eigenvalues of matrix $\Gamma$.

Proof. From Lemma 4, suppose that $q_{1}=\left(q_{11}^{T}, q_{12}^{T}\right)^{T}$ and $q_{2}=\left(q_{21}^{T}, q_{22}^{T}\right)^{T}$ are two linearly independent left eigenvectors of matrix $L$ corresponding to zero eigenvalues which satisfy $p_{1}^{T} q_{1}=1$ and $p_{2}^{T} q_{2}=1$.

For $\mu_{1}=\mu_{2}=0$, the corresponding eigenvalues of matrix $\Gamma$ are

$$
\begin{align*}
& \lambda_{11}=\lambda_{21}=\frac{-\alpha+\sqrt{\alpha^{2}-4 \beta}}{2} \\
& \lambda_{12}=\lambda_{22}=\frac{-\alpha-\sqrt{\alpha^{2}-4 \beta}}{2} \tag{10}
\end{align*}
$$

For convenience, one denotes $\lambda_{11}=\lambda_{21} \triangleq \lambda_{1}, \lambda_{12}=$ $\lambda_{22} \triangleq \lambda_{2}$.
Case $1\left(\alpha^{2}-4 \beta \neq 0\right)$. Let $w_{1 r}$ be the right eigenvector of matrix $\Gamma$ associated with eigenvalue $\lambda_{1}$. By Lemma 4 , one has $L p_{1}=$ 0.

Note that

$$
\begin{align*}
\Gamma\binom{p_{1}}{\lambda_{1} p_{1}} & =\binom{\lambda_{1} p_{1}}{-\beta p_{1}-\gamma_{0} L p_{1}-\alpha \lambda_{1} p_{1}-\gamma_{1} \lambda_{1} L p_{1}} \\
& =\binom{\lambda_{1} p_{1}}{-\beta p_{1}-\alpha \lambda_{1} p_{1}}  \tag{11}\\
\lambda_{1}\binom{p_{1}}{\lambda_{1} p_{1}} & =\binom{\lambda_{1} p_{1}}{\lambda_{1}^{2} p_{1}}=\binom{\lambda_{1} p_{1}}{-\beta p_{1}-\alpha \lambda_{1} p_{1}}
\end{align*}
$$

Thus,

$$
\begin{equation*}
\Gamma\binom{p_{1}}{\lambda_{1} p_{1}}=\lambda_{1}\binom{p_{1}}{\lambda_{1} p_{1}} \tag{12}
\end{equation*}
$$

It follows from (12) that $w_{1 r}=\left(p_{1}^{T}, \lambda_{1} p_{1}^{T}\right)^{T}$. It is straightforward to verify that $w_{2 r}=\left(p_{2}^{T}, \lambda_{1} p_{2}^{T}\right)^{T}$ is right
eigenvector of matrix $\Gamma$ associated with eigenvalue $\lambda_{1}$ which is linearly independent of $w_{1 r}$. Similarly, $\omega_{3 r}=\left(p_{1}^{T}, \lambda_{2} p_{1}^{T}\right)^{T}$ and $\omega_{4 r}=\left(p_{2}^{T}, \lambda_{2} p_{2}^{T}\right)^{T}$ can be verified to be linearly independent right eigenvectors of matrix $\Gamma$ corresponding to eigenvalue $\lambda_{2}$.

Denote

$$
\begin{align*}
& \lambda_{1}^{\prime}=\frac{\alpha+\sqrt{\alpha^{2}-4 \beta}}{2}=-\frac{\beta}{\lambda_{1}}  \tag{13}\\
& \lambda_{2}^{\prime}=\frac{\alpha-\sqrt{\alpha^{2}-4 \beta}}{2}=-\frac{\beta}{\lambda_{2}}
\end{align*}
$$

$$
\Gamma=P J P^{-1}=\left(\omega_{1 r}, \ldots, \omega_{2(n+m) r}\right)
$$

It can be verified that $\omega_{1 l}=\left(\lambda_{1}^{\prime} q_{1}^{T}, q_{1}^{T}\right)^{T}$ and $\omega_{2 l}=$ $\left(\lambda_{1}^{\prime} q_{2}^{T}, q_{2}^{T}\right)^{T}$ are two linearly independent left eigenvectors of matrix $\Gamma$ associated with eigenvalue $\lambda_{1}$. Similarly, one can obtain that $\omega_{3 l}=\left(1 / \sqrt{\alpha^{2}-4 \beta}\right)\left(\lambda_{2}^{\prime} q_{1}^{T}, q_{1}^{T}\right)^{T}$ and $\omega_{4 l}=$ $\left(1 / \sqrt{\alpha^{2}-4 \beta}\right)\left(\lambda_{2}^{\prime} q_{2}^{T}, q_{2}^{T}\right)^{T}$ are two linearly independent left eigenvectors of matrix $\Gamma$ corresponding to eigenvalue $\lambda_{2}$. It can be seen that $\omega_{i r}^{T} \omega_{i l}=1(i=1,2,3,4)$ after simple calculation.

Let $J$ be the Jordan canonical form associated with $\Gamma$. Then, there exists a nonsingular matrix $P$ such that

$$
\times\left(\begin{array}{ccccc}
\lambda_{1} & 0 & 0 & 0 & 0_{1 \times(2 n+2 m-4)}  \tag{14}\\
0 & \lambda_{1} & 0 & 0 & 0_{1 \times(2 n+2 m-4)} \\
0 & 0 & \lambda_{2} & 0 & 0_{1 \times(2 n+2 m-4)} \\
0 & 0 & 0 & \lambda_{2} & 0_{1 \times(2 n+2 m-4)} \\
0_{(2 n+2 m-4) \times 1} & 0_{(2 n+2 m-4) \times 1} & 0_{(2 n+2 m-4) \times 1} & 0_{(2 n+2 m-4) \times 1} & J_{1}
\end{array}\right)
$$

$$
\times\left(\begin{array}{c}
\omega_{1 l}^{T} \\
\omega_{2 l}^{T} \\
\omega_{3 l}^{T} \\
\vdots \\
\omega_{2(n+m) l}^{T}
\end{array}\right)
$$

where $J_{1}$ is the Jordan upper diagonal block matrix corresponding to the eigenvalues $\lambda_{i j}(i=3, \ldots, n+m ; j=1,2)$.

