

Research Article

The Explicit Identities for Spectral Norms of Circulant-Type Matrices Involving Binomial Coefficients and Harmonic Numbers

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The explicit formulae of spectral norms for circulant-type matrices are investigated; the matrices are circulant matrix, skew-circulant matrix, and g -circulant matrix, respectively. The entries are products of binomial coefficients with harmonic numbers. Explicit identities for these spectral norms are obtained. Employing these approaches, some numerical tests are listed to verify the results.

1. Introduction

The classical hypergeometric summation theorems are exploited to derive several striking identities on harmonic numbers [1]. In numerical analysis, circulant matrices (named “premultipliers” in numerical methods) are important because they are diagonalized by a discrete Fourier transform, and hence linear equations that contain them may be quickly solved using a fast Fourier transform. Furthermore, circulant, skew-circulant, and g -circulant matrices play important roles in various applications, such as image processing, coding, and engineering model. For more details, please refer to [2–13] and the references therein. The skew-circulant matrices were collected to construct preconditioners for LMF-based ODE codes; Hermitian and skew-Hermitian Toeplitz systems were considered in [14–17]; Lyness employed a skew-circulant matrix to construct s -dimensional lattice rules in [18]. Recently, there are lots of research on the spectral distribution and norms of circulant-type matrices. In [19], the authors pointed out the processes based on the eigenvalue of circulant-type matrices and the convergence to a Poisson random measure in vague topology. There were discussions about the convergence in probability and distribution of the spectral norm of circulant-type matrices in [20]. The authors in [21] listed the limiting spectral distribution for a class of circulant-type matrices with heavy tailed input sequence. Ngondiep et al. showed that

the singular values of g -circulants in [22]. Solak established the lower and upper bounds for the spectral norms of circulant matrices with classical Fibonacci and Lucas numbers entries in [23]. İpek investigated an improved estimation for spectral norms in [24].

In this paper, we derive some explicit identities of spectral norms for some circulant-type matrices with product of binomial coefficients with harmonic numbers.

The outline of the paper is as follows. In Section 2, the definitions and preliminary results are listed. In Section 3, the spectral norms of some circulant matrices are studied. In Section 4, the formulae of spectral norms for skew-circulant matrices are established. Section 5 is devoted to investigate the explicit formulae for g -circulant matrices. The numerical tests are given in Section 6.

2. Preliminaries

The binomial coefficients are defined by $\binom{n}{k}$ for all natural numbers k at once by

$$(1 + X)^n = \sum_{k \geq 0} \binom{n}{k} X^k. \quad (1)$$

Note that $\binom{n}{k}$ is the k th binomial coefficient of n . It is clear that $\binom{n}{0} = 1$, $\binom{n}{n} = 1$, and $\binom{n}{k} = 0$, for $k > n$.

The generalized harmonic numbers are defined to be partial sums of the harmonic series [1]:

$$H_0(x) = 0, \quad H_i(x) = \sum_{k=0}^i \frac{1}{x+k} \quad (i = 1, 2, \dots). \quad (2)$$

For $x = 0$ in particular, they reduce to classical harmonic numbers:

$$H_0 = 0, \quad H_i = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{i} \quad (i = 1, 2, \dots). \quad (3)$$

We recall the following harmonic number identities [1]:

$$\begin{aligned} & \sum_{i=0}^n \binom{n}{i}^2 \binom{2n+i}{i} (H_{2n+i} - H_i) \\ &= 2 \binom{2n}{n}^2 (H_{2n} - H_n), \\ & \sum_{i=0}^n \binom{n}{i} \binom{2n}{i} \binom{3n+i}{i} (H_{3n+i} - H_i) \\ &= \binom{3n}{n}^2 (2H_{3n} - H_{2n} - H_n). \end{aligned} \quad (4)$$

Definition 1 (see [6, 8]). A circulant matrix is an $n \times n$ complex matrix with the following form:

$$A_c = \begin{pmatrix} a_0 & a_1 & \dots & a_{n-1} \\ a_{n-1} & a_0 & \dots & a_{n-2} \\ a_{n-2} & a_{n-1} & \dots & a_{n-3} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \dots & a_0 \end{pmatrix}_{n \times n}. \quad (5)$$

The first row of A_c is $(a_0, a_1, \dots, a_{n-1})$; its $(j+1)$ th row is obtained by giving its j th row a right circular shift by one position.

