Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2014, Article ID 490692, 10 pages
http://dx.doi.org/10.1155/2014/490692

Research Article

Residual Probability Function, Associated Orderings, and Related Aging Classes

M. Kayid,[1,2] S. Izadkhah,[3] and S. Alshami[1]

[1] *Department of Statistics and Operations Research, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia*
[2] *Department of Mathematics and Computer Science, Faculty of Science, Suez University, Suez 41522, Egypt*
[3] *Department of Statistics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad 91779, Iran*

Correspondence should be addressed to M. Kayid; el_kayid2000@yahoo.com

Received 6 November 2014; Revised 29 November 2014; Accepted 3 December 2014; Published 30 December 2014

Academic Editor: Jinhu Lü

The concept of residual probability plays an important role in reliability and life testing. In this investigation, we study further the residual probability order and its related aging classes. Several characterizations and preservation properties of this order under some statistical and reliability operations of monotone transformation, mixture, weighted distributions, and order statistics are discussed. In addition, by comparing the original distribution with its associated equilibrium distribution with respect to the residual probability order, new aging classes of life distributions are proposed and studied. Finally, a test of exponentiality against such classes is derived and sets of real data are used as examples to elucidate the use of the proposed test for practical problems.

1. Introduction

The residual probability (RP) function is a well-known reliability measure which has applications in many disciplines such as reliability theory, survival analysis, and actuarial studies. Let X and Y be two random lifetimes representing the lifetimes of two systems with distribution functions F and G and survival functions $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$, respectively. The systems can be considered as the products of two different branches of a company. Then the quantity $P(X > Y)$ gives the reliability of X relative to Y. In case of both X and Y being distributed as Weibull, Brown and Rutemiller [1] have pointed out that to design as long-lived a product as possible one can consider the quantity $P(X > Y)$ and then choose X or Y when this probability is greater or less than 0.5, respectively. However, if the systems are known to have a survival age t, it is important to take into account the age, when we compare the remaining lifetimes. Let $X_t = [X - t \mid X > t]$ and $Y_t = [Y - t \mid Y > t]$ denote the additional residual lifetime of X and Y given that the systems have survived up to age t. The RP function is defined as

$$R(t) = P(X_t > Y_t), \quad t > 0. \tag{1}$$

The RP function uniquely determines the distribution function of F (and hence the distribution function of G), under the condition that the ratio of the hazard rates of X and Y is known. In addition, when the ratio of the hazard rates of X and Y is a monotone function of time, then RP function is also a monotone function of time. As a result, the study of the properties of RP function might be important for engineers and system designers to compare the lifetime of the products and, hence, to design better products. For example, consider a series system with two independent components. If X and Y denote the lifetime of the components, then clearly the lifetime of the system is $T = \min\{X, Y\}$. It is easily seen that $R(t) = P(Y = T \mid T > t)$, that is, the probability that the component with lifetime Y causes the system failure given that the system has survived up to time t (cf. Zardasht and Asadi [2] for several reliability properties, Tan and Lü [3] for some biological background, and Lü and Chen [4], Chen et al. [5], and Zhou et al. [6] for some real world applications).

One of the main objectives of statistics is the comparison of random quantities. These comparisons are mainly based on the comparison of some measures associated with these random quantities. In many cases the researcher can express various forms of properties about the underlying distributions in terms of their survival functions, hazard rate functions, reversed hazard functions, mean residual functions, and other suitable functions of probability distributions. Comparisons of random variables based on such different functions are usually establishing partial orders among them which is known as stochastic orders. Formally, in view of the RP function, the lifetime random variable X is said to be smaller than Y in the RP order (denoted by $X \leq_{\text{rp}} Y$) if and only if

$$R(t) \leq 0.5, \quad \forall t > 0. \tag{2}$$

This stochastic order states that, between two used items of the same age t having original lifetimes X and Y, the one that has lifetime Y has a greater chance of surviving after the time t than the item with lifetime X. On the other hand, statisticians and reliability analysts have shown a growing interest in modeling survival data using classifications of life distributions by means of various stochastic orders. These categories are useful for modeling situations, maintenance, inventory theory, and biometry (cf. Barlow and Proschan [7] and Lai and Xie [8]). The random variable X^* with distribution $F^*(x) = \int_0^x \overline{F}(u)du/\mu$, for all $x \geq 0$, where μ is the mean of X, is known in the literature as the equilibrium distribution associated with X. In literature, it is found that the equilibrium distribution can be used to characterize some aging properties (cf. Bhattacharjee et al. [9], Mi [10], Bon and Illayk [11], Mugdadi and Ahmad [12], Bon and Illayk [13], Li and Xu [14], and Kayid et al. [15]). The main motivation of our work is a recent paper written by Zardasht and Asadi [2]. Ordinarily, when a stochastic order is proposed in the literature, its further properties in different forms of statistical analysis become important to study.

The purpose of this paper is to achieve two goals. The first one is to provide some characterizations, preservation results, and applications for the RP order. The second goal is to propose and study a new aging notion based upon the RP comparison between a random life and its equilibrium version. The paper is organized as follows. In Section 2, some characterizations and implications regarding the RP order are provided. In that section, preservation properties under some reliability operations such as monotonic transformation and mixture are discussed. In Section 3, we investigate a new aging notion based upon the RP order and develop a nonparametric method to test exponentiality against such a strict aging property. Finally, in Section 4, we give a brief conclusion and some remarks of the current research and its future.

Throughout this paper, the term increasing is used instead of monotone nondecreasing and the term decreasing is used instead of monotone nonincreasing. It is always assumed that the nonnegative random variables X and Y will have f and g, respectively, as their respective density functions and $r_X(x) = f(x)/\overline{F}(x)$ and $r_Y(x) = g(x)/\overline{G}(x)$ as their hazard rate functions. All integrals, expectations, and derivatives are implicitly assumed to exist wherever they are given.

2. Definitions, Characterizations, and Implications

The objective of this section is to concentrate on the relations between RP order and other well-known stochastic orders. In addition, we discuss some preservation properties of the RP order under some well-known reliability operations such as mixture, monotone transformations, weighted distributions, and order statistics. For ease of reference, before stating our main results, we present some definitions and basic properties which will be used in the sequel. For an exhaustive monograph on definitions and properties of stochastic orders we refer to Shaked and Shanthikumar [16], Di Crescenzo [17], and Alzaid and Benkherouf [18] and for aging notions we refer to Barlow and Proschan [7] and Lai and Xie [8].