$$
\begin{aligned}
& \lim _{t \rightarrow \infty} e^{\Gamma t} \\
& =\lim _{t \rightarrow \infty} P e^{J t} P^{-1}
\end{aligned}
$$

$$
=\left(\begin{array}{cccc}
\left(e^{\lambda_{1} t} \lambda_{1}^{\prime}+e^{\lambda_{2} t} \lambda_{2}^{\prime}\right) 1_{n} q_{11}^{T} & \left(e^{\lambda_{1} t} \lambda_{1}^{\prime}+e^{\lambda_{2} t} \lambda_{2}^{\prime}\right) 1_{m} q_{12}^{T} & \left(e^{\lambda_{1} t}+e^{\lambda_{2} t}\right) 1_{n} q_{11}^{T} & \left(e^{\lambda_{1} t}+e^{\lambda_{2} t}\right) 1_{m} q_{12}^{T}  \tag{15}\\
\left(e^{\lambda_{1} t} \lambda_{1}^{\prime}+e^{\lambda_{2} t} \lambda_{2}^{\prime}\right) 1_{n} q_{21}^{T} & \left(e^{\lambda_{1} t} \lambda_{1}^{\prime}+e^{\lambda_{2} t} \lambda_{2}^{\prime}\right) 1_{m} q_{22}^{T} & \left(e^{\lambda_{1} t}+e^{\lambda_{2} t}\right) 1_{n} q_{21}^{T} & \left(e^{\lambda_{1} t}+e^{\lambda_{2} t}\right) 1_{m} q_{22}^{T} \\
2 \beta\left(e^{\lambda_{1} t}+e^{\lambda_{2} t}\right) 1_{n} q_{11}^{T} & 2 \beta\left(e^{\lambda_{1} t}+e^{\lambda_{2} t}\right) 1_{m} q_{12}^{T} & \left(e^{\lambda_{1} t} \lambda_{1}+e^{\lambda_{2} t} \lambda_{2}\right) 1_{n} q_{11}^{T} & \left(e^{\lambda_{1} t} \lambda_{1}+e^{\lambda_{2} t} \lambda_{2}\right) 1_{m} q_{12}^{T} \\
2 \beta\left(e^{\lambda_{1} t}+e^{\lambda_{2} t}\right) 1_{n} q_{21}^{T} & 2 \beta\left(e^{\lambda_{1} t}+e^{\lambda_{2} t}\right) 1_{m} q_{22}^{T} & \left(e^{\lambda_{1} t} \lambda_{1}+e^{\lambda_{2} t} \lambda_{2}\right) 1_{n} q_{21}^{T} & \left(e^{\lambda_{1} t} \lambda_{1}+e^{\lambda_{2} t} \lambda_{2}\right) 1_{m} q_{22}^{T}
\end{array}\right) .
$$

It follows from (5) that $[x(t), v(t)]^{T}=$ $e^{\Gamma t}[x(0), v(0)]^{T}$. Then, one has $\lim _{t \rightarrow \infty}[x(t), v(t)]^{T}=$ $\lim _{t \rightarrow \infty} e^{\Gamma t}[x(0), v(0)]^{T}$. After some calculations, one can obtain the consensus state.

Denote $x^{1}=\left(x_{1}, x_{2}, \ldots, x_{n}\right)^{T}, x^{2}=\left(x_{n+1}, x_{n+2}, \ldots\right.$, $\left.x_{n+m}\right)^{T}, v^{1}=\left(v_{1}, v_{2}, \ldots, v_{n}\right)^{T}$, and $v^{2}=\left(v_{n+1}, \ldots, v_{n+m}\right)^{T}$.

$$
\begin{gather*}
+\cos (\sqrt{\beta} t) q_{11}^{T} v^{1}(0)+\cos (\sqrt{\beta} t) q_{12}^{T} v^{2}(0), \\
\forall i \in l_{1}, \\
x_{i} \longrightarrow \sin (\sqrt{\beta} t) q_{21}^{T} x^{1}(0)+\sin (\sqrt{\beta} t) q_{22}^{T} x^{2}(0) \\
+\cos (\sqrt{\beta} t) q_{21}^{T} v^{1}(0)+\cos (\sqrt{\beta} t) q_{22}^{T} v^{2}(0), \\
\forall i \in l_{2},  \tag{17}\\
v_{i} \longrightarrow \\
\quad 2 \beta \cos (\sqrt{\beta} t) q_{11}^{T} x^{1}(0)+2 \beta \cos (\sqrt{\beta} t) q_{12}^{T} x^{2}(0) \\
\\
+\sin (\sqrt{\beta} t) q_{11}^{T} v^{1}(0)+\sin (\sqrt{\beta} t) q_{12}^{T} v^{2}(0), \\
\forall i \in l_{1}, \\
v_{i} \longrightarrow
\end{gather*} \begin{aligned}
& 2 \beta \cos (\sqrt{\beta} t) q_{21}^{T} x^{1}(0)+2 \beta \cos (\sqrt{\beta} t) q_{22}^{T} x^{2}(0)  \tag{16}\\
& +\sin (\sqrt{\beta} t) q_{21}^{T} v^{1}(0)+\sin (\sqrt{\beta} t) q_{22}^{T} v^{2}(0),
\end{aligned}
$$

as $t \rightarrow \infty$. It can be seen that system (1) reaches periodic multiconsensus from the convergence state.

$$
\Gamma=P J P^{-1}=\left(\omega_{1 r}, \omega_{2 r}, \ldots, \omega_{2(n+m) r}\right)
$$

For $\beta=0, \alpha>0$, it follows that $\lambda_{1}=0, \lambda_{2}=-\alpha, \lambda_{1}^{\prime}=\alpha$, and $\lambda_{1}^{\prime}=0$. Then from (15), one obtains

$$
\begin{array}{r}
x_{i} \longrightarrow \alpha q_{11}^{T} x^{1}(0)+\alpha q_{12}^{T} x^{2}(0)+q_{11}^{T} v^{1}(0)+q_{12}^{T} v^{2}(0), \\
\forall i \in l_{1}, \\
x_{i} \longrightarrow \alpha q_{21}^{T} x^{1}(0)+\alpha q_{22}^{T} x^{2}(0)+q_{21}^{T} v^{1}(0)+q_{22}^{T} v^{2}(0), \\
\forall i \in l_{2}, \\
v_{i} \longrightarrow 0, \quad \forall i \in l,
\end{array}
$$

as $t \rightarrow \infty$. It means that static multiconsensus of the system (1) can be reached.