Equivalently, a circulant matrix can be described with polynomial as

$$A_c = f(\eta_c) = \sum_{i=0}^{n-1} a_i \eta_c^i, \quad (6)$$

where

$$\eta_c = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}_{n \times n}. \quad (7)$$

Obviously, $\eta_c^n = I_n$.

Now, we discuss the eigenvalues of A_c . We declare that the eigenvalues of η_c are the corresponding eigenvalues of A_c with the function f in (6), which is

$$\lambda(A_c) = f(\lambda(\eta_c)) = \sum_{i=0}^{n-1} a_i \lambda(\eta_c)^i. \quad (8)$$

Whereas $\lambda_j(\eta_c) = \omega^j$, ($j = 0, 1, \dots, n-1$), then $\lambda_j(A_c)$ can be calculated by

$$\lambda_j(A_c) = \sum_{i=0}^{n-1} a_i (\omega^j)^i, \quad (9)$$

where $\omega = \cos(2\pi/n) + i \sin(2\pi/n)$.

Similarly, we recall a skew-circulant matrix.

Definition 2 (see [6, 8]). A skew-circulant matrix is an $n \times n$ complex matrix with the following form:

$$A_{sc} = \begin{pmatrix} a_0 & a_1 & \dots & a_{n-1} \\ -a_{n-1} & a_0 & \dots & a_{n-2} \\ -a_{n-2} & -a_{n-1} & \dots & a_{n-3} \\ \vdots & \vdots & \ddots & \vdots \\ -a_1 & -a_2 & \dots & a_0 \end{pmatrix}_{n \times n}. \quad (10)$$

Moreover, a skew-circulant matrix can be described with polynomial as

$$A_{sc} = f(\eta_{sc}) = \sum_{i=0}^{n-1} a_i \eta_{sc}^i, \quad (11)$$

where

$$\eta_{sc} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -1 & 0 & 0 & \dots & 0 \end{pmatrix}_{n \times n}. \quad (12)$$

Obviously, $\eta_{sc}^n = -I_n$.

Thus we have to calculate the eigenvalues of A_{sc} . For the same reason, we obtain that

$$\lambda(A_{sc}) = f(\lambda(\eta_{sc})) = \sum_{i=0}^{n-1} a_i \lambda^i(\eta_{sc}). \quad (13)$$

Whereas $\lambda_j(\eta_{sc}) = \omega^j \alpha$, ($j = 0, 1, \dots, n-1$) $\lambda_j(A_{sc})$ can be computed by

$$\lambda_j(A_{sc}) = \sum_{i=0}^{n-1} a_i (\omega^j \alpha)^i, \quad (14)$$

where $\omega = \cos(2\pi/n) + i \sin(2\pi/n)$, $\alpha = \cos(\pi/n) + i \sin(\pi/n)$.

Definition 3 (see [21, 25]). A g -circulant matrix is an $n \times n$ complex matrix with the following form:

$$A_g = \begin{pmatrix} a_0 & a_1 & \dots & a_{n-1} \\ a_{n-g} & a_{n-g+1} & \dots & a_{n-g-1} \\ a_{n-2g} & a_{n-2g+1} & \dots & a_{n-2g-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_g & a_{g+1} & \dots & a_{g-1} \end{pmatrix}_{n \times n}, \quad (15)$$

where g is a nonnegative integer and each of the subscripts is understood to be reduced modulo n .

The first row of A_g is $(a_0, a_1, \dots, a_{n-1})$; its $(j+1)$ th row is obtained by giving its j th row a right circular shift by g positions (equivalently, $g \bmod n$ positions). Note that $g = 1$ or $g = n + 1$ yields the standard *circulant matrix*. If $g = n - 1$, then we obtain the so-called *reverse circulant matrix* [21].

Definition 4 (see [26]). The spectral norm $\|\cdot\|_2$ of a matrix A with complex entries is the square root of the largest eigenvalue of the positive semidefinite matrix A^*A :

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^*A)}, \quad (16)$$

where A^* denotes the conjugate transpose of A . Therefore if A is an $n \times n$ real symmetric matrix or A is a normal matrix, then

$$\|A\|_2 = \max_{1 \leq i \leq n} |\lambda_i|, \quad (17)$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of A .

