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Research Article

Numerical Solution of Fractional Integro-Differential Equations by Least Squares Method and Shifted Chebyshev Polynomial

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We investigate the numerical solution of linear fractional integro-differential equations by least squares method with aid of shifted Chebyshev polynomial. Some numerical examples are presented to illustrate the theoretical results.

1. Introduction

Many problems can be modeled by fractional Integro-differential equations from various sciences and engineering applications. Furthermore most problems cannot be solved analytically, and hence finding good approximate solutions, using numerical methods, will be very helpful.

Recently, several numerical methods to solve fractional differential equations (FDEs) and fractional Integro-differential equations (FIDEs) have been given. The authors in [1, 2] applied collocation method for solving the following: nonlinear fractional Langevin equation involving two fractional orders in different intervals and fractional Fredholm Integro-differential equations. Chebyshev polynomials method is introduced in [3–5] for solving multiterm fractional orders differential equations and nonlinear Volterra and Fredholm Integro-differential equations of fractional order. The authors in [6] applied variational iteration method for solving fractional Integro-differential equations with the nonlocal boundary conditions. Adomian decomposition method is introduced in [7, 8] for solving fractional diffusion equation and fractional Integro-differential equations. References [9, 10] used homotopy perturbation method for solving nonlinear Fredholm Integro-differential equations of fractional order and system of linear Fredholm fractional Integro-differential equations. Taylor series method is introduced in [11] for solving linear integrofractional differential equations of Volterra type. The authors in [12, 13] give an application of nonlinear fractional differential equations and their approximations and existence and uniqueness theorem for fractional differential equations with integral boundary conditions.

In this paper least squares method with aid of shifted Chebyshev polynomial is applied to solving fractional Integro-differential equations. Least squares method has been studied in [14–18].

In this paper, we are concerned with the numerical solution of the following linear fractional Integro-differential equation:

$$D^{\alpha}\varphi(x) = f(x) + \int_0^1 K(x,t)\varphi(t)\,dt, \quad 0 \le x, t \le 1, \tag{1}$$

with the following supplementary conditions:

$$\varphi^{(i)}(0) = \delta_i, \quad n-1 < \alpha \le n, \quad n \in \mathbf{N}, \tag{2}$$

where $D^{\alpha}\varphi(x)$ indicates the αth Caputo fractional derivative of $\varphi(x)$; $f(x)$, $K(x,t)$ are given functions, x and t are real variables varying in the interval $[0,1]$, and $\varphi(x)$ is the unknown function to be determined.

2. Basic Definitions of Fractional Derivatives

In this section some basic definitions and properties of fractional calculus theory which are necessary for the formulation of the problem are given.

Definition 1. A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in \mathbf{R}$, if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, 1)$.

Definition 2. A function $f(x)$, $x > 0$, is said to be in the space C_μ^m, $m \in \mathbf{N} \cup \{0\}$, if $f^{(m)} \in C_\mu$.

Definition 3. The left sided Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\mu$, $\mu \geq -1$, is defined as [19]

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad \alpha > 0, \; x > 0, \tag{3}$$

$$J^0 f(x) = f(x). \tag{4}$$

Definition 4. Let $f \in C_{-1}^m 1$, $m \in \mathbf{N} \cup \{0\}$. Then the Caputo fractional derivative of $f(x)$ is defined as [20–22]

$$D^\alpha f(x) = \begin{cases} J^{m-\alpha} f^m(x), & m-1 < \alpha \leq m, \; m \in \mathbf{N}, \\ \dfrac{D^m f(x)}{Dx^m}, & \alpha = m. \end{cases} \tag{5}$$

Hence, we have the following properties:

(1) $J^\alpha J^\nu f = J^{\alpha+\nu} f, \quad \alpha, \nu > 0, \; f \in C_\mu, \; \mu > 0,$

(2) $J^\alpha x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}, \quad \alpha > 0, \; \gamma > -1, \; x > 0,$

