Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2014, Article ID 539215, 13 pages
http://dx.doi.org/10.1155/2014/539215

Research Article

(Anti-)Hermitian Generalized (Anti-)Hamiltonian Solution to a System of Matrix Equations

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Received 12 August 2013; Revised 14 November 2013; Accepted 14 November 2013; Published 25 March 2014

Academic Editor: Masoud Hajarian

We mainly solve three problems. Firstly, by the decomposition of the (anti-)Hermitian generalized (anti-)Hamiltonian matrices, the necessary and sufficient conditions for the existence of and the expression for the (anti-)Hermitian generalized (anti-)Hamiltonian solutions to the system of matrix equations $AX = B$, $XC = D$ are derived, respectively. Secondly, the optimal approximation solution $\min_{X\in K}\|\widehat{X} - X\|$ is obtained, where K is the (anti-)Hermitian generalized (anti-)Hamiltonian solution set of the above system and $\widehat{X}$ is the given matrix. Thirdly, the least squares (anti-)Hermitian generalized (anti-)Hamiltonian solutions are considered. In addition, algorithms about computing the least squares (anti-)Hermitian generalized (anti-)Hamiltonian solution and the corresponding numerical examples are presented.

1. Introduction

Throughout this paper, the set of all $m \times n$ complex matrices, the set of all $n \times n$ Hermitian matrices, the set of all $n \times n$ anti-Hermitian matrices, the set of all $n \times n$ unitary matrices, and the set of all $n \times n$ antisymmetric orthogonal matrices are denoted, respectively, by $\mathbb{C}^{m\times n}$, $H\mathbb{C}^{n\times n}$, $AH\mathbb{C}^{n\times n}$, $U\mathbb{C}^{n\times n}$, and $ASO\mathbb{R}^{n\times n}$. The symbol I_n represents an identity matrix of order n and $r(A)$, $A^{\dagger}$, and A^*, respectively, stand for the rank, the Moore-Penrose inverse, and the conjugate transpose of matrix A. For two matrices $A, B \in \mathbb{C}^{m\times n}$, the inner product is defined by $\langle A, B\rangle = \operatorname{tr}(B^* A)$. Obviously, $\mathbb{C}^{m\times n}$ is a complete inner product space. The norm $\|\cdot\|$, induced by the inner product, is called the Frobenius norm. $A * B$ stands for the Hadamard product of two matrices A and B. For $A \in \mathbb{C}^{m\times n}$, two matrices L_A and R_A, respectively, represent two orthogonal projectors $L_A = I_n - A^{\dagger}A$ and $R_A = I_m - AA^{\dagger}$, both of which satisfy

$$L_A = (L_A)^2 = (L_A)^* = (L_A)^{\dagger},$$
$$R_A = (R_A)^2 = (R_A)^* = (R_A)^{\dagger}. \tag{1}$$

The Hamiltonian matrices defined as in [1] are very important in engineering (see [2] and the references therein). Moreover, using Hamiltonian matrices to solve algebraic matrix Riccati equation is a very effective method in optimal control theory [3–5]. As the extension of the Hamiltonian matrices, the following four definitions, which can also be found in [1, 6, 7], are given. Without special statement, we in this paper always assume that $J \in ASO\mathbb{R}^{2k\times 2k}$ satisfies

$$J^T = -J, \qquad J^T J = JJ^T = I_n. \tag{2}$$

Definition 1. A matrix $X \in HH\mathbb{C}^{2k\times 2k}$ is said to be a Hermitian generalized Hamiltonian matrix if $X = X^*$ and $JXJ = X^*$.

Definition 2. A matrix $X \in HAH\mathbb{C}^{2k\times 2k}$ is said to be a Hermitian generalized anti-Hamiltonian matrix if $X = X^*$ and $JXJ = -X^*$.

Definition 3. A matrix $X \in AHAH\mathbb{C}^{2k\times 2k}$ is said to be an anti-Hermitian generalized anti-Hamiltonian matrix if $X = -X^*$ and $JXJ = -X^*$.

Definition 4. A matrix $X \in AHH\mathbb{C}^{2k\times 2k}$ is said to be an anti-Hermitian generalized Hamiltonian matrix if $X = -X^*$ and $JXJ = X^*$.