3. Spectral Norms of Some Circulant Matrices

Now, we will analyse spectral norms of some given circulant matrices, whose entries are binomial coefficients combined with harmonic numbers.

Our main results for those matrices are stated as follows.

Theorem 5. Let $(n+1) \times (n+1)$ -circulant matrix B_1 be as in (5), and the first row of B_1 is

$$\begin{aligned} & \left(\binom{n}{0}^2 \binom{2n}{0} (H_{2n} - H_0), \right. \\ & \left. \binom{n}{1}^2 \binom{2n+1}{1} (H_{2n+1} - H_1), \dots, \right. \\ & \left. \binom{n}{n}^2 \binom{2n+n}{n} (H_{2n+n} - H_n) \right), \end{aligned} \quad (18)$$

where $a_i = \binom{n}{i}^2 \binom{2n+i}{i} (H_{2n+i} - H_i)$. Then one has

$$\|B_1\|_2 = 2 \binom{2n}{n}^2 (H_{2n} - H_n). \quad (19)$$

Proof. Since circulant matrix B_1 is normal, employing Definition 4, we claim that the spectral norm of B_1 is equal to its spectral radius. Furthermore, applying the irreducible and entrywise nonnegative properties, we claim that $\|B_1\|_2$ (i.e., its spectral norm) is equal to its Perron value. We select an $(n+1)$ -dimensional column vector $v = (1, 1, \dots, 1)^T$; then

$$B_1 v = \left(\sum_{i=0}^n \binom{n}{i}^2 \binom{2n+i}{i} (H_{2n+i} - H_i) \right) v. \quad (20)$$

Obviously, $\sum_{i=0}^n \binom{n}{i}^2 \binom{2n+i}{i} (H_{2n+i} - H_i)$ is an eigenvalue of B_1 associated with v , which is necessarily the Perron value of B_1 . Employing (4), we obtain

$$\|B_1\|_2 = 2 \binom{2n}{n}^2 (H_{2n} - H_n). \quad (21)$$

This completes the proof. \square

Hence, employing the same approaches, we get the following corollary.

Corollary 6. Let $(n+1) \times (n+1)$ -circulant matrix B_2 be as in (5), and the first row of B_2 is

$$\begin{aligned} & \left(\binom{n}{0} \binom{2n}{0} \binom{3n}{0} (H_{3n} - H_0), \right. \\ & \left. \binom{n}{1} \binom{2n}{1} \binom{3n+1}{1} (H_{3n+1} - H_1), \dots, \right. \\ & \left. \binom{n}{n} \binom{2n}{n} \binom{4n}{n} (H_{4n} - H_n) \right), \end{aligned} \quad (22)$$

where $a_i = \binom{n}{i} \binom{2n}{i} \binom{3n+i}{i} (H_{3n+i} - H_i)$. Then

$$\|B_2\|_2 = \binom{3n}{n}^2 (2H_{3n} - H_{2n} - H_n). \quad (23)$$

Now, we investigate some even-order alternative as follows, where m is odd (i.e., $m+1$ is even).

Theorem 7. Let $(m+1) \times (m+1)$ -circulant matrix B_3 be as in (5), and the first row of B_3 is

$$\begin{aligned} & \left(\binom{m}{0}^2 \binom{2m}{0} (H_{2m} - H_0), \right. \\ & \left. - \binom{m}{1}^2 \binom{2m+1}{1} (H_{2m+1} - H_1), \dots, \right. \\ & \left. - \binom{m}{m}^2 \binom{2m+m}{m} (H_{2m+m} - H_m) \right), \end{aligned} \quad (24)$$

where $a_i = (-1)^i \binom{m}{i}^2 \binom{2m+i}{i} (H_{2m+i} - H_i)$. Then there holds the following identity:

$$\|B_3\|_2 = 2 \binom{2m}{m}^2 (H_{2m} - H_m). \quad (25)$$

Proof. Noticing (9) and (17), it is clear that the spectral norm of B_3 can be calculated by

$$\begin{aligned} \|B_3\|_2 &= \max_{0 \leq t \leq m} |\lambda_t(B_3)| = \max_{0 \leq t \leq m} \left| \sum_{i=0}^m a_i (\omega^t)^i \right| \\ &\leq \max_{0 \leq t \leq m} \left\{ \sum_{i=0}^m |a_i| \cdot |(\omega^t)^i| \right\} = \sum_{i=0}^m |a_i|, \end{aligned} \quad (26)$$

where $a_i = (-1)^i \binom{m}{i}^2 \binom{2m+i}{i} (H_{2m+i} - H_i)$, and we employed that all circulant matrices are normal.

