Supplementary Materials for EVD-Dualdating based Online Subspace Learning

1. Proof of Theorem 1

Proof.

$$F^TF = \begin{bmatrix} C^TC & C^TD & C^TB \\ D^TC & D^TD & D^TB \\ B^TC & B^TD & B^TB \end{bmatrix} = X_F + \sigma^2 I_{n+m}.$$

Then,

$$X_F = F^TF - \sigma^2 I_{n+m} = \begin{bmatrix} C^TC - \sigma^2 I_{n-p} & C^TD & C^TB \\ D^TC & D^TD - \sigma^2 I_p & D^TB \\ B^TC & B^TD & B^TB - \sigma^2 I_m \end{bmatrix}.$$

Since $rank(X_F) = k$, it follows that:

$$\begin{aligned} rank(C^TC - \sigma^2 I_{n-p}) &= k_C \leq k, \\ rank(D^TD - \sigma^2 I_p) &= k_D \leq k, \\ rank(B^TB - \sigma^2 I_m) &= k_B \leq k. \end{aligned}$$

Let the EVDs of $C^TC - \sigma^2 I_{n-p}$, $D^TD - \sigma^2 I_p$, and $B^TB - \sigma^2 I_m$ are:

$$\begin{aligned} C^TC - \sigma^2 I_{n-p} &= V_C \begin{bmatrix} \Lambda'_C & \\ & 0 \end{bmatrix} V_C^T, \\ D^TD - \sigma^2 I_p &= V_D \begin{bmatrix} \Lambda'_D & \\ & 0 \end{bmatrix} V_D^T, \\ B^TB - \sigma^2 I_m &= V_B \begin{bmatrix} \Lambda'_B & \\ & 0 \end{bmatrix} V_B^T, \end{aligned}$$

where $\Lambda'_C = \Sigma_C^2 - \sigma^2 I_{k_C}$, $\Lambda'_D = \Sigma_D^2 - \sigma^2 I_{k_D}$, $\Lambda'_B = \Sigma_B^2 - \sigma^2 I_{k_B}$ and $\Sigma_C \in \mathbb{R}^{k_C \times k_C}$, $\Sigma_D \in \mathbb{R}^{k_D \times k_D}$, $\Sigma_B \in \mathbb{R}^{k_B \times k_B}$ are from the singular values of C, D and B.

So the SVDs of C, D, and B can be written as:

$$\begin{aligned} C &= U_C \begin{bmatrix} \Sigma_C & \\ & \sigma I_{t_C} \end{bmatrix} V_C^T = \begin{bmatrix} U_{Ck_C} & U_{Cl} \end{bmatrix} \begin{bmatrix} \Sigma_C & \\ & \sigma I_{t_C} \end{bmatrix} \begin{bmatrix} V_{Ck_C} & V_{Cl} \end{bmatrix}^T, \\ D &= U_D \begin{bmatrix} \Sigma_D & \\ & \sigma I_{t_D} \end{bmatrix} V_D^T = \begin{bmatrix} U_{Dk_D} & U_{Dl} \end{bmatrix} \begin{bmatrix} \Sigma_D & \\ & \sigma I_{t_D} \end{bmatrix} \begin{bmatrix} V_{Dk_D} & V_{Dl} \end{bmatrix}^T, \\ B &= U_B \begin{bmatrix} \Sigma_B & \\ & \sigma I_{t_B} \end{bmatrix} V_B^T = \begin{bmatrix} U_{Bk_B} & U_{Bl} \end{bmatrix} \begin{bmatrix} \Sigma_B & \\ & \sigma I_{t_B} \end{bmatrix} \begin{bmatrix} V_{Bk_B} & V_{Bl} \end{bmatrix}^T, \end{aligned}$$

where $U_{Ck_C} \in \mathbb{R}^{d \times k_C}$, $U_{Cl} \in \mathbb{R}^{d \times t_C}$, $U_{Dk_D} \in \mathbb{R}^{d \times k_D}$, $U_{Dl} \in \mathbb{R}^{d \times t_D}$, $U_{Bk_B} \in \mathbb{R}^{d \times k_B}$, $U_{Bl} \in \mathbb{R}^{d \times t_B}$, $t_C = rank(C) - k_C$, $t_D = rank(D) - k_D$, $t_B = rank(B) - k_B$.