Case $2\left(\alpha^{2}-4 \beta=0\right)$. It follows from (10) that $\lambda_{1}=\lambda_{2}=-\alpha / 2$. It implies that matrix $\Gamma$ has an eigenvalue $-\alpha / 2$ with algebraic multiplicity four. Similar to Case 1, we need to solve the eigenvectors and generalized eigenvectors of $\Gamma$ corresponding to eigenvalues $-\alpha / 2$.

One can easily obtain two right eigenvectors $\omega_{1 r}=\left(p_{1}^{T}\right.$, $\left.-(\alpha / 2) p_{1}^{T}\right)^{T}$ and $\omega_{3 r}=\left(p_{2}^{T},-(\alpha / 2) p_{2}^{T}\right)^{T}$ and two generalized right eigenvectors $\omega_{2 r}=\left[\alpha p_{1}^{T},\left(1-\left(\alpha^{2} / 2\right)\right) p_{1}^{T}\right]^{T}$ and $\omega_{4 r}=$ $\left[\alpha p_{2}^{T},\left(1-\left(\alpha^{2} / 2\right)\right) p_{2}^{T}\right]^{T}$ of matrix $\Gamma$ associated with eigenvalues $-\alpha / 2$. Accordingly, $\omega_{1 l}=\left[\left(1+\left(\alpha^{2} / 2\right)\right) q_{1}^{T}, \alpha q_{1}^{T}\right]^{T}, \omega_{3 l}=[(1+$ $\left.\left.\left(\alpha^{2} / 2\right)\right) q_{2}^{T}, \alpha q_{2}^{T}\right]^{T}$ and $\omega_{2 l}=\left[(\alpha / 2) q_{1}^{T}, q_{1}^{T}\right]^{T}, \omega_{4 l}=\left[(\alpha / 2) q_{2}^{T}\right.$, $\left.q_{2}^{T}\right]^{T}$ are the generalized left eigenvectors and the left eigenvectors of matrix $\Gamma$ associated with eigenvalues $-\alpha / 2$. Obviously, it can be seen that $\omega_{i r}^{T} \omega_{i l}=1, i=1,2,3,4$. Then, $-\alpha / 2$ is an eigenvalue of matrix $\Gamma$ with geometric multiplicity 2 .

Note that $\Gamma$ can be written in Jordan canonical form as

$$
\times\left(\begin{array}{ccccc}
\lambda_{1} & 1 & 0 & 0 & 0_{1 \times(2 n+2 m-4)} \\
0 & \lambda_{1} & 0 & 0 & 0_{1 \times(2 n+2 m-4)} \\
0 & 0 & \lambda_{1} & 1 & 0_{1 \times(2 n+2 m-4)} \\
0 & 0 & 0 & \lambda_{1} & 0_{1 \times(2 n+2 m-4)} \\
0_{(2 n+2 m-4) \times 1} & 0_{(2 n+2 m-4) \times 1} & 0_{(2 n+2 m-4) \times 1} & 0_{(2 n+2 m-4) \times 1} & J_{1}
\end{array}\right)
$$

$$
\binom{\hat{x}}{\hat{k}}
$$

where $\lambda_{1}=-\alpha / 2$ and $J_{1}$ is the Jordan upper diagonal block matrix corresponding to the eigenvalues $\lambda_{i j}(i=3, \ldots, n+m$; $j=1,2$ ).

Then,

$$
\begin{align*}
\lim _{t \rightarrow \infty} e^{\Gamma t}= & P \lim _{t \rightarrow \infty} e^{J t} P^{-1} \\
= & e^{\lambda_{1} t}\left(\omega_{1 r} \omega_{1 l}^{T}+t \omega_{1 r} \omega_{2 l}^{T}+\omega_{2 r} \omega_{2 l}^{T}\right.  \tag{19}\\
& \left.\quad+\omega_{3 r} \omega_{3 l}^{T}+t \omega_{3 r} \omega_{4 l}^{T}+\omega_{4 r} \omega_{4 l}^{T}\right) .
\end{align*}
$$

For the special case of $\alpha=0$, by noting that $\alpha^{2}-4 \beta=0$, it is clear that $\beta=0$. Then, it follows from (19) that

$$
\begin{array}{r}
x_{i} \longrightarrow q_{11}^{T} x^{1}(0)+q_{12}^{T} x^{2}(0)+t q_{11}^{T} v^{1}(0)+t q_{12}^{T} v^{2}(0), \\
\forall i \in l_{1}, \\
x_{i} \longrightarrow q_{21}^{T} x^{1}(0)+q_{22}^{T} x^{2}(0)+t q_{21}^{T} v^{1}(0)+t q_{22}^{T} v^{2}(0), \\
\forall i \in l_{2}, \\
v_{i} \longrightarrow q_{11}^{T} v^{1}(0)+q_{12}^{T} v^{2}(0), \quad \forall i \in l_{1}, \\
v_{i} \longrightarrow q_{21}^{T} v^{1}(0)+q_{22}^{T} v^{2}(0), \quad \forall i \in l_{2} \tag{20}
\end{array}
$$

as $t \rightarrow \infty$. It implies that the systems (1) reach dynamic multiconsensus.

For $\alpha>0$, one has $e^{\lambda_{1} t} \rightarrow 0(t \rightarrow \infty)$. Then, it follows from (19) that $x_{i} \rightarrow 0$ and $v_{i} \rightarrow 0$ for all $i \in l$ as $t \rightarrow \infty$.

Remark 6. In Theorem 5, we mainly analyze the eigenvectors and generalized eigenvectors of matrix $\Gamma$ associated with eigenvalues $\lambda_{1}$ and $\lambda_{2}$. Then, we obtain the ultimate consensus state based on matrix theory.