Note that, if m is odd, then $m+1$ is even, and $\lambda_{t_0}(\eta_c) = \omega^{t_0} = -1$ is an eigenvalue of η_c , so

$$\|B_3\|_2 = \sum_{i=0}^m |a_i|. \quad (27)$$

Combining (4) and (27) yields

$$\|B_3\|_2 = 2 \binom{2m}{m}^2 (H_{2m} - H_m). \quad (28)$$

This completes the proof. \square

Employing the same approaches, we get the following corollary.

Corollary 8. Let $(m+1) \times (m+1)$ -circulant matrix B_4 be as in (5), and the first row of B_4 is

$$\begin{aligned} & \left(\binom{m}{0} \binom{2m}{0} \binom{3m}{0} (H_{3m} - H_0), \right. \\ & \left. - \binom{m}{1} \binom{2m}{1} \binom{3m+1}{1} (H_{3m+1} - H_1), \dots, \right. \\ & \left. \binom{m}{m} \binom{2m}{m} \binom{4m}{m} (H_{4m} - H_m) \right), \end{aligned} \quad (29)$$

where $a_i = (-1)^i \binom{m}{i} \binom{2m}{i} \binom{3m+i}{i} (H_{3m+i} - H_i)$. Then one has the following identity:

$$\|B_4\|_2 = \left(\binom{3m}{m} \right)^2 (2H_{3m} - H_{2m} - H_m). \quad (30)$$

Similarly, we set $\tilde{B}_3 = -B_3$, $\tilde{B}_4 = -B_4$.

Corollary 9. Let \tilde{B}_3 , \tilde{B}_4 be as above, respectively, and m is odd. Then

$$\begin{aligned} \|\tilde{B}_3\|_2 &= 2 \left(\binom{2m}{m} \right)^2 (H_{2m} - H_m), \\ \|\tilde{B}_4\|_2 &= \left(\binom{3m}{m} \right)^2 (2H_{3m} - H_{2m} - H_m). \end{aligned} \quad (31)$$

4. Spectral Norms of Skew-Circulant Matrices

An odd-order alternative skew-circulant matrix is defined as follows, where s is even.

Theorem 10. Let $(s+1) \times (s+1)$ -circulant matrix B_5 be as in (10), and the first row of B_5 is

$$\begin{aligned} & \left(\binom{s}{0} \binom{2s}{0} (H_{2s} - H_0), \right. \\ & \left. - \binom{s}{1} \binom{2s+1}{1} (H_{2s+1} - H_1), \dots, \right. \\ & \left. \binom{s}{s} \binom{2s+s}{s} (H_{2s+s} - H_s) \right), \end{aligned} \quad (32)$$

where $a_i = (-1)^i \binom{s}{i} \binom{2s+i}{i} (H_{2s+i} - H_i)$. Then one obtains

$$\|B_5\|_2 = 2 \left(\binom{2s}{s} \right)^2 (H_{2s} - H_s). \quad (33)$$

Proof. We employ (14) and (17) to calculate the spectral norm of B_5 as follows, for all $t = 0, 1, \dots, s$:

$$\begin{aligned} |\lambda_t(B_5)| &= \left| \sum_{i=0}^s a_i (\omega^t \alpha)^i \right| \leq \sum_{i=0}^s |a_i| \cdot |(\omega^t \alpha)^i| \\ &= \sum_{i=0}^s |a_i| = \sum_{i=0}^s \binom{s}{i}^2 \binom{2s+i}{i} (H_{2s+i} - H_i), \end{aligned} \quad (34)$$

where $a_i = (-1)^i \binom{s}{i} \binom{2s+i}{i} (H_{2s+i} - H_i)$.