Then,

$$\begin{bmatrix} V_C & & \\ & V_D & \\ & & V_B \end{bmatrix}^T X_F \begin{bmatrix} V_C & & \\ & V_D & \\ & & V_B \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} \Lambda_C' & \\ & 0 \end{bmatrix} & V_C^T C^T D V_D & V_C^T C^T B V_B \\ V_D^T D^T C V_C & \begin{bmatrix} \Lambda_D' & \\ & 0 \end{bmatrix} & V_D^T D^T B V_B \\ V_B^T B^T C V_C & V_B^T B^T D V_D & \begin{bmatrix} \Lambda_B' & \\ & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \Lambda_C' & 0 & S_1 & S_2 & S_5 & S_6 \\ 0 & 0 & S_3 & S_4 & S_7 & S_8 \\ S_1^T & S_3^T & \Lambda_D' & 0 & S_9 & S_{10} \\ S_2^T & S_4^T & 0 & 0 & S_{11} & S_{12} \\ S_5^T & S_7^T & S_9^T & S_{11}^T & \Lambda_B' & 0 \\ S_6^T & S_8^T & S_{10}^T & S_{12}^T & 0 & 0 \end{bmatrix}.$$

Because X_F is symmetric positive semidefinite and $rank(X_F) = k$, according the determination condition of a symmetric positive semidefinite matrix, there exists a k-order principal minor is positive, and the determinant of arbitrary $(k+1)$-order principal minor is zero. It follows that $S_2 = 0$, $S_3 = 0$, $S_4 = 0$, $S_6 = 0$, $S_7 = 0$, $S_8 = 0$, $S_{10} = 0$, $S_{11} = 0$, $S_{12} = 0$, and $k_C + k_D + k_B = k$.

Thus, we have

$$\begin{bmatrix} U_{Ck_C}\Sigma_C & \sigma U_{Cl} \end{bmatrix}^T \begin{bmatrix} U_{Dk_D}\Sigma_D & \sigma U_{Dl} \end{bmatrix} = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} U_{Ck_C}\Sigma_C & \sigma U_{Cl} \end{bmatrix}^T \begin{bmatrix} U_{Uk_B}\Sigma_B & \sigma U_{Bl} \end{bmatrix} = \begin{bmatrix} S_5 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} U_{Dk_D}\Sigma_C & \sigma U_{Dl} \end{bmatrix}^T \begin{bmatrix} U_{Bk_B}\Sigma_B & \sigma U_{Bl} \end{bmatrix} = \begin{bmatrix} S_9 & 0 \\ 0 & 0 \end{bmatrix},$$

which lead to

$$\begin{aligned} &U_{Ck_C} \perp U_{Dl}, U_{Cl} \perp U_{Dk_D}, U_{Cl} \perp U_{Dl}, \\ &U_{Ck_C} \perp U_{Bl}, U_{Cl} \perp U_{Bk_B}, U_{Cl} \perp U_{Bl}, \\ &U_{Dk_D} \perp U_{Bl}, U_{Dl} \perp U_{Bk_B}, U_{Dl} \perp U_{Bl}. \end{aligned}$$

Let $\hat{U}_A$ be an orthonormal basis of

$$\mathcal{L}\left(\begin{bmatrix} U_{Ck_C} & U_{Dk_D} \end{bmatrix}\right) \cap \mathcal{L}^\perp\left(\begin{bmatrix} U_{Cl} & U_{Dl} \end{bmatrix}\right),$$

where $\mathcal{L}(\cdot)$ denotes the column space of a matrix, and $\mathcal{L}^\perp(\cdot)$ denotes the orthonormal complement.

Obviously, $rank(\hat{U}_A) \le k$, so in order to construct the first k eigen vectors, we append a few mutually orthogonal vectors from U_{Cl} and U_{Dl} to $\hat{U}_A$. Divide U_{Cl} and U_{Dl} into two parts, $U_{Cl} = \begin{bmatrix} U_{Cl}^{A1} & U_{Cl}^{A2} \end{bmatrix}$, $U_{Dl} = \begin{bmatrix} U_{Dl}^{A1} & U_{Dl}^{A2} \end{bmatrix}$, satisfying that the first k eigen vectors of A are

$$U_{Ak} = \begin{bmatrix} \hat{U}_A & U_{Cl}^{A1} & U_{Dl}^{A1} \end{bmatrix}.$$