Definition 1. The random variable X is said to be smaller than Y in the following:

(i) hazard rate order (denoted by $X \leq_{\text{hr}} Y$) if

$$\frac{f(t)}{\overline{F}(t)} \geq \frac{g(t)}{\overline{G}(t)}, \quad \forall t > 0, \tag{3}$$

(ii) mean residual life order (denoted by $X \leq_{\text{mrl}} Y$) if

$$\frac{\int_t^\infty \overline{F}(x)\,dx}{\overline{F}(t)} \leq \frac{\int_t^\infty \overline{G}(x)\,dx}{\overline{G}(t)}, \quad \forall t \geq 0, \tag{4}$$

(iii) dual order (denoted by $X \leq_d Y$) if for some $R \geq 1$

$$\overline{G}(t) \leq R\overline{F}(t), \quad \forall t \geq 0, \tag{5}$$

(iv) residual probability order (denoted by $X \leq_{\text{rp}} Y$) if

$$\int_t^\infty \left[f(x)\overline{G}(x) - g(x)\overline{F}(x)\right]dx \geq 0, \quad \forall t \geq 0, \tag{6}$$

(v) probability order (denoted by $X \leq_{\text{pr}} Y$) if $P(X \geq Y) \leq 0.5$.

Definition 2. Let X be a lifetime random variable having survival function $\overline{F}$. We say that the random variable X is

(i) decreasing (increasing) in mean residual life, written as DMRL (IMRL), whenever

$$\frac{\int_t^\infty \overline{F}(x)\,dx}{\overline{F}(t)} \tag{7}$$

is decreasing (increasing) for $t \geq 0$ or equivalently if $X^* \leq_{\text{hr}} (\geq_{\text{hr}}) X$;

(ii) decreasing (increasing) in variance residual life, written as DVRL (IVRL), whenever

$$\frac{\int_t^\infty \int_x^\infty \overline{F}(u)\,du\,dx}{\int_t^\infty \overline{F}(x)\,dx} \tag{8}$$

is decreasing (increasing) for $t \geq 0$ or equivalently if $X^* \leq_{\text{mrl}} (\geq_{\text{mrl}}) X$.

The following lemma which is due to Barlow and Proschan [7] is essential in deriving our main results.

Lemma 3. *Let W be a real valued measure which is not necessarily nonnegative. Let h be a nonnegative real valued function. If $\int_t^\infty dW(x) \geq 0$ for all $t \geq 0$ and if h is an increasing function, then*

$$\int_0^\infty h(x)\, dW(x) \geq 0. \tag{9}$$

The first result shows that the RP order is stronger than the dual order.

Theorem 4. *Let $X \leq_{rp} Y$. Then $X \leq_d Y$.*

Proof. For any $x \geq 0$, set

$$W(x) = \int_x^\infty \overline{F}(u)\overline{G}(u)\left[r_Y(u) - r_X(u)\right] du, \tag{10}$$

from which we get, for any $t \geq 0$,

$$\int_t^\infty dW(x) = \int_t^\infty \overline{F}(x)\overline{G}(x)\left[r_X(x) - r_Y(x)\right] dx. \tag{11}$$

According to Remark 3 in Zardasht and Asadi [2], $\int_t^\infty dW(x) \geq 0$, for all $t \geq 0$. For a fixed $t \geq 0$, take

$$h(x) = \begin{cases} \dfrac{1}{\overline{F}(x)\overline{G}(x)}, & x > t \\ 0, & x \leq t. \end{cases} \tag{12}$$

We see that h is a nonnegative and increasing function in x, for any $t \geq 0$. So, because of Lemma 3, the nonnegativity of $\int_0^\infty h(x)dW(x)$ is guaranteed. That is,

$$\begin{aligned} \int_0^\infty h(x)\, dW(x) &= \int_t^\infty \frac{1}{\overline{F}(x)\overline{G}(x)} dW(x) \\ &= \int_t^\infty \left[r_X(x) - r_Y(x)\right] dx \\ &= \ln\left[\lim_{x\to\infty}\frac{\overline{G}(x)}{\overline{F}(x)}\right] - \ln\left[\frac{\overline{G}(t)}{\overline{F}(t)}\right] \geq 0, \\ &\qquad \text{for any } t \geq 0. \end{aligned} \tag{13}$$

By taking $R = \lim_{x\to\infty}\overline{G}(x)/\overline{F}(x)$, it follows that $R \geq \overline{G}(t)/\overline{F}(t)$, for all $t \geq 0$. Note that if we put $t = 0$ in the recent inequality we see that $R \geq 1$. Hence there exists an $R \geq 1$ for which $\overline{G}(t) \leq R\overline{F}(t)$, for all $t \geq 0$. This completes the proof. □

Mixture models are widely used as computational convenient representations for modeling complex probability distributions. In practical situations, it often happens that data from several populations are mixed and information about which subpopulation gave rise to individual data points is unavailable. Mixture models are used to model such data sets in nature. The next result states the preservation property of the RP order under mixture. This result strengthens the result of Theorem 8 of Zardasht and Asadi [2] to a more general setting.

Theorem 5. *Let X, Y, and Θ be random variables such that $[X \mid \Theta = \theta] \leq_{rp} [Y \mid \Theta = \theta']$ for all θ and θ' in the support of χ. Then $X \leq_{rp} Y$.*

Proof. Denote by $f(x \mid \theta)$, $g(x \mid \theta')$, $\overline{F}(x \mid \theta)$, and $\overline{G}(x \mid \theta')$ the densities and the survivals of $[X \mid \Theta = \theta]$ and $[Y \mid \Theta = \theta']$, respectively. In view of the assumption we have, for all $\theta, \theta' \in \chi$,

$$\int_t^\infty \left[\overline{G}\left(x \mid \theta'\right) f(x \mid \theta) - \overline{F}(x \mid \theta)\, g\left(x \mid \theta'\right)\right] dx \geq 0, \qquad \forall t \geq 0. \tag{14}$$

Let Θ have distribution function H. Then, integrating both sides of (14) with respect to $H(\theta)$ and $H(\theta')$ yields

$$\begin{aligned} &\int_{\theta'\in\chi}\int_{\theta\in\chi}\int_t^\infty \overline{G}\left(x \mid \theta'\right) f(x \mid \theta)\, dx\, dH(\theta)\, dH\left(\theta'\right) \\ &\quad \geq \int_{\theta'\in\chi}\int_{\theta\in\chi}\int_t^\infty \overline{F}(x \mid \theta)\, g\left(x \mid \theta'\right) dx\, dH(\theta)\, dH\left(\theta'\right), \end{aligned} \tag{15}$$

which gives

$$\begin{aligned} &\int_t^\infty \left[\left(\int_{\theta'\in\chi} \overline{G}\left(x \mid \theta'\right) dH\left(\theta'\right)\right) \times \left(\int_{\theta\in\chi} f(x \mid \theta)\, dH(\theta)\right)\right] dx \\ &\quad \geq \int_t^\infty \left[\left(\int_{\theta\in\chi} \overline{F}(x \mid \theta)\, dH(\theta)\right) \times \left(\int_{\theta'\in\chi} g\left(x \mid \theta'\right) dH\left(\theta'\right)\right)\right] dx. \end{aligned} \tag{16}$$

This means that $\int_t^\infty [\overline{G}(x)f(x) - \overline{F}(x)g(x)]dx \geq 0$, for all $t > 0$. The result now immediately follows. □

In the following result, we show that if the lifetimes of two series systems with i.i.d. components are RP ordered then their components are also RP ordered.