Lemma 7 (see [9]). Assume that $\operatorname{Re}(\mu)<0, \gamma_{0} \geq 0$, and $\alpha \geq 0$, the two roots of the polynomial

$$
\begin{equation*}
f_{\mu}(\lambda)=\lambda^{2}+\left(\alpha-\gamma_{1} \mu\right) \lambda+\beta-\gamma_{0} \mu \tag{21}
\end{equation*}
$$

lie in the open left-half complex plane if and only if

$$
\begin{align*}
\gamma_{0}> & \frac{\beta \operatorname{Re}(\mu)}{\operatorname{Re}^{2}(\mu)+\operatorname{Im}^{2}(\mu)}, \\
\gamma_{1}> & \frac{\alpha}{\operatorname{Re}(\mu)} \\
& +\frac{\gamma_{0}|\operatorname{Im}(\mu)|\left(\alpha|\operatorname{Im}(\mu)|+\sqrt{\alpha^{2} \operatorname{Im}^{2}(\mu)+4 \beta^{*} \operatorname{Re}(\mu)}\right)}{2 \beta^{*} \operatorname{Re}(\mu)} \tag{22}
\end{align*}
$$

where $\beta^{*}=\beta \operatorname{Re}\left(\mu_{i}\right)-\gamma_{0}\left(\operatorname{Re}^{2}\left(\mu_{i}\right)+\operatorname{Im}^{2}\left(\mu_{i}\right)\right)$.

Theorem 8. Suppose that - L has two simple zero eigenvalues and all the other eigenvalues have negative real parts, system (1) achieves multiconsensus if

$$
\begin{align*}
\gamma_{0}>\max _{i=3, \ldots, n+m} & \frac{\beta \operatorname{Re}\left(\mu_{i}\right)}{\operatorname{Re}^{2}\left(\mu_{i}\right)+\operatorname{Im}^{2}\left(\mu_{i}\right)}, \\
\gamma_{1}>\max _{i=3, \ldots, n+m}( & \frac{\alpha}{\operatorname{Re}\left(\mu_{i}\right)} \\
& +\gamma_{0}\left|\operatorname{Im}\left(\mu_{i}\right)\right| \\
& \times\left(\alpha\left|\operatorname{Im}\left(\mu_{i}\right)\right|+\sqrt{\alpha^{2} \operatorname{Im}^{2}\left(\mu_{i}\right)+4 \beta^{*} \operatorname{Re}\left(\mu_{i}\right)}\right) \\
& \left.\times\left(2 \beta^{*} \operatorname{Re}\left(\mu_{i}\right)\right)^{-1}\right), \tag{23}
\end{align*}
$$

where $u_{i}$ are the nonzero eigenvalues of $-L$ and $\beta^{*}=\beta \operatorname{Re}\left(\mu_{i}\right)-$ $\gamma_{0}\left(\operatorname{Re}^{2}\left(\mu_{i}\right)+\operatorname{Im}^{2}\left(\mu_{i}\right)\right)$.

Proof. From Lemma 7, it can be seen that $\operatorname{Re}(\mu)<0$ and (22) holds if and only if the roots of $\lambda^{2}+\left(\alpha-\gamma_{1} \mu\right) \lambda+\beta-\gamma_{0} \mu=0$ have negative real parts. Therefore, one knows that $\operatorname{Re}\left(\lambda_{i j}\right)<0$ ( $i=3, \ldots, n+m, j=1,2$ ) if and only if $\operatorname{Re}\left(\mu_{i}\right)<0$ and (9) holds, where $\lambda_{i j}$ and $\mu_{i}$ are eigenvalues of $\Gamma$ and $L$, respectively. By Theorem 5, it is easy to get the conclusion.

Corollary 9. For $\alpha=\beta=0, \gamma_{0}>0$, and $\gamma_{1}>0$, suppose that $-L$ has two simple zero eigenvalues and all the other eigenvalues have negative real parts, second-order dynamic multiconsensus of system (1) can be achieved if

$$
\begin{equation*}
\gamma_{1}>\max _{3 \leq i \leq n+m} \frac{\sqrt{\gamma_{0}}\left|\operatorname{Im}\left(\mu_{i}\right)\right|}{\left|\mu_{i}\right| \sqrt{-\operatorname{Re}\left(\mu_{i}\right)}}, \tag{24}
\end{equation*}
$$

where $u_{i}$ are the nonzero eigenvalues of $-L$. In addition, if the second-order dynamic multiconsensus is reached, one has

$$
\begin{array}{r}
x_{i} \longrightarrow q_{11}^{T} x^{1}(0)+q_{12}^{T} x^{2}(0)+t q_{11}^{T} v^{1}(0)+t q_{12}^{T} v^{2}(0), \\
\\
\forall i \in l_{1}, \\
x_{i} \longrightarrow q_{21}^{T} x^{1}(0)+q_{22}^{T} x^{2}(0)+t q_{21}^{T} v^{1}(0)+t q_{22}^{T} v^{2}(0),  \tag{25}\\
\forall i \in l_{2}, \\
v_{i} \longrightarrow q_{11}^{T} v^{1}(0)+q_{12}^{T} v^{2}(0), \quad \forall i \in l_{1}, \\
v_{i} \longrightarrow q_{21}^{T} v^{1}(0)+q_{22}^{T} v^{2}(0), \quad \forall i \in l_{2},
\end{array}
$$

as $t \rightarrow \infty$, where $q_{1}=\left(q_{11}^{T}, q_{12}^{T}\right)^{T}$ and $q_{2}=\left(q_{21}^{T}, q_{22}^{T}\right)^{T}$ are the two linearly independent left eigenvectors of matrix $L$ associated with zero eigenvalues which satisfy $p_{1}^{T} q_{1}=1$ and $p_{2}^{T} q_{2}=1$.