Then, we have

$$AA^T = \hat{A}\hat{A}^T + \sigma^2 U_{Cl}^{A2} {U_{Cl}^{A2}}^T + \sigma^2 U_{Dl}^{A2} {U_{Dl}^{A2}}^T,$$

which leads to

$$EE^T = \hat{E}\hat{E}^T + \sigma^2 U_{Cl}^{A2} {U_{Cl}^{A2}}^T + \sigma^2 U_{Dl}^{A2} {U_{Dl}^{A2}}^T,$$

Similarly, let $\hat{U}_E$ be an orthonormal basis of $\mathcal{L}\left(\left[\begin{array}{cc} U_{Ck_C} & U_{Bk_B} \end{array}\right]\right) \mathcal{L}^{\perp}\left(\left[\begin{array}{cc} U_{Cl} & U_{Bl} \end{array}\right]\right)$, divide U_{Cl} and U_{Bl} into two parts, $U_{Cl} = \left[\begin{array}{cc} U_{Cl}^{E1} & U_{Cl}^{E2} \end{array}\right]$, $U_{Bl} = \left[\begin{array}{cc} U_{Bl}^{E1} & U_{Bl}^{E2} \end{array}\right]$, then the first k singular vectors of E are

$$U_{Ek} = \left[\begin{array}{ccc} \hat{U}_E & U_{Cl}^{E1} & U_{Bl}^{E1} \end{array}\right].$$

Assuming that the intersection of of U_{Cl}^{A2} and U_{Cl}^{E1} is U_{Cl}^{I} (null or non-null), it can be written that $U_{Cl}^{A2} = \left[\begin{array}{cc} U_{Cl}^{I} & U_{Cl}^{A} \end{array}\right]$, $U_{Cl}^{E1} = \left[\begin{array}{cc} U_{Cl}^{I} & U_{Cl}^{E} \end{array}\right]$. Let $rank(U_{Cl}^{C}) = k_s$, the following holds

$$\begin{aligned}
& U_{Ek}^T U_{Cl}^{A2} {U_{Cl}^{A2}}^T U_{Ek} \\
= & \left[\begin{array}{ccc} \hat{U}_E & U_{Cl}^{E1} & U_{Bl}^{E1} \end{array}\right]^T U_{Cl}^{A2} {U_{Cl}^{A2}}^T \left[\begin{array}{ccc} \hat{U}_E & U_{Cl}^{E1} & U_{Bl}^{E1} \end{array}\right] \\
= & \left[\begin{array}{ccc} 0 & \left[\begin{array}{c} {U_{Cl}^{I}}^T \\ {U_{Cl}^{A}}^T \end{array}\right] \left[\begin{array}{cc} U_{Cl}^{I} & U_{Cl}^{E} \end{array}\right] & 0 \end{array}\right]^T \left[\begin{array}{ccc} 0 & \left[\begin{array}{c} {U_{Cl}^{I}}^T \\ {U_{Cl}^{A}}^T \end{array}\right] \left[\begin{array}{cc} U_{Cl}^{I} & U_{Cl}^{E} \end{array}\right] & 0 \end{array}\right] \\
= & \left[\begin{array}{ccc} 0 & & \\ & I_{k_s} & \\ & & 0 \end{array}\right]_{k\times k},
\end{aligned}$$

$$\begin{aligned}
& U_{Ek}^T U_{Dl}^{A2} {U_{Dl}^{A2}}^T U_{Ek} \\
= & \left[\begin{array}{ccc} \hat{U}_E & U_{Cl}^{E1} & U_{Bl}^{E1} \end{array}\right]^T U_{Dl}^{A2} {U_{Dl}^{A2}}^T \left[\begin{array}{ccc} \hat{U}_E & U_{Cl}^{E1} & U_{Bl}^{E1} \end{array}\right] = 0_{k\times k}.
\end{aligned}$$

So

$$U_{Ek}^T EE^T U_{Ek} = U_{Ek}^T \hat{E}\hat{E}^T U_{Ek} + \sigma^2 \left[\begin{array}{ccc} 0 & & \\ & I_{k_s} & \\ & & 0 \end{array}\right]_{k\times k}.$$

Finally, we can conclude that

$$best_k(S_E) = best_k(\hat{S_E}).$$

completing the proof.

□