Theorem 6. *Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be two random samples from survival (density) functions $\overline{F}(f)$ and $\overline{G}(g)$, respectively. Then*

$$\min\{X_1, X_2, \ldots, X_n\} \leq_{rp} \min\{Y_1, Y_2, \ldots, Y_n\} \Longrightarrow X_1 \leq_{rp} Y_1. \tag{17}$$

Proof. We know that $\min\{X_1, X_2, \ldots, X_n\} \leq_{\text{rp}} \min\{Y_1, Y_2, \ldots, Y_n\}$, if and only if

$$\int_t^\infty \left[\overline{F}(x)\overline{G}(x)\right]^{n-1}\left[\overline{G}(x) f(x) - \overline{F}(x) g(x)\right] dx \geq 0, \quad \forall t \geq 0. \tag{18}$$

Let us denote, for all $x \geq 0$,

$$W(x) = \int_x^\infty \left[\overline{F}(u)\overline{G}(u)\right]^{n-1}\left[\overline{F}(u) g(u) - \overline{G}(u) f(u)\right] du. \tag{19}$$

Thus, for any $x \geq 0$

$$\begin{aligned} dW(x) &= W'(x)\,dx \\ &= \left[\overline{F}(x)\overline{G}(x)\right]^{n-1}\left[\overline{G}(x) f(x) - \overline{F}(x) g(x)\right]. \end{aligned} \tag{20}$$

Take

$$h(x) = \begin{cases} \dfrac{1}{\left[\overline{F}(x)\overline{G}(x)\right]^{n-1}}, & x > t \\ 0, & x \leq t, \end{cases} \tag{21}$$

which is nonnegative and increasing in x, for all $t \geq 0$. We have, for any $t \geq 0$,

$$\begin{aligned} \int_0^\infty h(x)\,dW(x) &= \int_t^\infty \frac{1}{\left[\overline{F}(x)\overline{G}(x)\right]^{n-1}} dW(x) \\ &= \int_t^\infty \left[\overline{G}(x) f(x) - \overline{F}(x) g(x)\right] dx. \end{aligned} \tag{22}$$

Because of (18), $\int_t^\infty dW(x) \geq 0$, for all $t \geq 0$. So, by Lemma 3 we can obtain that $\int_t^\infty h(x) dW(x) \geq 0$, for all $t \geq 0$, which means that $X_1 \leq_{\text{rp}} Y_1$. □

Weighted distributions are useful in the context of model selection. Rao [19] introduced the family of weighted distributions in a unified way. One of the problems that has recently received much attention by many researchers is the preservation of stochastic orders by weighted distributions (cf. Bartoszewicz and Skolimowska [20], Misra et al. [21], Izadkhah and Kayid [22], and Belzunce et al. [23]). For two weight functions w_1 and w_2, assume that X_{w_1} and Y_{w_2} denote two random variables with respective density functions:

$$\begin{aligned} f_1(x) &= \frac{w_1(x) f(x)}{c_1}, \quad \text{for } x \geq 0, \\ g_2(x) &= \frac{w_2(x) g(x)}{c_2}, \quad \text{for } x \geq 0, \end{aligned} \tag{23}$$

where $0 < c_1 = E[w_1(X)] < \infty$ and $0 < c_2 = E(w_2(Y)) < \infty$. The random variables X_{w_1} and Y_{w_2} are weighted versions of X and Y, respectively. Let $\beta_1(x) = E[w_1(X) \mid X > x]$, and $\beta_2(x) = E[w_2(Y) \mid Y > x]$. Then the survival functions of X_{w_1} and Y_{w_2} are, respectively, given by

$$\begin{aligned} \overline{F}_1(x) &= \frac{\beta_1(x)\overline{F}(x)}{c_1} \quad \text{for } x \geq 0, \\ \overline{G}_2(x) &= \frac{\beta_2(x)\overline{G}(x)}{c_2} \quad \text{for } x \geq 0. \end{aligned} \tag{24}$$

The next result provides a preservation property of the RP order under weighted distributions.

Theorem 7. *Let $\beta_2(x)w_1(x) \geqslant \beta_1(x)w_2(x)$, for all $x \geq 0$, such that $\beta_1(x)w_2(x)$ is increasing for $x \geq 0$. Then*

$$X \leq_{rp} Y \Longrightarrow X_{w_1} \leq_{rp} Y_{w_2}. \tag{25}$$

Proof. Because of the first stated assumption we can write, for all $t \geq 0$,

$$\begin{aligned} &\int_t^\infty \left[\overline{G}_2(x) f_1(x) - \overline{F}_1(x) g_2(x)\right] dx \\ &= \int_t^\infty \left[\frac{B_2(x) w_1(x) \overline{G}(x) f(x)}{c_1 c_2} - \frac{B_1(x) w_2(x) f(x) \overline{F}(x)}{c_1 c_2}\right] dx \\ &\geq \int_t^\infty \frac{B_1(x) w_2(x)}{c_1 c_2}\left[\overline{G}(x) f(x) - \overline{F}(x) g(x)\right] dx \\ &= \int_0^\infty \frac{B_1(x) w_2(x) I_{(t,\infty)}(x)}{c_1 c_2} \times \left[\overline{G}(x) f(x) - \overline{F}(x) g(x)\right] dx \\ &= \int_0^\infty h_t(x)\,dW(x), \end{aligned} \tag{26}$$

where $h_t(x) = (c_1c_2)^{-1}B_1(x)w_2(x)I_{(t,\infty)}(x)$, which is increasing in $x \geq 0$, for all $t \geq 0$, because of the second assumption, and

$$dW(x) = \left[\overline{G}(x) f(x) - \overline{F}(x) g(x)\right] dx. \tag{27}$$

We have

$$\int_t^\infty dW(x) = \int_t^\infty \left[\overline{G}(x) f(x) - \overline{F}(x) g(x)\right] dx, \tag{28}$$

which is nonnegative, for any $t \geq 0$, by assumption. So, since $h_t(x)$ is increasing in x, for any fixed $t \geq 0$, thus by Lemma 3 we deduce that $\int_0^\infty h_t(x) dW(x) \geq 0$, for all $t \geq 0$. Hence this completes the proof of the theorem. □

The next example shows that the RP order is preserved under the model of proportional hazard rates (PHR).

Example 8. Suppose that X and Y are two nonnegative random variables with survival functions $\overline{F}$ and $\overline{G}$, respectively, such that $X \leq_{\text{rp}} Y$ holds. Assume that X_{w_1} and Y_{w_2} are weighted versions of X and Y, with weights $w_1(x) = [\overline{F}(x)]^{\theta_1 - 1}, w_2(x) = [\overline{G}(x)]^{\theta_2 - 1}$, for any $0 < \theta_1 \leq \theta_2 \leq 1$. We easily see that $B_1(x) = [\overline{F}(x)]^{\theta_1 - 1}/\theta_1$ and $B_2(x) = [\overline{G}(x)]^{\theta_2 - 1}/\theta_2$. Now, we have that $w_1(x)/B_1(x) = \theta_1 \leq \theta_2 = w_2(x)/B_2(x)$ and that $B_1(x) w_2(x) = [\overline{F}(x)]^{\theta_1 - 1}[\overline{G}(x)]^{\theta_2 - 1}/\theta_1$ are increasing in $x \geq 0$. Thus, assumptions of Theorem 7 hold and hence $X_{w_1} \leq_{\text{rp}} Y_{w_2}$.