(3) $J^\alpha D^\alpha f(x) = f(x) - \displaystyle\sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!},$

$$x > 0, \quad m-1 < \alpha \leq m, \tag{6}$$

(4) $D^\alpha J^\alpha f(x) = f(x), \quad x > 0, \; m-1 < \alpha \leq m,$

(5) $D^\alpha C = 0, \quad C$ is a constant,

(6) $D^\alpha x^\beta = \begin{cases} 0, & \beta \in \mathbf{N}_0, \; \beta < [\alpha], \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, & \beta \in \mathbf{N}_0, \; \beta \geq [\alpha], \end{cases}$

where $[\alpha]$ denoted the smallest integer greater than or equal to α and $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$.

3. Solution of Linear Fractional Integro-Differential Equation

In this section the least squares method with aid of shifted Chebyshev polynomial is applied to study the numerical solution of the fractional Integro-differential (1).

This method is based on approximating the unknown function $\varphi(x)$ as

$$\varphi_n(x) \cong \sum_{i=0}^{n} a_i T_i^*(x), \quad 0 \leq x \leq 1, \tag{7}$$

where $T_i^*(x)$ is shifted Chebyshev polynomial of the first kind which is defined in terms of the Chebyshev polynomial $T_n(x)$ by the following relation [23]:

$$T_n^*(x) = T_n(2x-1), \tag{8}$$

and the following recurrence formulae:

$$T_n^*(x) = 2(2x-1) T_{n-1}^*(x) - T_{n-2}^*(x), \quad n = 2, 3, \ldots, \tag{9}$$

with initial conditions

$$T_0^*(x) = 1, \qquad T_1^*(x) = 2x - 1, \tag{10}$$

a_i, $i = 0, 1, 2, \ldots$, are constants.

Substituting (7) into (1) we obtain

$$D^\alpha \left(\sum_{i=0}^{n} a_i T_i^*(x) \right) = f(x) + \int_0^1 K(x,t) \left[\sum_{i=0}^{n} a_i T_i^*(t) \right] dt. \tag{11}$$

Hence the residual equation is defined as

$$R(x, a_0, a_1, \ldots, a_n) = \sum_{i=0}^{n} a_i D^\alpha T_i^*(x) - f(x) - \int_0^1 K(x,t) \left[\sum_{i=0}^{n} a_i T_i^*(t) \right] dt. \tag{12}$$

Let

$$S(a_0, a_1, \ldots, a_n) = \int_0^1 \left[R(x, a_0, a_1, \ldots, a_n) \right]^2 w(x) \, dx, \tag{13}$$

where $w(x)$ is the positive weight function defined on the interval $[0, 1]$. In this work we take $w(x) = 1$ for simplicity. Thus

$$S(a_0, a_1, \ldots, a_n) = \int_0^1 \left\{ \sum_{i=0}^{n} a_i D^\alpha T_i^*(x) - f(x) - \int_0^1 K(x,t) \left[\sum_{i=0}^{n} a_i T_i^*(t) \right] dt \right\}^2 dx. \tag{14}$$

So, finding the values of a_i, $i = 0, 1, \ldots, n$, which minimize S is equivalent to finding the best approximation for the solution of the fractional Integro-differential equation (1).

The minimum value of S is obtained by setting

$$\frac{\partial S}{\partial a_j} = 0, \quad j = 0, 1, \ldots, n. \tag{15}$$

Applying (15) to (14) we obtain

$$\int_0^1 \left\{ \sum_{i=0}^{n} a_i D^\alpha T_i^*(x) - f(x) - \int_0^1 K(x,t) \left[\sum_{i=0}^{n} a_i T_i^*(t) \right] dt \right\} \times \left\{ D^\alpha T_j^*(x) - \int_0^1 K(x,t) T_j^*(t) \, dt \right\} dx. \tag{16}$$