Corollary 10. For $\beta=0, \gamma_{1}>0, \gamma_{0}=1$, and $\alpha>0$, suppose that $-L$ has two simple zero eigenvalues and all the
other eigenvalues have negative real parts, second-order static multiconsensus of system (1) is achieved if
$\gamma_{1}$

$$
\begin{align*}
>\max _{3 \leq i \leq n+m}( & \frac{\alpha}{\operatorname{Re}\left(\mu_{i}\right)} \\
& \left.+\frac{\alpha\left|\operatorname{Im}^{2}\left(\mu_{i}\right)\right| \sqrt{\alpha^{2}\left|\operatorname{Im}^{2}\left(\mu_{i}\right)\right|-4 \operatorname{Re}\left(\mu_{i}\right)\left\|u_{i}\right\|^{2}}}{-2 \operatorname{Re}\left(\mu_{i}\right)\left\|\mu_{i}\right\|^{2}}\right) \tag{26}
\end{align*}
$$

where $u_{i}$ are the nonzero eigenvalues of $-L$. Moreover, if the static second-order multiconsensus is achieved, one obtains

$$
\begin{array}{r}
x_{i} \longrightarrow \alpha q_{11}^{T} x^{1}(0)+\alpha q_{12}^{T} x^{2}(0)+q_{11}^{T} v^{1}(0)+q_{12}^{T} v^{2}(0), \\
\forall i \in l_{1}, \\
x_{i} \longrightarrow \alpha q_{21}^{T} x^{1}(0)+\alpha q_{22}^{T} x^{2}(0)+q_{21}^{T} v^{1}(0)+q_{22}^{T} v^{2}(0), \\
\forall i \in l_{2}, \\
v_{i} \longrightarrow 0, \quad \forall i \in l, \tag{27}
\end{array}
$$

ast $\rightarrow \infty$, where $q_{1}=\left(q_{11}^{T}, q_{12}^{T}\right)^{T}$ and $q_{2}=\left(q_{21}^{T}, q_{22}^{T}\right)^{T}$ are the two linearly independent left eigenvectors of $L$ corresponding to eigenvalues 0 which satisfy $p_{1}^{T} q_{1}=1$ and $p_{2}^{T} q_{2}=1$.

Remark 11. In Corollary 9, inequality (24) is equivalent to the following form:

$$
\begin{equation*}
\frac{\gamma_{1}^{2}}{\gamma_{0}}>\max _{i=3, \ldots, n+m} \frac{\operatorname{Im}^{2}\left(\mu_{i}\right)}{\left|\mu_{i}\right|^{2}\left(-\operatorname{Re}\left(\mu_{i}\right)\right)} \tag{28}
\end{equation*}
$$

which is consistent with the results of Theorem 1 in [16]. For the special case of $\beta=0, \gamma_{1}=0, \gamma_{0}=1$, and $\alpha>0$, one obtains a sufficient condition from Corollary 10:

$$
\begin{equation*}
\alpha>\max _{i=3, \ldots, n+m} \frac{\left|\operatorname{Im}\left(\mu_{i}\right)\right|}{\sqrt{-\operatorname{Re}\left(\mu_{i}\right)}} \tag{29}
\end{equation*}
$$

which is consistent with Theorem 3 in [16]. Therefore, the case in [16] can be seen as a special case, and Theorem 8 presents more general results for multiconsensus of secondorder multiagent systems in this paper.

Corollary 12. For $\alpha=0$ and $\beta>0$, if $-L$ has two simple zero eigenvalues and all the other eigenvalues have negative real parts, system (1) achieves periodic multiconsensus if

$$
\begin{gather*}
\gamma_{0}>\max _{i=3, \ldots, n+m} \frac{\beta \operatorname{Re}\left(\mu_{i}\right)}{\operatorname{Re}^{2}\left(\mu_{i}\right)+\operatorname{Im}^{2}\left(\mu_{i}\right)}, \\
\gamma_{1}>\max _{i=3, \ldots, n+m} \frac{\gamma_{0}\left|\operatorname{Im}\left(\mu_{i}\right)\right|}{\sqrt{\left[\beta-\gamma_{0}\left(\operatorname{Re}^{2}\left(\mu_{i}\right)+\operatorname{Im}^{2}\left(\mu_{i}\right)\right)\right] \operatorname{Re}\left(\mu_{i}\right)}}, \tag{30}
\end{gather*}
$$

where $u_{i}$ are the nonzero eigenvalues of $-L$. In addition, if the second-order periodic multiconsensus is reached, one can obtain

$$
\begin{align*}
& x_{i} \longrightarrow \sin (\sqrt{\beta} t) q_{11}^{T} x^{1}(0)+\sin (\sqrt{\beta} t) q_{12}^{T} x^{2}(0) \\
&+\cos (\sqrt{\beta} t) q_{11}^{T} v^{1}(0)+\cos (\sqrt{\beta} t) q_{12}^{T} v^{2}(0), \\
& \forall i \in l_{1}, \\
& x_{i} \longrightarrow \sin (\sqrt{\beta} t) q_{21}^{T} x^{1}(0)+\sin (\sqrt{\beta} t) q_{22}^{T} x^{2}(0) \\
&+\cos (\sqrt{\beta} t) q_{21}^{T} v^{1}(0)+\cos (\sqrt{\beta} t) q_{22}^{T} v^{2}(0), \\
& \forall i \in l_{2}, \\
& v_{i} \longrightarrow 2 \beta \cos (\sqrt{\beta} t) q_{11}^{T} x^{1}(0)+2 \beta \cos (\sqrt{\beta} t) q_{12}^{T} x^{2}(0) \\
&+\sin (\sqrt{\beta} t) q_{11}^{T} v^{1}(0)+\sin (\sqrt{\beta} t) q_{12}^{T} v^{2}(0), \\
& \forall i \in l_{1}, \\
& v_{i} \longrightarrow 2 \beta \cos (\sqrt{\beta} t) q_{21}^{T} x^{1}(0)+2 \beta \cos (\sqrt{\beta} t) q_{22}^{T} x^{2}(0) \\
&+\sin (\sqrt{\beta} t) q_{21}^{T} v^{1}(0)+\sin (\sqrt{\beta} t) q_{22}^{T} v^{2}(0), \\
& \forall i \in l_{2}, \tag{31}
\end{align*}
$$

as $t \rightarrow \infty$, where $q_{1}=\left(q_{11}^{T}, q_{12}^{T}\right)^{T}$ and $q_{2}=\left(q_{21}^{T}, q_{22}^{T}\right)^{T}$ are the two linearly independent left eigenvectors of matrix $L$ associated with zero eigenvalues which satisfy $p_{1}^{T} q_{1}=1$ and $p_{2}^{T} q_{2}=1$.

Corollary 13. For $\alpha>0, \beta>0, \gamma_{1}>0$, and $\gamma_{2}>0$, if $-L$ has two simple zero eigenvalues and all the other eigenvalues have negative real parts, one has $x_{i} \rightarrow 0$ and $v_{i} \rightarrow 0$ as $t \rightarrow \infty$ for all $i \in l$.