The next result shows that the RP order is preserved under monotone transformation.

Theorem 9. *Let ϕ be a nonnegative strictly increasing and differentiable function. Then*

$$X \leq_{rp} Y \Longrightarrow \phi(X) \leq_{rp} \phi(Y). \tag{29}$$

Proof. We know that $X \leq_{\text{rp}} Y$ implies that

$$\int_{\phi^{-1}(t)}^{\infty} \left\{ f(x) \overline{G}(x) - g(x) \overline{F}(x) \right\} dx \geq 0, \quad \forall t \geq 0. \tag{30}$$

Denote by f_ϕ, $\overline{F}_\phi$, g_ϕ, and $\overline{G}_\phi$ the density and the survival functions of X and Y, respectively. We can see that

$$\begin{aligned} &\int_t^{\infty} \left\{ f_\phi(x) \overline{G}_\phi(x) - g_\phi(x) \overline{F}_\phi(x) \right\} dx \\ &= \int_t^{\infty} \frac{f\left(\phi^{-1}(x)\right) \overline{G}\left(\phi^{-1}(x)\right) - g\left(\phi^{-1}(x)\right) \overline{F}\left(\phi^{-1}(x)\right)}{\phi'\left\{\phi^{-1}(x)\right\}} dx \\ &= \int_{\phi^{-1}(t)}^{\infty} \left\{ f(u) \overline{G}(u) - g(u) \overline{F}(u) \right\} du, \end{aligned} \tag{31}$$

where the change of variable $u = \phi^{-1}(x)$ is imposed. Appealing to (30) the result follows. □

The concept of residual life plays an important role in reliability, survival analysis, economics, and actuarial sciences. Let X be a nonnegative random variable with survival function $\overline{F}$. The random variable $X_t = (X - t \mid X > t)$, for $t : \overline{F}(t) > 0$, is well known in the literature as the residual lifetime variable associated with X. The random variable X_t represents the lifetime of a used device of age t. Similarly, define $Y_t = (Y - t \mid Y > t)$, for $t : \overline{G}(t) > 0$, in which $\overline{G}$ is the survival function of Y. In the literature, the concept of residual life has been extended to the case where t is random. Suppose that T is a nonnegative random variable with distribution function H such that $E[\overline{F}(T)] > 0$. Then the random variable $X_T = (X - T \mid X > T)$ is known as the residual lifetime of X at the random time T (cf. Marshall [24], Nanda and Kundu [25], and Abouammoh et al. [26]). The following result establishes two characterizations of the RP order by means of the concepts of residual life and residual life at random time.

Theorem 10. *For the two lifetime distributions X and Y, we have*

(i) *$X \leq_{rp} Y$ if and only if $X_t \leq_{rp} Y_t$, for all $t \geq 0$;*

(ii) *$X \leq_{rp} Y$ if and only if $X_T \leq_{rp} Y_T$, for all nonnegative random variables T that are independent from X and Y.*

Proof.

(i) Denote by f_t, $\overline{F}_t$, g_t, and $\overline{G}_t$ the density and the survival functions of X_t and Y_t, respectively. We know that $X_t \leq_{\text{rp}} Y_t$, for any $s \geq 0$, if and only if $\int_s^{\infty} [\overline{G}_t(x) f_t(x) - \overline{F}_t(x) g_t(x)] \geq 0$, for all $s \geq 0$, and for any $t \geq 0$. We can see that

$$\begin{aligned} &\int_s^{\infty} \left[\overline{G}_t(x) f_t(x) - \overline{F}_t(x) g_t(x) \right] \\ &= \int_s^{\infty} \left[\frac{\overline{G}(t+x) f(t+x)}{\overline{G}(t) \overline{F}(t)} - \frac{\overline{F}(t+x) g(t+x)}{\overline{F}(t) \overline{G}(t)} \right] dx \\ &= \frac{1}{\overline{F}(t) \overline{G}(t)} \int_{t+s}^{\infty} \left[f(x) \overline{G}(x) - g(x) \overline{F}(x) \right] dx, \end{aligned} \tag{32}$$

which is nonnegative for all $t \geq 0$, and $s \geq 0$, if and only if $X \leq_{\text{rp}} Y$.

(ii) First assume that $X \leq_{\text{rp}} Y$. From Zardasht and Asadi [2] we know that $X \leq_{\text{rp}} Y$ is equivalent to $R(t) = P[X_t \geq Y_t] \leq 0.5$, for all $t \geq 0$. Also, it is derived that $R(t) = P[X > Y > t]/[\overline{F}(t)\overline{G}(t)]$, for any $t \geq 0$. Suppose that T has distribution H. Because T is independent from X and Y, thus we can write

$$\begin{aligned} P\left(X_T \geq Y_T\right) &= P\left(X - T > Y - T \mid X > T, Y > T\right) \\ &= \frac{\int_0^{\infty} P[X > Y > t]\, dH(t)}{\int_0^{\infty} \overline{F}(t) \overline{G}(t)\, dH(t)} \\ &= \frac{\int_0^{\infty} R(t) \overline{F}(t) \overline{G}(t)\, dH(t)}{\int_0^{\infty} \overline{F}(t) \overline{G}(t)\, dH(t)} \\ &\geq \frac{\int_0^{\infty} (1/2) \overline{F}(t) \overline{G}(t)\, dH(t)}{\int_0^{\infty} \overline{F}(t) \overline{G}(t)\, dH(t)} \\ &= 0.5, \end{aligned} \tag{33}$$

which means that $X_T \leq_{\text{rp}} Y_T$. Conversely, if we assume that $X_T \leq_{\text{rp}} Y_T$ for all nonnegative random variables T, independent from X and Y, then by taking T as degenerate random variables the result immediately follows.

□

3. Related Aging Classes

In reliability, various aging classes of life distributions have been introduced to describe several types of deterioration (improvement) that accompany aging. The past decades witnessed some aging notions based on a stochastic comparison between a random life X and its equilibrium version X^* which are introduced and studied (cf. Li and Xu [14], Bhattacharjee et al. [9], and Ahmad et al. [27]). This section investigates the following new aging notion.