By evaluating the above equation for $j = 0, 1, \ldots, n$ we can obtain a system of $(n + 1)$ linear equations with $(n + 1)$ unknown coefficients a_i 's. This system can be formed by using matrices form as follows:

$$A = \begin{pmatrix} \int_0^1 R(x, a_0) h_0 dx & \int_0^1 R(x, a_1) h_0 dx & \cdots & \int_0^1 R(x, a_n) h_0 dx \\ \int_0^1 R(x, a_0) h_1 dx & \int_0^1 R(x, a_1) h_1 dx & \cdots & \int_0^1 R(x, a_n) h_1 dx \\ \vdots & \vdots & \ddots & \vdots \\ \int_0^1 R(x, a_0) h_n dx & \int_0^1 R(x, a_1) h_n dx & \cdots & \int_0^1 R(x, a_n) h_n dx \end{pmatrix},$$

$$B = \begin{pmatrix} \int_0^1 f(x) h_0 dx \\ \int_0^1 f(x) h_1 dx \\ \vdots \\ \int_0^1 f(x) h_n dx \end{pmatrix}, \tag{17}$$

where

$$h_j = D^\alpha T_j^*(x) - \int_0^1 K(x, t) T_j^*(t) dt, \quad j = 0, 1, \ldots, n,$$

$$R(x, a_i) = \sum_{i=0}^{n} a_i D^\alpha T_i^*(x) - \int_0^1 K(x, t) \left[\sum_{i=0}^{n} a_i T_i^*(t)\right] dt, \quad i = 0, 1, \ldots, n. \tag{18}$$

By solving the above system we obtain the values of the unknown coefficients and the approximate solution of (1).

4. Numerical Examples

In this section, some numerical examples of linear fractional Integro-differential equations are presented to illustrate the above results. All results are obtained by using Maple 15.

Example 1. Consider the following fractional Integro-differential equation:

$$D^{1/2}\varphi(x) = \frac{(8/3)x^{3/2} - 2x^{1/2}}{\sqrt{\pi}} + \frac{x}{12} + \int_0^1 xt\varphi(t) dt, \quad 0 \le x, t \le 1, \tag{19}$$

subject to $\varphi(0) = 0$ with the exact solution $\varphi(x) = x^2 - x$.

Applying the least squares method with aid of shifted Chebyshev polynomial of the first kind $T_i^*(x)$, $i = 0, 1, \ldots, n$ at $n = 5$, to the fractional Integro-differential equation (19) we obtain a system of (6) linear equations with (6) unknown coefficients a_i, $i = 0, 1, \ldots, 5$. This system can be transformed into a matrix equation and by solving this matrix equation we obtain the inverse which is given in Figure 1 and we obtain the values of the coefficients. Substituting the values of the coefficients into (7) we obtain the approximate solution which is the same as the exact solution and the results are shown in Figure 2.

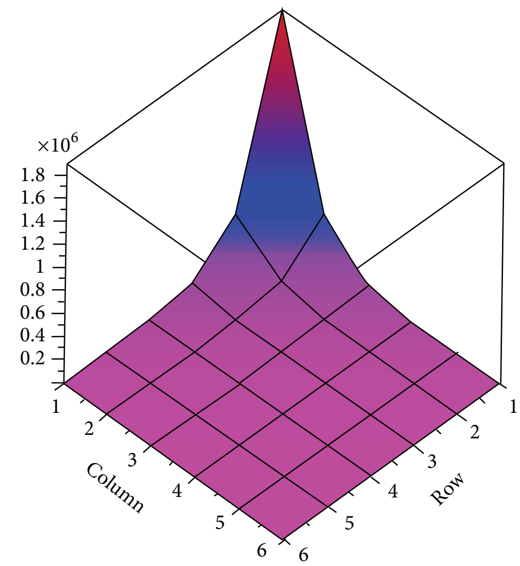

Figure 1: The matrix inverse of Example 1.