Since the skew-circulant matrix is normal, we deduce that

$$\|B_5\|_2 = \max_{0 \leq t \leq s} |\lambda_t(B_5)|. \quad (35)$$

If s is even, then $s+1$ is odd. We declare that $\lambda_{sc} = -1$ is an eigenvalue of η_{sc} ; then we calculate the corresponding eigenvalue of B_5 as follows:

$$\begin{aligned} \lambda_{\tilde{t}}(B_5) &= \sum_{i=0}^s a_i \lambda_{sc}^i = \sum_{i=0}^s a_i (-1)^i \\ &= \sum_{i=0}^s \binom{s}{i}^2 \binom{2s+i}{i} (H_{2s+i} - H_i), \end{aligned} \quad (36)$$

where we had employed (14).

Noticing (34), we claim that $\lambda_{\tilde{t}}(B_5)$ is the maximum of $|\lambda_t(B_5)|$, which means

$$\|B_5\|_2 = \sum_{i=0}^s \binom{s}{i}^2 \binom{2s+i}{i} (H_{2s+i} - H_i). \quad (37)$$

Thus, from (4) we obtain

$$\|B_5\|_2 = 2 \left(\binom{2s}{s} \right)^2 (H_{2s} - H_s). \quad (38)$$

This completes the proof. \square

Similarly, we can calculate the identity for B_6 .

Corollary 11. Let $(s+1) \times (s+1)$ -circulant matrix B_6 be as in (10), and the first row of B_6 is

$$\begin{aligned} & \left(\binom{s}{0} \binom{2s}{0} \binom{3s}{0} (H_{3s} - H_0), \right. \\ & \left. - \binom{s}{1} \binom{2s}{1} \binom{3s+1}{1} (H_{3s+1} - H_1), \dots, \right. \\ & \left. \binom{s}{s} \binom{2s}{s} \binom{3s+s}{s} (H_{3s+s} - H_s) \right), \end{aligned} \quad (39)$$

where $a_i = (-1)^i \binom{s}{i} \binom{2s}{i} \binom{3s+i}{i} (H_{3s+i} - H_i)$. Then there holds

$$\|B_6\|_2 = \left(\binom{3s}{s} \right)^2 (2H_{3s} - H_{2s} - H_s). \quad (40)$$

Corollary 12. Let $\tilde{B}_5 = -B_5$ and $\tilde{B}_6 = -B_6$, and s is even. Then one has the identities for spectral norm

$$\begin{aligned}\|\tilde{B}_5\|_2 &= 2 \binom{2s}{s}^2 (2H_{2s} - H_s), \\ \|\tilde{B}_6\|_2 &= \binom{3s}{s}^2 (2H_{3s} - H_{2s} - H_s).\end{aligned}\quad (41)$$

5. Spectral Norms of g -Circulant Matrices

Inspired by the above propositions, we analyse spectral norms of some given g -circulant matrices in this section.

Lemma 13 (see [25]). The $(n+1) \times (n+1)$ matrix Q_g is unitary if and only if

$$(n+1, g) = 1, \quad (42)$$

where Q_g is a g -circulant matrix with the first row $e^* = [1, 0, \dots, 0]$.

Lemma 14 (see [25]). A is a g -circulant matrix with the first row $[a_0, a_1, \dots, a_n]$ if and only if

$$A = Q_g C, \quad (43)$$

where

$$C = \text{circ}(a_0, a_1, \dots, a_n). \quad (44)$$

In the following part, we set $(n+1, g) = 1$.

Theorem 15. Let $(n+1) \times (n+1)$ -circulant matrix B_7 be as in (15), and the first row of B_7 is

$$\begin{aligned}&\left(\binom{n}{0}^2 \binom{2n}{0} (H_{2n} - H_0), \right. \\ &\left. \binom{n}{1}^2 \binom{2n+1}{1} (H_{2n+1} - H_1), \dots, \right. \\ &\left. \binom{n}{n}^2 \binom{2n+n}{n} (H_{2n+n} - H_n) \right),\end{aligned}\quad (45)$$

where $a_i = \binom{n}{i}^2 \binom{2n+i}{i} (H_{2n+i} - H_i)$. Then

$$\|B_7\|_2 = 2 \binom{2n}{n}^2 (H_{2n} - H_n). \quad (46)$$

Proof. With the help of Lemmas 13 and 14, we know that the g -circulant matrix B_7 is normal; then we claim that the spectral norm of B_7 is equal to its spectral radius. Furthermore, applying the irreducible and entrywise nonnegative properties, we claim that $\|B_7\|_2$ (i.e., its spectral norm) is equal to its Perron value. We select a $(n+1)$ -dimensional column vector $v = (1, 1, \dots, 1)^T$; then

$$B_7 v = \left(\sum_{i=0}^n \binom{n}{i}^2 \binom{2n+i}{i} (H_{2n+i} - H_i) \right) v. \quad (47)$$