Remark 14. If $\alpha>0, \beta=\alpha^{2} / 4>0, \gamma_{0}>0$, and $\gamma_{1}>0$, all the eigenvalues of matrix $\Gamma$ have negative parts, which implies that $\lim _{t \rightarrow \infty} e^{\Gamma t}=0_{2(n+m) \times 2(n+m)}$. Therefore, the system is asymptotically stabilized.
3.2. Second-Order Multiconsensus with Input Delays. In this subsection, the second-order multiconsensus with input delays is considered. We study the following system:

$$
\begin{gather*}
\dot{x}_{i}=v_{i},  \tag{32}\\
\dot{v}_{i}=u_{i}(t-\tau),
\end{gather*} \quad i=1, \ldots, n+m,
$$

where $\tau>0$ is the time-delay constant.


Figure 1: Topology graph of a network with seven agents.

Denote $z=\left[x^{T}, v^{T}\right]^{T}$, with algorithm (4), systems (32) can be rewritten in matrix form as follows:

$$
\begin{equation*}
\dot{z}(t)=\Gamma_{1} z+\Gamma_{2} z(t-\tau), \tag{33}
\end{equation*}
$$

where $\Gamma_{1}=\left(\begin{array}{cc}0_{(n+m) \times(n+m)} & I_{(n+m)} \\ 0_{(n+m) \times(n+m)} & 0_{(n+m) \times(n+m)}\end{array}\right)$ and $\Gamma_{2}=$ $\left(\begin{array}{c}0_{(n+m) \times(n+m)} \\ -\beta I_{(n+m)}-\gamma_{0} L \\ 0_{(n+m) \times(n+m)} \\ -\alpha I_{(n+m)}-\gamma_{1} L\end{array}\right)$.

The characteristic equation of system (33) is $\operatorname{det}\left(\lambda I_{2(n+m)}-\right.$ $\left.\Gamma_{1}-e^{-\lambda \tau} \Gamma_{2}\right)=0$

$$
\begin{align*}
\operatorname{det} & \left(\lambda I_{2(n+m)}-\Gamma_{1}-e^{-\lambda \tau} \Gamma_{2}\right) \\
= & \operatorname{det}\left(\lambda^{2} I_{2(n+m)}+\left(\left(\alpha I+\gamma_{1} L\right) \lambda+\left(\beta I+\gamma_{0} L\right)\right) e^{-\lambda \tau}\right) \\
= & \prod_{i=1}^{n+m}\left\{\lambda^{2}+\left[\left(\alpha-\gamma_{1} \mu_{i}\right) \lambda+\left(\beta-\gamma_{0} \mu_{i}\right)\right] e^{-\lambda \tau}\right\} \tag{34}
\end{align*}
$$

Let $g_{i}(\lambda)=\lambda^{2}+\left[\left(\alpha-\gamma_{1} \mu_{i}\right) \lambda+\left(\beta-\gamma_{0} \mu_{i}\right)\right] e^{-\lambda \tau}$ and $g(\lambda)=$ $\prod_{i=1}^{n+m} g_{i}(\lambda)$.

In order to obtain the maximal allowable upper bound of the delays, stability theory and Hopf bifurcation analysis are introduced. Then, we give the following lemmas.

Lemma 15. Suppose that $-L$ has two simple zero eigenvalues and all the other eigenvalues have negative real parts, then $g(\lambda)=0$ has a purely imaginary root if

$$
\begin{equation*}
\tau \in \psi=\left\{\left.\frac{2 N \pi+\theta_{i}}{w_{i}} \right\rvert\, i=3, \ldots, n+m, N=0,1, \ldots\right\}, \tag{35}
\end{equation*}
$$

where $\theta_{i} \in[0,2 \pi]$, which satisfies

$$
\begin{align*}
& \sin \theta_{i}=\frac{\alpha w_{i}-\gamma_{0} \operatorname{Im}\left(\mu_{i}\right)-\gamma_{1} \operatorname{Re}\left(\mu_{i}\right) w_{i}}{w_{i}^{2}},  \tag{36}\\
& \cos \theta_{i}=\frac{\beta-\gamma_{0} \operatorname{Re}\left(\mu_{i}\right)+\gamma_{1} \operatorname{Im}\left(\mu_{i}\right) w_{i}}{w_{i}^{2}}, \\
& w_{i}^{4}= \\
& =\left\{\gamma_{1} \operatorname{Im}^{2}\left(\mu_{i}\right)+\left[\alpha-\gamma_{1} \operatorname{Re}\left(\mu_{i}\right)\right]^{2}\right\} w_{i}^{2}  \tag{37}\\
& \\
& +2\left(\beta \gamma_{1}-\alpha \gamma_{0}\right) \operatorname{Im}\left(\mu_{i}\right) w_{i}+\left[\beta-\gamma_{0} \operatorname{Re}\left(\mu_{i}\right)\right]^{2} \\
& \\
& +\gamma_{0}^{2} \operatorname{Im}^{2}\left(\mu_{i}\right),
\end{align*}
$$

and $\mu_{i}(i=3, \ldots, n+m)$ are the nonzero eigenvalues of $-L$.
Proof. Let $\lambda=i w_{i}\left(w_{i} \neq 0\right)$, from $g_{i}(\lambda)=0$, one has

$$
\begin{equation*}
w_{i}^{2}=\left(\alpha-\gamma_{1} \mu_{i}\right) e^{-i w_{i} \tau} w_{i} i+\left(\beta-\gamma_{0} \mu_{i}\right) e^{-i w_{i} \tau} \tag{38}
\end{equation*}
$$