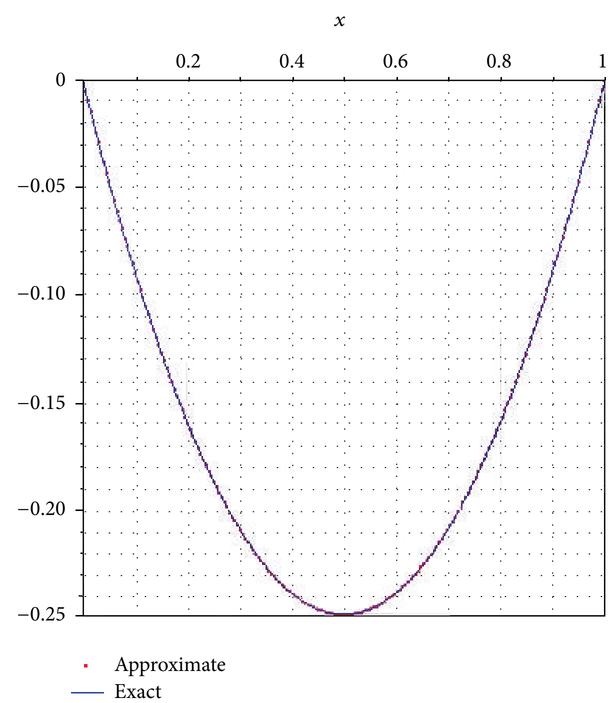

Figure 2: Numerical results of Example 1.

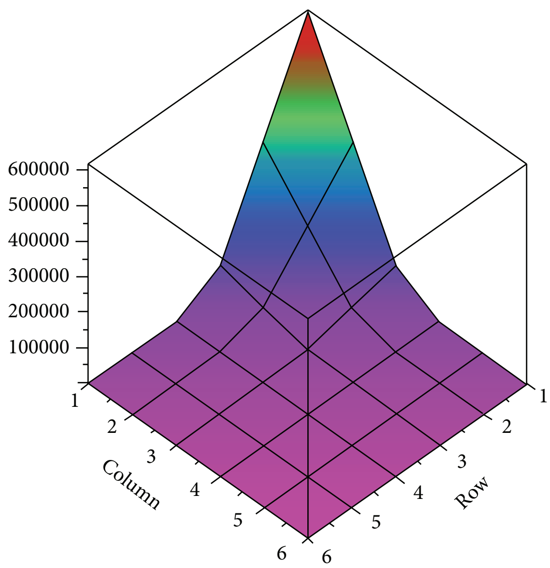

Figure 3: The matrix inverse of Example 2.

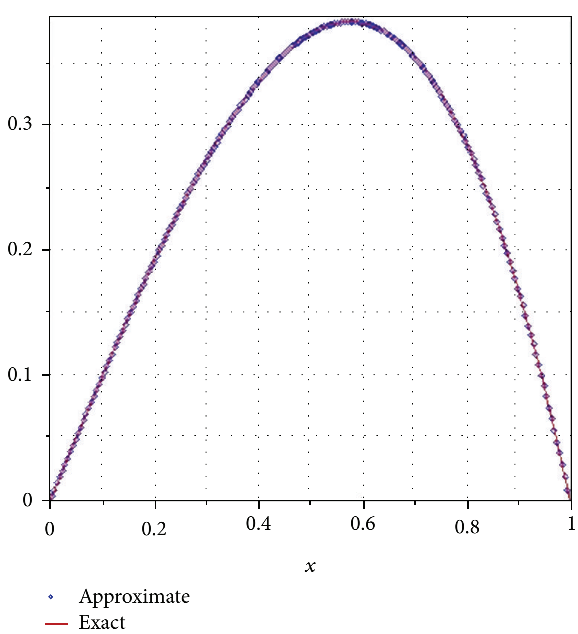

Figure 4: Numerical results of Example 2.

Example 2. Consider the following fractional Integro-differential equation:

$$D^{5/6}\varphi(x) = f(x) + \int_0^1 xe^t\varphi(t)\,dt, \quad 0 \le x,\ t \le 1, \tag{20}$$

subject to $\varphi(0) = 0$, where

$$f(x) = -\frac{3}{91}\frac{x^{1/6}\Gamma(5/6)\left(-91 + 216x^2\right)}{\pi} + (5 - 2e)\,x \tag{21}$$

with the exact solution $\varphi(x) = x - x^3$.