Obviously, $\sum_{i=0}^n \binom{n}{i}^2 \binom{2n+i}{i} (H_{2n+i} - H_i)$ is an eigenvalue of B_7 associated with v , which is necessarily the Perron value of B_7 . Employing (4), we obtain

$$\|B_7\|_2 = 2 \binom{2n}{n}^2 (H_{2n} - H_n). \quad (48)$$

This completes the proof. \square

Corollary 16. Let $(n+1) \times (n+1)$ -circulant matrix B_8 be as in (15), and the first row of B_8 is

$$\begin{aligned}&\left(\binom{n}{0} \binom{2n}{0} \binom{3n}{0} (H_{3n} - H_0), \right. \\ &\left. \binom{n}{1} \binom{2n}{1} \binom{3n+1}{1} (H_{3n+1} - H_1), \dots, \right. \\ &\left. \binom{n}{n} \binom{2n}{n} \binom{4n}{n} (H_{4n} - H_n) \right),\end{aligned}\quad (49)$$

where $a_i = \binom{n}{i} \binom{2n}{i} \binom{3n+i}{i} (H_{3n+i} - H_i)$. Then one obtains

$$\|B_8\|_2 = \binom{3n}{n}^2 (2H_{3n} - H_{2n} - H_n). \quad (50)$$

6. Numerical Examples

Example 1. In this example, we give the numerical results for B_1 and B_2 .

Comparing the data in Table 1, we declare that the identities of spectral norms for B_i ($i = 1, 2$) hold.

Example 2. In this example, we list the numerical results for B_i, \tilde{B}_i ($i = 3, 4$).

With the help of data in Table 2, it is clear that the identities of spectral norms for B_i, \tilde{B}_i ($i = 3, 4$) hold.

Example 3. In this example, we reveal the numerical results for alternative skew-circulant matrices B_i, \tilde{B}_i ($i = 5, 6$).

Combining the data in Table 3, we deduce that the identities of spectral norms for B_i, \tilde{B}_i ($i = 5, 6$) hold.

Example 4. In this example, we show numerical results for B_7 and B_8 .

Considering the data in Table 4, we deduce that the identities of spectral norms for B_i ($i = 7, 8$) hold.

The above results demonstrate that the identities of spectral norms for the given matrices hold.

7. Conclusion

This paper had discussed the explicit formulae for identical estimations of spectral norms for circulant, skew-circulant and g -circulant matrices, whose entries are binomial coefficients combined with harmonic numbers. Furthermore, it is easy to take other entries to obtain more interesting identities, and the same approaches can be used to verify those identities. Furthermore, explicit formulas for both

TABLE 1: Spectral norms of B_i ($i = 1, 2$), $C_{1,n} = 2\binom{2n}{n}^2(H_{2n} - H_n)$, and $C_{2,n} = \binom{3n}{n}^2(2H_{3n} - H_{2n} - H_n)$.

n	0	1	2	3	4	5	6
$\ B_1\ _2$	0	4	42	$4.93e + 2$	$6.22e + 3$	$8.20e + 4$	$1.12e + 6$
$\ B_2\ _2$	0	$1.05e + 1$	$2.96e + 2$	$9.69e + 3$	$3.44e + 5$	$1.28e + 7$	$4.95e + 8$
$C_{1,n}$	0	4	42	$4.93e + 2$	$6.22e + 3$	$8.20e + 4$	$1.12e + 6$
$C_{2,n}$	0	$1.05e + 1$	$2.96e + 2$	$9.69e + 3$	$3.44e + 5$	$1.28e + 7$	$4.95e + 8$

TABLE 2: Spectral norms of B_i , \tilde{B}_i ($i = 3, 4$), $C_{1,m} = 2\binom{2m}{m}^2(H_{2m} - H_m)$, and $C_{2,m} = \binom{3m}{m}^2(2H_{3m} - H_{2m} - H_m)$.