Definition 11. A random life X is said to be new better than renewal used in the RP order (NBRU$_{\text{rp}}$), if $X^* \leq_{\text{rp}} X$, or, equivalently,

$$\int_t^\infty \left[\overline{F}^2(x) - f(x)\int_x^\infty \overline{F}(u)\,du\right]dx \geq 0, \quad \forall t \geq 0. \quad (34)$$

As the dual version, new worse than renewal used in the RP order (NWRU$_{\text{rp}}$) may be defined through $X^* \geq_{\text{rp}} X$. The next result states some relationships among aging classes of life distributions.

Theorem 12. *We have the following assertions.*

(i) *If X is DMRL then X is NBRU$_{rp}$.*

(ii) *If X is IMRL, then X is NWRU$_{rp}$.*

(iii) *If X is NWRU$_{rp}$, then X is IVRL.*

Proof. To prove (i) and (ii) note that from definition X is DMRL (IMRL) if and only if $X^* \leq_{\text{hr}} (\geq_{\text{hr}})X$. By Theorem 6 of Zardasht and Asadi [2] it follows that $X^* \leq_{\text{rp}} (\geq_{\text{rp}})X$, which means that X is NBRU$_{\text{rp}}$ (NWRU$_{\text{rp}}$). To prove (iii) we first need to recall from Zardasht and Asadi [2] that if Y has a decreasing density function, then $X \leq_{\text{rp}} Y$ implies $X \leq_{\text{mrl}} Y$. In addition, we know from Shaked and Shanthikumar [16] that the hazard rate order implies the mean residual life order. Now, if X is NWRU$_{\text{rp}}$ then $X^* \geq_{\text{rp}} X$, where X^* has a decreasing density. Consequently, $X^* \geq_{\text{mrl}} X$, which is equivalent to X being IVRL. □

Next we prove the preservation of NWRU$_{\text{rp}}$ under monotone transformations.

Theorem 13. *Let ϕ be a nonnegative strictly increasing and convex function which is differentiable. If X is NWRU$_{rp}$, then $\phi(X)$ is also NWRU$_{rp}$.*

Proof. First note that $\phi(X)$ is NWRU$_{\text{rp}}$ if and only if

$$\int_t^\infty \left[f_\phi(x)\int_x^\infty \overline{F}_\phi(u)\,du - \overline{F}_\phi^2(x)\right]dx \geq 0, \quad \forall t \geq 0. \quad (35)$$

Because of assumptions, ϕ' is a nonnegative increasing function which provides that, for all $t \geq 0$,

$$\begin{aligned}
&\int_t^\infty \left[f_\phi(x)\int_x^\infty \overline{F}_\phi(u)\,du - \overline{F}_\phi^2(x)\right]dx \\
&= \int_t^\infty \left[\frac{f\left(\phi^{-1}(x)\right)}{\phi'\left(\phi^{-1}(x)\right)} \times \int_x^\infty \overline{F}\left(\phi^{-1}(u)\right)du - \overline{F}^2\left(\phi^{-1}(x)\right)\right]dx \\
&= \int_t^\infty \left[\frac{f\left(\phi^{-1}(x)\right)}{\phi'\left(\phi^{-1}(x)\right)} \times \int_{\phi^{-1}(x)}^\infty \phi'(u)\,\overline{F}(u)\,du - \overline{F}^2\left(\phi^{-1}(x)\right)\right]dx \\
&\geq \int_t^\infty \left[f\left(\phi^{-1}(x)\right)\int_{\phi^{-1}(x)}^\infty \overline{F}(u)\,du - \overline{F}^2\left(\phi^{-1}(x)\right)\right]dx \\
&= \int_{\phi^{-1}(t)}^\infty \phi'(y)\left[f(y)\int_y^\infty \overline{F}(y)\,du - \overline{F}^2(y)\right]dy.
\end{aligned} \quad (36)$$

Note also that if X is NWRU$_{\text{rp}}$ then

$$\int_t^\infty \left[f(y)\int_y^\infty \overline{F}(u)\,du - \overline{F}^2(y)\right]dy \geq 0, \quad \forall t \geq 0. \quad (37)$$

In addition, since $\phi'(y)$ is nonnegative and increasing in y, thus by taking $\rho(y) = \phi'(y)I[y > \phi^{-1}(t)]$, for any fixed $t \geq 0$, as a nonnegative increasing function in y, in Lemma 3 we deduce that the integral given in (36) is nonnegative which validates the proof. □

Before stating other results we recall the following lemma from Ahmad et al. [27].

Lemma 14. *Assume that $W(x)$ is a Lebesgue-Stieltjes measure, not necessarily positive. If h_1 and h_2 are two nonnegative increasing functions, and*

$$\int_t^\infty \int_x^\infty dW(u)\,dx \geq 0, \quad \forall t \geq 0, \quad (38)$$

then $\int_t^\infty \int_x^\infty h_1(u)h_2(x)dW(u)dx \geq 0$, for all $t \geq 0$.

The following result states that the NBRU$_{\text{rp}}$ (NWRU$_{\text{rp}}$) property passes from the lifetime of a parallel system into its i.i.d. lifetime components.

Theorem 15. *Let $\min\{X_1, X_2, \ldots, X_n\}$ be NBRU$_{rp}$ (NWRU$_{rp}$). Then, X_1 is also NBRU$_{rp}$ (NWRU$_{rp}$).*

Proof. We only prove the theorem for the case of $\mathrm{NBRU_{rp}}$. The other case is quite similar. Because of the identity

$$\int_t^\infty \int_x^\infty \left[f(u)\overline{F}(x) - f(x)\overline{F}(u)\right] du\, dx = \int_t^\infty \left[\overline{F}^2(x) - f(x)\int_x^\infty \overline{F}(u)\, du\right] dx, \tag{39}$$

from (36) we have that X is $\mathrm{NBRU_{rp}}$ if and only if

$$\int_t^\infty \int_x^\infty \left[f(u)\overline{F}(x) - f(x)\overline{F}(u)\right] du\, dx \geq 0, \quad \forall t \geq 0. \tag{40}$$

Note that

$$\int_t^\infty \int_x^\infty \left[f(u)\overline{F}(x) - f(x)\overline{F}(u)\right] du\, dx = \int_t^\infty \int_x^\infty \frac{1}{\overline{F}^n(u)\overline{F}^n(x)} dW(u)\, dx, \tag{41}$$

where $dW(u) = (\overline{F}(u)\overline{F}(x))^n[f(u)\overline{F}(x) - f(x)\overline{F}(u)]du$. We can see that if $\min\{X_1, X_2, \ldots, X_n\}$ is $\mathrm{NBRU_{rp}}$, then $\int_t^\infty \int_x^\infty dW(u)dx \geq 0$, for all $t \geq 0$. Taking $h_1(x) = h_2(x) = 1/\overline{F}^n(x)$ in Lemma 14 provides the proof. □

Below we discuss the preservation property of $\mathrm{NBRU_{rp}}$ ($\mathrm{NWRU_{rp}}$) class under weighted distributions. Assume that X_w is the weighted version of random variable X that has the density $f_w(x) = w(x)f(x)/c$ and the survival $\overline{F}_w(x) = B(x)\overline{F}(x)/c$, in which $c = E[w(X)]$ and $B(x) = E(w(X) \mid X > x)$.