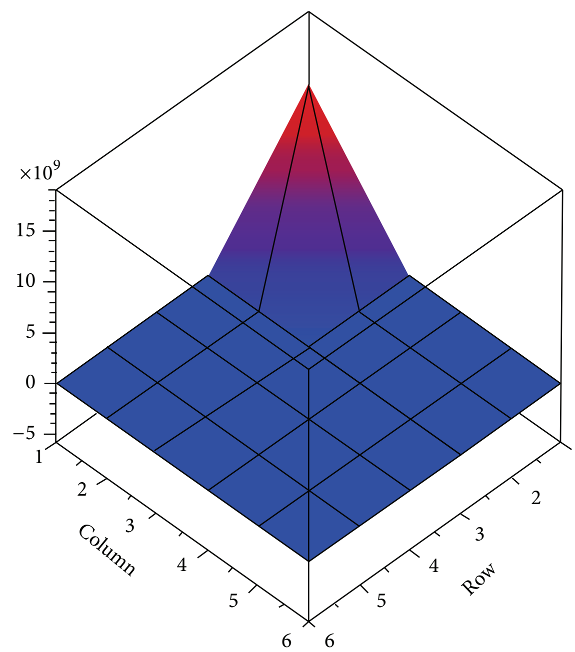

Figure 5: The matrix inverse of Example 3.

Similarly as in Example 1 applying the least squares method with aid of shifted Chebyshev polynomial of the first kind $T_i^*(x)$, $i = 0, 1, \ldots, n$ at $n = 5$, to the fractional Integro-differential equation (20) the numerical results are shown in Figures 3 and 4 and we obtain the approximate solution which is the same as the exact solution.

Example 3. Consider the following fractional Integro-differential equation:

$$D^{5/3}\varphi(x) = \frac{3\sqrt{3}\Gamma(2/3)\,x^{1/3}}{\pi} - \frac{1}{5}x^2 - \frac{1}{4}x + \int_0^1 \left(xt + x^2t^2\right)\varphi(t)\,dt, \quad 0 \le x, t \le 1, \tag{22}$$

subject to $\varphi(0) = \dot{\varphi}(0) = 0$ with the exact solution $\varphi(x) = x^2$.

Similarly as in Examples 1 and 2 applying the least squares method with aid of shifted Chebyshev polynomial of the first kind $T_i^*(x)$, $i = 0, 1, \ldots, n$ at $n = 5$, to the fractional Integro-differential equation (22) the numerical results are shown in Figures 5 and 6 and we obtain the approximate solution which is the same as the exact solution.

5. Conclusion

In this paper we study the numerical solution of three examples by using least squares method with aid of shifted Chebyshev polynomial which derives a good approximation. We show that this method is effective and has high convergency rate.