Let $\mu_{i}=\operatorname{Re}\left(\mu_{i}\right)+\operatorname{Im}\left(\mu_{i}\right) i$ and $e^{-i w_{i} \tau}=\cos \left(w_{i} \tau\right)+i \sin \left(w_{i} \tau\right)$, taking modulus on both sides of (38) and separating the real and imaginary parts of (38), we obtain

$$
\begin{align*}
w_{i}^{4}=[\beta & \left.-\gamma_{0} \operatorname{Re}\left(\mu_{i}\right)+\gamma_{1} \operatorname{Im}\left(\mu_{i}\right) w_{i}\right]^{2} \\
+ & {\left[\alpha w_{i}-\gamma_{0} \operatorname{Im}\left(\mu_{i}\right)-\gamma_{1} \operatorname{Re}\left(\mu_{i}\right) w_{i}\right]^{2}, } \\
w_{i}^{2}= & \cos \left(w_{i} \tau\right)\left[\beta-\gamma_{0} \operatorname{Re}\left(\mu_{i}\right)+\gamma_{1} \operatorname{Im}\left(\mu_{i}\right) w_{i}\right]  \tag{39}\\
& +\sin \left(w_{i} \tau\right)\left[\alpha w_{i}-\gamma_{0} \operatorname{Im}\left(\mu_{i}\right)-\gamma_{1} \operatorname{Re}\left(\mu_{i}\right) w_{i}\right], \\
0= & \cos \left(w_{i} \tau\right)\left[\alpha w_{i}-\gamma_{0} \operatorname{Im}\left(\mu_{i}\right)-\gamma_{1} \operatorname{Re}\left(\mu_{i}\right) w_{i}\right] \\
& -\sin \left(w_{i} \tau\right)\left[\beta-\gamma_{0} \operatorname{Re}\left(\mu_{i}\right)+\gamma_{1} \operatorname{Im}\left(\mu_{i}\right) w_{i}\right] .
\end{align*}
$$

By simple calculations, one obtains

$$
\begin{align*}
& \cos \left(w_{i} \tau\right)=\frac{\alpha w_{i}-\gamma_{0} \operatorname{Im}\left(\mu_{i}\right)-\gamma_{1} \operatorname{Re}\left(\mu_{i}\right) w_{i}}{w_{i}^{2}}, \\
& \sin \left(w_{i} \tau\right)=\frac{\beta-\gamma_{0} \operatorname{Re}\left(\mu_{i}\right)+\gamma_{1} \operatorname{Im}\left(\mu_{i}\right) w_{i}}{w_{i}^{2}},  \tag{40}\\
& w_{i}^{4}= \\
& \quad\left\{\gamma_{1} \operatorname{Im}^{2}\left(\mu_{i}\right)+\left[\alpha-\gamma_{1} \operatorname{Re}\left(\mu_{i}\right)\right]^{2}\right\} w_{i}^{2} \\
& \\
& +2\left(\beta \gamma_{1}-\alpha \gamma_{0}\right) \operatorname{Im}\left(\mu_{i}\right) w_{i} \\
& \\
& +\left[\beta-\gamma_{0} \operatorname{Re}\left(\mu_{i}\right)\right]^{2}+\gamma_{0}^{2} \operatorname{Im}^{2}\left(\mu_{i}\right)
\end{align*}
$$

For $i=3, \ldots, n+m$, we can conclude that there exists a single $w_{i}>0$ such that (37) is satisfied. Then, $\tau>0, w_{i} \tau=$ $\theta_{i}+2 N \pi$, and $N=1,2, \ldots$, for $\theta_{i} \in[0,2 \pi]$, which satisfies (36). We have that $g_{i}(\lambda)=0$ has purely imaginary toots if (35) is satisfied. This completes the proof.


Figure 2: Position and velocity states of agents in a network, where $\gamma_{1}=0.48$ and $\gamma_{1}=0.49$.

Lemma 16 (see [21]). Consider the exponential polynomial

$$
\begin{align*}
p\left(\lambda, e^{-\lambda \tau_{1}}, \ldots, e^{-\lambda \tau_{m}}\right)= & \lambda^{n}+p_{1}^{(0)} \lambda^{n-1}+\cdots+p_{n}^{(0)} \\
& +\left[p_{1}^{(1)} \lambda^{n-1}+\cdots+p_{n}^{(1)}\right] e^{-\lambda \tau_{1}} \\
& +\cdots+\left[p_{1}^{(m)} \lambda^{n-1}+\cdots+p_{n}^{(m)}\right] e^{-\lambda \tau_{m}} \tag{41}
\end{align*}
$$

where $\tau_{i} \geq 0(i=1, \ldots, m)$ and $p_{j}^{(i)}(i=0,1, \ldots, m ; j=$ $1, \ldots, n)$ are constants. As $\left(\tau_{1}, \ldots, \tau_{m}\right)$ vary, the sum of the orders of the zero of $p\left(\lambda, e^{-\lambda \tau_{1}}, \ldots, e^{-\lambda \tau_{m}}\right)$ on the open right-half plane can change only if a zero appears or crosses the imaginary.

Theorem 17. Suppose that $-L$ has two simple zero eigenvalues and all the other eigenvalues have negative real parts and (24) is satisfied. Then, second-order multiconsensus in system (32) is achieved if

$$
\begin{equation*}
\tau<\tau_{0}=\min _{3 \leq i \leq n+m}\left\{\frac{\theta_{i}}{w_{i}}\right\} \tag{42}
\end{equation*}
$$

where $\theta_{i} \in[0,2 \pi]$, which satisfies

$$
\begin{align*}
& \sin \theta_{i}=\frac{\alpha w_{i}-\gamma_{0} \operatorname{Im}\left(\mu_{i}\right)-\gamma_{1} \operatorname{Re}\left(\mu_{i}\right) w_{i}}{w_{i}^{2}} \\
& \cos \theta_{i}=\frac{\beta-\gamma_{0} \operatorname{Re}\left(\mu_{i}\right)+\gamma_{1} \operatorname{Im}\left(\mu_{i}\right) w_{i}}{w_{i}^{2}} \tag{43}
\end{align*}
$$

$$
\begin{aligned}
w_{i}^{4}= & \left\{\gamma_{1} \operatorname{Im}^{2}\left(\mu_{i}\right)+\left[\alpha-\gamma_{1} \operatorname{Re}\left(\mu_{i}\right)\right]^{2}\right\} w_{i}^{2} \\
& +2\left(\beta \gamma_{1}-\alpha \gamma_{0}\right) \operatorname{Im}\left(\mu_{i}\right) w_{i}+\left[\beta-\gamma_{0} \operatorname{Re}\left(\mu_{i}\right)\right]^{2} \\
& +\gamma_{0}^{2} \operatorname{Im}^{2}\left(\mu_{i}\right)
\end{aligned}
$$

and $\mu_{i}(i=3, \ldots, n+m)$ are the nonzero eigenvalues of $-L$.
Proof. Since $-L$ has two simple zero eigenvalues and all the other eigenvalues have negative real parts and inequality (24) holds, it follows from Theorem 8 that the second-order consensus can be achieved in system (32) when $\tau=0$, where all the roots of $\prod_{i=3}^{n+m} g_{i}(\lambda)=\prod_{i=3}^{n+m}\left\{\lambda^{2}+\left(\alpha-\gamma_{1} \mu_{i}\right) \lambda+\right.$ $\left.\left(\beta-\gamma_{1} \mu_{i}\right)\right\}=0$ have negative real parts and the roots


Figure 3: Position and velocity states of agents in a network, where $\alpha=0.68$ and $\alpha=0.66$.