Theorem 16. *Let X be $NBRU_{rp}$ ($NWRU_{rp}$) and let $w(x)/B(x)$ be increasing in x such that $B(x)$ is increasing in x. Then, X_w is also $NBRU_{rp}$ ($NWRU_{rp}$).*

Proof. We prove the theorem in the case of $\mathrm{NBRU_{rp}}$ because the other case can analogously be derived. We know that X_w is $\mathrm{NBRU_{rp}}$ if and only if

$$\int_t^\infty \int_x^\infty \left[f_w(u)\overline{F}_w(x) - f_w(x)\overline{F}_w(u)\right] du\, dx \geq 0, \quad \forall t \geq 0, \tag{42}$$

which is equivalent to

$$\int_t^\infty \int_x^\infty \left[w(u)f(u)B(x)\overline{F}(x) - w(x)f(x)B(u)\overline{F}(u)\right] du\, dx \geq 0, \quad \forall t \geq 0. \tag{43}$$

By assumption, for all $t \geq 0$, one obtains that

$$\int_t^\infty \int_x^\infty \left[w(u)f(u)B(x)\overline{F}(x) - w(x)f(x)B(u)\overline{F}(u)\right] du\, dx \geq \int_t^\infty \int_x^\infty B(u)w(x)\, dW(u)\, dx, \tag{44}$$

where $dW(u) = [f(u)\overline{F}(x) - f(x)\overline{F}(u)]du$. We know that if X is $\mathrm{NBRU_{rp}}$, then $\int_t^\infty \int_x^\infty dW(u) \geq 0$, for all $t \geq 0$. Because $h_1 = B$ and $h_2 = w$ are two increasing functions by assumption, thus applying Lemma 14 to (44) completes the proof. □

The exponential distribution represents the lifetime of the units that never ages due to wear and tear. On the other hand, some nonparametric classes have come up in the literature testifying to how a lifetime component or/and a system ages over the time. A natural question to ask is which aging class a real data set belongs to. Thus, the problem of testing exponentiality against various nonparametric classes may be of some interest in reliability or survival analysis (cf. Lai and Xie [8]). In the rest of this section, we proceed to construct a test for exponential distribution within the $\mathrm{NBRU_{rp}}$ class of life distribution. Formally, we test H: X is exponential versus K: X is $\mathrm{NBRU_{rp}}$ but not exponential. The following result is substantially useful in the sequel to formalize an appropriate test statistic as a departure measure from H in favor of K.

Lemma 17. *Let X_1, X_2, X_3 be i.i.d. copies of X which is $NBRU_{rp}$. Then*

$$\Delta = E[\min\{X_1, X_2\}] - \frac{3}{4}E[\min\{X_1, X_2, X_3\}] - \frac{1}{4}E(X_1) \geq 0. \tag{45}$$

Proof. Using the well-known Fubini theorem that F is $\mathrm{NBRU_{rp}}$ implies that

$$\begin{aligned}\int_t^\infty \overline{F}^2(x)\, dx &\geq \int_t^\infty \left[f(x)\int_x^\infty \overline{F}(u)\, du\right] dx \\ &= \int_t^\infty \int_t^u f(x)\overline{F}(u)\, dx\, du \\ &= \int_t^\infty \left\{\overline{F}(t)\overline{F}(u) - \overline{F}^2(u)\right\} du, \quad \forall t \geq 0,\end{aligned} \tag{46}$$

which gives

$$2\int_t^\infty \overline{F}^2(x)\, dx \geq \overline{F}(t)\int_t^\infty \overline{F}(x)\, dx, \quad \forall t \geq 0. \tag{47}$$

Therefore, it follows that

$$2\int_0^\infty \left[\int_t^\infty \overline{F}^2(x)\, dx\right] dF(t) \geq \int_0^\infty \left[\overline{F}(t)\int_t^\infty \overline{F}(x)\, dx\right] dF(t). \tag{48}$$

For the left hand side of (48) we have

$$\begin{aligned}2\int_0^\infty\left[\int_t^\infty \overline{F}^2(x)\,dx\right]dF(t) &= 2\int_0^\infty\left[\overline{F}^2(x)\int_0^x dF(t)\right]dx \\ &= 2\int_0^\infty \overline{F}^2(x)\left(1-\overline{F}(x)\right)dx \\ &= 2E\left[\min\{X_1,X_2\}\right]-2E\left[\min\{X_1,X_2,X_3\}\right].\end{aligned} \tag{49}$$

For the right hand side of the inequality in (48) we get

$$\begin{aligned}\int_0^\infty\left[\overline{F}(t)\int_t^\infty \overline{F}(x)\,dx\right]dF(t) &= \int_0^\infty \overline{F}(x)\left[\int_0^x \overline{F}(t)\,dF(t)\right]dx \\ &= \int_0^\infty \overline{F}(x)\left[\frac{1}{2}-\frac{\overline{F}^2(x)}{2}\right]dx \\ &= \frac{1}{2}E(X_1)-\frac{1}{2}E\left[\min\{X_1,X_2,X_3\}\right].\end{aligned} \tag{50}$$

The result now immediately follows. □

By considering Δ given in Lemma 17 as a measure of departure from H in favor of K, it is noticeable that, under H, $\Delta = 0$ and it is positive under K. Denote

$$\phi(X_1,X_2,X_3) = \min\{X_1,X_2\}-\frac{3}{4}\min\{X_1,X_2,X_3\} - \frac{1}{4}X_1. \tag{51}$$

Given that $X_1, X_2, \ldots, X_n$ is a random sample from F, an unbiased estimation of Δ is

$$\widehat{\Delta} = \frac{1}{n(n-1)(n-2)}\sum_{i\neq j\neq k}\phi\left(X_i,X_j,X_k\right), \tag{52}$$

which serves as an unbiased estimate of the parameter $\Delta = E[\phi(X_1,X_2,X_3)]$. Note that, to make Δ scale free, we only need to divide it by μ. Hence, $\delta = \Delta/\mu$ is a scale-free test against K, and the test statistic $\widehat{\delta} = \widehat{\Delta}/\overline{X}$ is asymptotically distribution-free as we show in the following result.