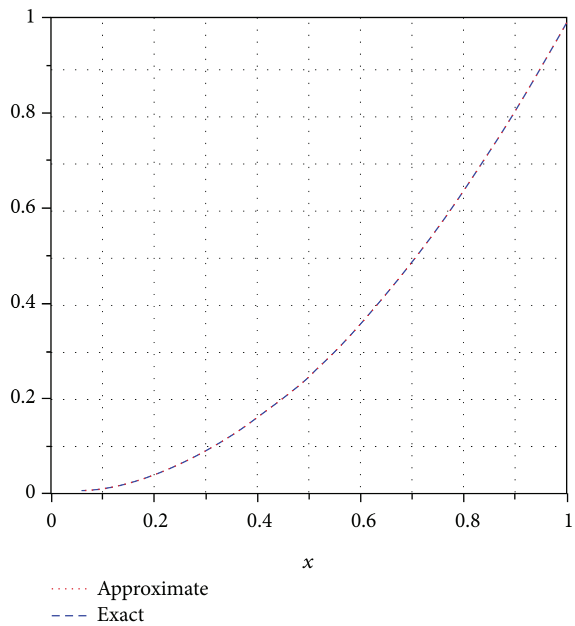

FIGURE 6: Numerical results of Example 3.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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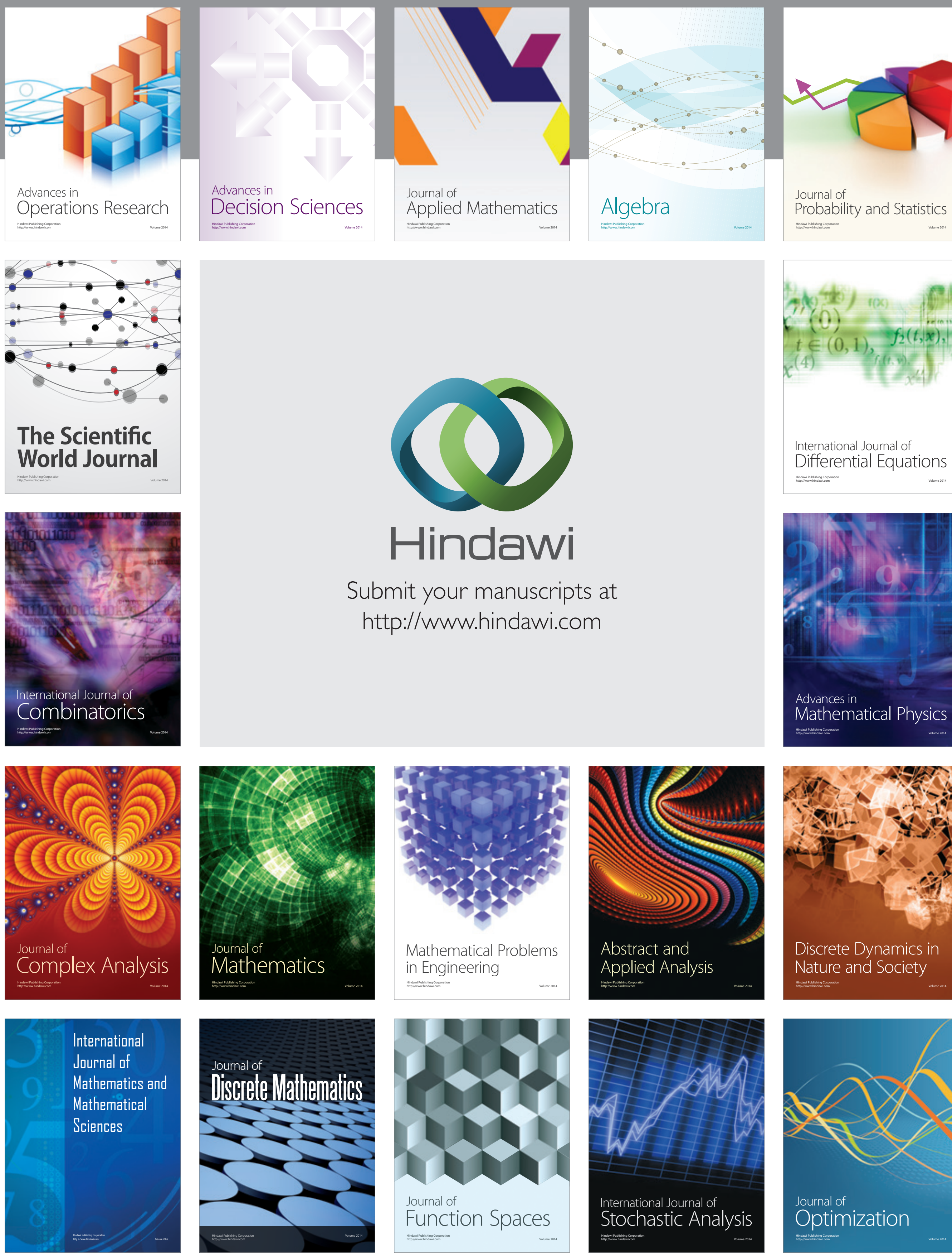
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