m	1	3	5	7	9	11
$\ B_3\ _2$	4	$4.93e + 2$	$8.20e + 4$	$1.55e + 7$	$3.15e + 9$	$6.68e + 11$
$\ B_4\ _2$	10.5	$9.69e + 3$	$1.28e + 7$	$1.96e + 10$	$3.20e + 13$	$5.49e + 16$
$\ \tilde{B}_3\ _2$	4	$4.93e + 2$	$8.20e + 4$	$1.55e + 7$	$3.15e + 9$	$6.68e + 11$
$\ \tilde{B}_4\ _2$	10.5	$9.69e + 3$	$1.28e + 7$	$1.96e + 10$	$3.20e + 13$	$5.49e + 16$
$C_{1,m}$	4	$4.93e + 2$	$8.20e + 4$	$1.55e + 7$	$3.15e + 9$	$6.68e + 11$
$C_{2,m}$	10.5	$9.69e + 3$	$1.28e + 7$	$1.96e + 10$	$3.20e + 13$	$5.49e + 16$

TABLE 3: Spectral norms of B_i , \tilde{B}_i ($i = 5, 6$), $C_{1,s} = 2\binom{2s}{s}^2(H_{2s} - H_s)$, and $C_{2,s} = \binom{3s}{s}^2(2H_{3s} - H_{2s} - H_s)$.

s	0	2	4	6	8	10
$\ B_5\ _2$	0	42	$6.22e + 3$	$1.12e + 6$	$2.20e + 8$	$4.57e + 10$
$\ B_6\ _2$	0	$2.96e + 2$	$3.44e + 5$	$4.95e + 5$	$7.86e + 11$	$1.32e + 15$
$\ \tilde{B}_5\ _2$	0	42	$6.22e + 3$	$1.12e + 6$	$2.20e + 8$	$4.57e + 10$
$\ \tilde{B}_6\ _2$	0	$2.96e + 2$	$3.44e + 5$	$4.95e + 5$	$7.86e + 11$	$1.32e + 15$
$C_{1,s}$	0	42	$6.22e + 3$	$1.12e + 6$	$2.20e + 8$	$4.57e + 10$
$C_{2,s}$	0	$2.96e + 2$	$3.44e + 5$	$4.95e + 5$	$7.86e + 11$	$1.32e + 15$

TABLE 4: Spectral norms of B_i ($i = 7, 8$), $C_{1,n} = 2\binom{2n}{n}^2(H_{2n} - H_n)$, and $C_{2,n} = \binom{3n}{n}^2(2H_{3n} - H_{2n} - H_n)$.

$n + 1$	5				6				7			
g	2	3	4	5	2	3	4	5	2	3	4	5
$\ B_7\ _2$	$6.22e + 3$	$6.22e + 3$	$6.22e + 3$	$8.20e + 4$	$1.12e + 6$	$1.12e + 6$	$1.12e + 6$	$1.12e + 6$	$1.12e + 6$	$1.12e + 6$	$1.12e + 6$	$1.12e + 6$
$\ B_8\ _2$	$3.44e + 5$	$3.44e + 5$	$3.44e + 5$	$1.28e + 7$	$4.95e + 8$	$4.95e + 8$	$4.95e + 8$	$4.95e + 8$	$4.95e + 8$	$4.95e + 8$	$4.95e + 8$	$4.95e + 8$
$C_{1,n}$	$6.22e + 3$	$6.22e + 3$	$6.22e + 3$	$8.20e + 4$	$1.12e + 6$	$1.12e + 6$	$1.12e + 6$	$1.12e + 6$	$1.12e + 6$	$1.12e + 6$	$1.12e + 6$	$1.12e + 6$
$C_{2,n}$	$3.44e + 5$	$3.44e + 5$	$3.44e + 5$	$1.28e + 7$	$4.95e + 8$	$4.95e + 8$	$4.95e + 8$	$4.95e + 8$	$4.95e + 8$	$4.95e + 8$	$4.95e + 8$	$4.95e + 8$

norms $\|A\|$ and $\|A^{-1}\|$ help us to estimate the so-called condition number. It is an interesting problem to investigate the properties of B_i ($i = 1, 2, \dots, 8$), such as the explicit formulations for determinants and inverses, by just using the entries in the first row.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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