Theorem 18. *As $n \to \infty$, $\sqrt{n}(\widehat{\delta}-\delta)$ is asymptotically normal with zero mean and variance σ^2 given in (54). Under H, the variance $\sigma^2 = 43/120$.*

Proof. Using the general theory of U-statistics and von Mises statistics (see Lee [28]) as $n \to \infty$, $\sqrt{n}(\widehat{\delta}-\delta)$ is asymptotically normal with mean 0 and variance σ^2, where

$$\begin{aligned}\sigma^2 &= \operatorname{Var}\{E[\phi(X_1,X_2,X_3)\mid X_1] + E[\phi(X_2,X_1,X_3)\mid X_1] + E[\phi(X_2,X_3,X_1)\mid X_1]\} \\ &= \operatorname{Var}[h_1(X_1)+h_2(X_1)+h_3(X_1)], \\ h_1(X_1) &= E\left[\min\{X_1,X_2\}-\frac{3}{4}\min\{X_1,X_2,X_3\} - \frac{1}{4}X_1\mid X_1\right] \\ &= \int_0^{X_1}\overline{F}(x)\,dx - \frac{3}{4}\int_0^{X_1}\overline{F}^2(x)\,dx-\frac{1}{4}X_1, \\ h_2(X_1) &= E\left[\min\{X_2,X_1\}-\frac{3}{4}\min\{X_2,X_1,X_3\} - \frac{1}{4}X_2\mid X_1\right] \\ &= \int_0^{X_1}\overline{F}(x)\,dx - \frac{3}{4}\int_0^{X_1}\overline{F}^2(x)\,dx-\frac{1}{4}E(X_2), \\ h_3(X_1) &= E\left[\min\{X_2,X_3\}-\frac{3}{4}\min\{X_2,X_3,X_1\} - \frac{1}{4}X_2\mid X_1\right] \\ &= \int_0^{\infty}\overline{F}^2(x)\,dx - \frac{3}{4}\int_0^{X_1}\overline{F}^2(x)\,dx-\frac{1}{4}E(X_2).\end{aligned} \tag{53}$$

Thus, we have

$$\sigma^2 = \operatorname{Var}\left\{2\int_0^{X_1}\overline{F}(x)\,dx - \frac{9}{4}\int_0^{X_1}\overline{F}^2(x)\,dx + \int_0^\infty \overline{F}^2(x)\,dx - \frac{1}{2}E(X_2)\right\}. \tag{54}$$

Under H, F in the above identity is exponential with mean 1. After some calculation, the assertion follows. □

In practice, we can evaluate $\sqrt{n}\widehat{\delta}/\sqrt{43/120}$ and reject H if the observed value exceeds the $1-\alpha$ quantile of the standard normal distribution $N(0,1)$. To assess the goodness, we evaluate the Pitman's asymptotic efficacy of the test:

$$\operatorname{PAE}(\delta_\theta) = \frac{120}{43}\left[\frac{d}{d\theta}\delta_\theta\right]_{\theta\to\theta_0}. \tag{55}$$

Three of the most commonly used alternatives are as follows.

(i) The linear failure rate, $\overline{F}_1(t) = \exp\{-t-(\theta/2)t^2\}$, for $t,\theta \geq 0$;

TABLE 1: The upper percentile of $\widehat{\delta}$ with 10000 replications.

n	95%	98%	99%
5	0.130012	0.155101	0.166202
10	0.082154	0.088568	0.098781
15	0.064102	0.071754	0.078857
20	0.055231	0.055911	0.065263
25	0.048901	0.051425	0.058352
30	0.039227	0.046873	0.051917
35	0.035716	0.042391	0.047124
40	0.032451	0.039520	0.045151
45	0.028109	0.036786	0.041209
50	0.027630	0.034917	0.039127
55	0.027174	0.033101	0.037674
60	0.025409	0.030827	0.036160
65	0.023914	0.030151	0.034928
70	0.022772	0.028762	0.031660
75	0.021331	0.027168	0.031251
80	0.021159	0.025607	0.028449
85	0.020615	0.025120	0.028292
90	0.020124	0.024597	0.028127
95	0.019302	0.024161	0.027342
100	0.018917	0.023252	0.026550

(ii) the Makeham family, $\overline{F}_2(t) = \exp\{-t - \theta(e^{-t} + t - 1)\}$, for $t, \theta \geq 0$;

(iii) the Weibull family, $\overline{F}_3(t) = \exp\{-t^{\theta}\}$, for $t, \theta \geq 0$.

The null is at $\theta = 0$ in (i) and (ii) and at $\theta = 1$ in (iii). Direct calculation for the above three alternatives gives the values 0.5708, 0.2681, and 1.4263, respectively.

3.1. Monte Carlo Null Distribution Critical Values. In practice, simulated percentiles for small samples are commonly used by applied statisticians and reliability analysts. We have simulated the upper percentile values for 95, 98, and 99. Table 1 presents these percentile values of the statistic $\widehat{\delta}_F$ and the calculations are based on 10,000 simulated samples of sizes $n = 5(5)100$.

3.2. The Power of the Proposed Test. We calculate the power of the proposed test at a significance level α using simulated number of sample 10000 for sample size $n = 10$, 20, and 30 and $\theta = 1, 2$, and 3 with respect to the alternatives F_1, F_2, and F_3. Table 2 below shows the power of the test at different values of θ and significance level $\alpha = 0.05$.

From Table 2, it is noted that the power of the test increases when the values of the parameter θ and sample size n increase, and it is clear that our test exhibits rather good powers.

3.3. Numerical Results. To demonstrate the test method above, we apply it to the data sets.

Example 19. Consider the following data in Bhattacharjee et al. [9]; these data represent set of 40 patients suffering from blood cancer (leukemia) from one of the Ministry of Health hospitals in Saudi Arabia and the ordered values (in days) are 115, 181, 255, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1024, 1062, 1063, 1165, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549, 1578, 1578, 15999, 1603, 1605, 1696, 1735, 1799, 1815, and 1852.

TABLE 2: Alternative distributions: LFR, Makhem, and Weibull.

n	θ	LFR	Makhem	Weibull
10	1	0.99848	0.99487	0.96429
	2	0.99503	0.99831	0.99510
	3	1.00000	1.00000	1.00000
20	1	0.99621	0.99885	0.94399
	2	0.99874	0.99997	0.99887
	3	1.00000	0.99999	1.00000
30	1	0.99881	0.99887	0.96875
	2	0.99998	0.99998	0.99989
	3	1.00000	1.00000	1.00000

It was found that $\widehat{\delta}_F = 0.0892$ which is greater than the critical value of Table 1. Then we reject the null hypothesis of exponentiality and accept K which states that the data set has NBRU_{rp} property.

Example 20. In an experiment at Florida State University to study the effect of methyl mercury poisoning on the life lengths of fish, goldfish were subjected to various dosages of methyl mercury (cf. Kochar [29]). At one dosage level the ordered times to death in week are 6.000, 6.143, 7.286, 8.714, 9.429, 9.857, 10.143, 11.571, 11.714, and 11.714.

It was found that $\widehat{\delta}_F = 0.01254$ which is less than the critical value of Table 1. Then we accept the null hypothesis of exponentiality property.