Figure 4: State trajectories of seven agents with $\alpha=0, \gamma_{0}=1, \gamma_{1}=1$, and $\beta=1$.


Figure 5: Position and velocity states of agents with $\alpha=4, \beta=1, \gamma_{0}=1$, and $\gamma_{1}=0.5$.


Figure 6: Position and velocity states of agents in a network with input delays, where $\tau=0.20$ and $\tau=0.21$.


Figure 7: Position and velocity states of agents in a network with input delays, where $\tau=0.27$ and $\tau=0.28$.
of $\lambda^{2}+\alpha \lambda+\beta=0(\alpha \geq 0, \beta \geq 0)$ are located in the closed left-half plane. It means that all the roots of $g(\lambda)=0$ are located in the closed left-half plane when $\tau=0$. When $\tau$ varies from 0 to $\tau_{0}$, by Lemmas 15 and 16 , a purely imaginary root emerges and the sum of order of zero of $g(\lambda)$ on the open right-half plane can change. Therefore, the stability of system (32) is not satisfied and multiconsensus cannot be achieved when $\tau \geq \tau_{0}$.

Remark 18. The idea of delays has been inspired by the idea used in [8, 21, 22]. Consensus problem for double-integrator multiagent systems under delays has been also investigated in $[8,22]$, where the input delays were regarded as bifurcation parameters. It can be seen that Hopf bifurcation occurs when time delays pass through some critical values where the conditions for local asymptotical stability of the equilibrium are not satisfied.

## 4. Simulation

In this section, we present numerical simulations to illustrate the effectiveness of the proposed theoretical analysis.
4.1. Second-Order Dynamical Multiconsensus. Consider multiagent systems with the directed communication graph shown in Figure 1. It can be seen that information exchange exists between two subnetworks.

The Laplacian matrix is given by

$$
L=\left[\begin{array}{ccccccc}
3 & -3 & 0 & 1 & 0 & 0 & -1  \tag{44}\\
-2 & 3 & -1 & -1 & 0 & 0 & 1 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 2 & 0 & 0 & -2 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 \\
-1 & 1 & 0 & -2 & 0 & -1 & 3
\end{array}\right]
$$

The eigenvalues of the matrix $-L$ are

$$
\begin{gather*}
\mu_{1}=\mu_{2}=0, \quad \mu_{3}=-6.9987, \quad \mu_{4}=-2.8370 \\
\mu_{5}=-1.8014, \quad \mu_{6}=-1.1815+0.7303 i  \tag{45}\\
\mu_{7}=-1.1815-0.7303 i
\end{gather*}
$$

Let $\alpha=\beta=0$ and $\gamma_{0}=1$; from Corollary 9, it can be found that the dynamic multiconsensus can be


FIGURE 8: Position and velocity states of agents in a network with input delays, where $\tau=0.19$ and $\tau=0.21$.
achieved if $\gamma_{1}>0.4837$. The position and velocity of all the agents are shown in Figure 2, where systems (1) can achieve dynamic multiconsensus when $\gamma_{1}=0.49$ but cannot achieve multiconsensus when $\gamma_{1}=0.48$.

Let $\beta=0, \gamma_{0}=1$, and $\gamma_{1}=0$; from Corollary 10, one can calculate $\alpha>0.6719$. Therefore, the static multiconsensus can be achieved when $\alpha=0.68$ but cannot be achieved when $\alpha=0.66$ as shown in Figure 3.

When $\alpha=0, \gamma_{0}=1, \gamma_{1}=1$, and $\beta=1$, one has that the inequality in Corollary 12 holds. Then, the periodic multiconsensus can be achieved as shown in Figure 4. When $\alpha=4, \beta=1, \gamma_{0}=1$, and $\gamma_{1}=0.5$, the condition in Corollary 13 is satisfied. The consensus of multiagent systems (1) can be reached, which are shown in Figure 5.
4.2. Second-Order Statical Multiconsensus with Input Delays. When $\alpha=\beta=0, \gamma_{0}=1$, and $\gamma_{1}=1$, with simple computations, one has that the inequality (24) holds and multiconsensus can be reached if $\tau<0.2023$. The position and velocity states of all the agents are shown in Figure 6, where systems (1) can achieve multiconsensus when $\tau=0.20$ but cannot achieve multiconsensus when $\tau=0.21$.

Let $\beta=0, \gamma_{0}=1, \gamma_{1}=0$, and $\alpha=3$; it can be verified that inequality (24) holds and multiconsensus can be achieved
if $\tau<0.2774$. Figure 7 shows that multiconsensus can be reached when $\tau=0.27$ but cannot be reached when $\tau=0.28$.

Let $\alpha=0, \beta=1, \gamma_{0}=1$, and $\gamma_{1}=1$; from Theorem 17, one can obtain that multiconsensus can be achieved if $\tau<0.1990$. Therefore, multiconsensus can be achieved when $\tau=0.19$ but cannot be achieved when $\tau=0.21$ as shown in Figure 8.

## 5. Conclusions

In this paper, we consider multiconsensus problems for multiple agents with double integrator. A sufficient condition is obtained for ensuring multiconsensus, in which different multiconsensus including the dynamic multiconsensus, the static consensus, and the periodic multiconsensus can be reached by choosing different gains. Furthermore, by using Hopf bifurcation theory, we obtain a sufficient condition of multiconsensus with a delayed multiagent system. Finally, some simulation results are presented to illustrate the theoretical results. In future, we should further consider the effects of time-varying delays and switching interactions graphs.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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