4. Conclusions

The residual probability function is a new meaningful quantity in reliability theory to evaluate the lifetime of systems by taking their age into account. In this paper, we focused on two purposes. The first one was to concentrate on the residual probability order that firstly initiated by Zardasht and Asadi [2], to refine and complement some of the results given in this context. Specifically, it was obtained that the RP order directly implies the dual order which states that the survival function of the greater random variable is bounded. In addition, we considered a general preservation property of the RP order under mixture of distributions. Another result which indicates that the RP order passes from the lifetime of two series systems into the lifetime of their i.i.d. components is derived. We concluded our first goal by giving some preservation properties of the RP order under weighted distributions and increasing transformations and giving some characterization properties of the RP order via the concepts of the residual life at fixed age and the residual life at random age. The second purpose of the paper was to introduce a new aging process using the concept of residual probability order. We first gave some of its properties and then provided a test of exponentiality against such aging notion. Some numerical results were given in order to indicate the usefulness and proficiency of the established testing procedure. Further properties and applications of the new stochastic order and the new proposed class can be considered in the future of this research. In particular, the closure properties of the PR order and the $\mathrm{NBRU}_{\mathrm{rp}}$ class under convolution and coherent structures are interesting and still remain as open problems.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to thank the anonymous reviewers for their valuable comments and suggestions, which were helpful in improving the paper. The authors also would like to extend their sincere appreciation to the Deanship of Scientific Research at King Saud University for funding this research group no. RG-1435-036.

References

[1] G. G. Brown and H. C. Rutemiller, "Evaluation of P[X>Y] when both X and Y are from three-parameter Weibull distributions," *IEEE Transactions on Reliability*, vol. 22, pp. 78–82, 1973.

[2] V. Zardasht and M. Asadi, "Evaluation of $P(X_t > Y_t)$ when both X_t and Y_t are residual lifetimes of two systems," *Statistica Neerlandica*, vol. 64, no. 4, pp. 460–481, 2010.

[3] S. Tan and J. Lü, "Characterizing the effect of population heterogeneity on evolutionary dynamics on complex networks," *Scientific Reports*, vol. 4, p. 5034, 2014.

[4] J. Lü and G. Chen, "A time-varying complex dynamical network model and its controlled synchronization criteria," *IEEE Transactions on Automatic Control*, vol. 50, no. 6, pp. 841–846, 2005.

[5] Y. Chen, J. Lü, X. Yu, and Z. Lin, "Consensus of discrete-time second-order multiagent systems based on infinite products of general stochastic matrices," *SIAM Journal on Control and Optimization*, vol. 51, no. 4, pp. 3274–3301, 2013.

[6] J. Zhou, J. Lü, and J. Lü, "Adaptive synchronization of an uncertain complex dynamical network," *IEEE Transactions on Automatic Control*, vol. 51, no. 4, pp. 652–656, 2006.

[7] R. E. Barlow and F. Proschan, *Statistical Theory of Reliability and Life Testing*, To Begin With, Silver Spring, Md, USA, 1981.

[8] C.-D. Lai and M. Xie, *Stochastic Ageing and Dependence for Reliability*, Springer, New York, NY, USA, 2006.

[9] M. C. Bhattacharjee, A. M. Abouammoh, A. N. Ahmed, and A. M. Barry, "Preservation results for life distributions based on comparisons with asymptotic remaining life under replacements," *Journal of Applied Probability*, vol. 37, no. 4, pp. 999–1009, 2000.

[10] J. Mi, "Some comparison results of system availability," *Naval Research Logistics*, vol. 45, no. 2, pp. 205–218, 1998.

[11] J.-L. Bon and A. Illayk, "A note on some new renewal ageing notions," *Statistics & Probability Letters*, vol. 57, no. 2, pp. 151–155, 2002.

[12] A. R. Mugdadi and I. A. Ahmad, "Moment inequalities derived from comparing life with its equilibrium form," *Journal of Statistical Planning and Inference*, vol. 134, no. 2, pp. 303–317, 2005.

[13] J.-L. Bon and A. Illayk, "Ageing properties and series systems," *Journal of Applied Probability*, vol. 42, no. 1, pp. 279–286, 2005.

[14] X. Li and M. Xu, "Reversed hazard rate order of equilibrium distributions and a related aging notion," *Statistical Papers*, vol. 49, no. 4, pp. 749–767, 2008.

[15] M. Kayid, I. A. Ahmad, S. Izadkhah, and A. M. Abouammoh, "Further results involving the mean time to failure order, and the decreasing mean time to failure class," *IEEE Transactions on Reliability*, vol. 62, no. 3, pp. 670–678, 2013.

[16] M. Shaked and J. G. Shanthikumar, *Stochastic Orders*, Springer, New York, NY, USA, 2007.

[17] A. Di Crescenzo, "Dual stochastic orderings describing ageing properties of devices of unknown age," *Communications in Statistics—Stochastic Models*, vol. 15, no. 3, pp. 561–576, 1999.

[18] A. A. Alzaid and L. Benkherouf, "Characterization of the *D*-ordering," *Pakistan Journal of Statistics*, vol. 10, no. 3, pp. 593–596, 1994.

[19] C. R. Rao, "Weighted distributions arising out of methods of ascertainment: what population does a sample represent?" in *A Celebration of Statistics: The ISI Centenary Volume*, A. C. Atkinson and S. E. Fienberg, Eds., pp. 543–569, Springer, New York, NY, USA, 1985.

[20] J. Bartoszewicz and M. Skolimowska, "Preservation of classes of life distributions and stochastic orders under weighting," *Statistics & Probability Letters*, vol. 76, no. 6, pp. 587–596, 2006.

[21] N. Misra, N. Gupta, and I. D. Dhariyal, "Preservation of some aging properties and stochastic orders by weighted distributions," *Communications in Statistics. Theory and Methods*, vol. 37, no. 3–5, pp. 627–644, 2008.

[22] S. Izadkhah and M. Kayid, "Reliability analysis of the harmonic mean inactivity time order," *IEEE Transactions on Reliability*, vol. 62, no. 2, pp. 329–337, 2013.

[23] F. Belzunce, T. Hu, and B.-E. Khaledi, "Dispersion-type variability orders," *Probability in the Engineering and Informational Sciences*, vol. 17, no. 3, pp. 305–334, 2003.

[24] K. T. Marshall, "Bounds for some generalizations of the GI/G/1 queue," *Operation Research Letters*, vol. 16, pp. 841–848, 1968.

[25] A. K. Nanda and A. Kundu, "On generalized stochastic orders of dispersion-type," *Calcutta Statistical Association Bulletin*, vol. 61, pp. 155–182, 2009.

[26] A. M. Abouammoh, S. A. Abdulghani, and I. S. Qamber, "On partial orderings and testing of new better than renewal used classes," *Reliability Engineering and System Safety*, vol. 43, no. 1, pp. 37–41, 1994.

[27] I. A. Ahmad, A. Ahmed, I. Elbatal, and M. Kayid, "An aging notion derived from the increasing convex ordering: the NBUCA class," *Journal of Statistical Planning and Inference*, vol. 136, no. 3, pp. 555–569, 2006.

[28] A. J. Lee, *U-Statistics*, Marcel Dekker, New York, NY, USA, 1989.

[29] S. C. Kochar, "Testing exponentiality against monotone failure rate average," *Communications in Statistics. A. Theory and Methods*, vol. 14, no. 2, pp. 381–392, 1985.