The well-known system of matrix equations

$$AX = B, \qquad XC = D, \tag{3}$$

with unknown matrix X, has attracted much attention and has been widely and deeply studied by many authors. For example, Khatri and Mitra [8] in 1976 established the Hermitian and nonnegative definite solution to the system (3). Mitra [9] in 1984 gave the system (3) the minimal rank solution over the complex field $\mathbb{C}$. Wang in [10] and Wang et al. [11], respectively, investigated the bisymmetric and centrosymmetric solutions over the quaternion algebra and obtained the bisymmetric nonnegative definite solutions with extremal ranks and inertias to the system (3). Xu in [12] studied the common Hermitian and positive solutions to the adjointable operator equations (3). Yuan in [13] presented the least squares solutions to the system (3). Some other results concerning the system (3) can be found in [14–23].

As special cases of the system (3), the classical matrix equations $AX = B$ and $XC = D$ have also been investigated (see, e.g., [1, 2, 5–7, 24–31]). For instance, Dai [24], by means of the singular value decomposition, derived the symmetric solution to equation $AX = B$. Guan and Jiang [6], using the decomposition of the anti-Hermitian generalized anti-Hamiltonian matrices, derived the least squares solution to equation $AX = B$. Zhang et al. in [29] and [1], respectively, obtained the general expression of the least squares Hermitian generalized Hamiltonian solutions to equation $XC = D$ and got the unite optimal approximation solution in the least squares solutions set and gave the solvable conditions and the general representation of the Hermitian generalized Hamiltonian solutions to equation $AX = B$, by using the singular value decomposition and the properties of Hermitian generalized Hamiltonian matrices.

As far as we know, there has been little information on studying the (anti-)Hermitian generalized (anti-)Hamiltonian solution to the system (3) over $\mathbb{C}^{2k\times 2k}$. So, motived by the work mentioned above, especially the work in [6, 7, 26, 29, 30], we, in this paper, are mainly concerned with the following three problems.

Problem 5. Given $A, B \in \mathbb{C}^{m\times 2k}$, $C, D \in \mathbb{C}^{2k\times q}$, find $X \in HH\mathbb{C}^{2k\times 2k}$ ($HAH\mathbb{C}^{2k\times 2k}$, $AHH\mathbb{C}^{2k\times 2k}$, or $AHAH\mathbb{C}^{2k\times 2k}$) such that the system (3) holds.

Problem 6. Given $\widehat{X} \in \mathbb{C}^{2k\times 2k}$, find $\widetilde{X} \in K$ such that

$$\left\|\widehat{X} - \widetilde{X}\right\| = \min_{X\in K}\left\|\widehat{X} - X\right\|, \tag{4}$$

where K is the solution set of Problem 5.

Problem 7. Let $A, B \in \mathbb{C}^{m\times 2k}$, $C, D \in \mathbb{C}^{2k\times q}$. Find $X \in HH\mathbb{C}^{2k\times 2k}$ ($HAH\mathbb{C}^{2k\times 2k}$, $AHH\mathbb{C}^{2k\times 2k}$, or $AHAH\mathbb{C}^{2k\times 2k}$) such that

$$\min_X = \|AX - B\|^2 + \|XC - D\|^2. \tag{5}$$

The remainder of this paper is arranged as follows. In Section 2, some lemmas will be introduced, which will be useful for us to obtain the solutions to Problems 5–7. In Section 3, by applying the decomposition of the (anti-)Hermitian generalized (anti-)Hamiltonian matrices, the solvability condition and the explicit expression of the solution to Problem 5 will be derived. In Section 4, the optimal approximation solution to Problem 6 will be established. In Section 5, the solution to Problem 7 will be investigated and meanwhile the minimum norm of the solution will be obtained. In Section 6, algorithms and numerical examples about computing the solution to Problem 7 will be provided. Finally, in Section 7, some conclusions will be made.

2. Preliminaries

In this section, we focus on introducing some lemmas, which will play key roles in solving Problems 5–7.

Taking into account Definitions 1–4 and the eigenvalue decomposition of the matrix $J \in ASO\mathbb{R}^{2k\times 2k}$, it is not difficult to conclude that the following decompositions of the (anti-)Hermitian generalized (anti-)Hamiltonian matrices hold, some of which can also be seen in [6, 26, 29, 30].

Lemma 8. *Let the eigenvalue decomposition of matrix $J \in ASO\mathbb{R}^{2k\times 2k}$ be*

$$J = P\begin{pmatrix} iI_k & 0 \\ 0 & -iI_k \end{pmatrix}P^*, \tag{6}$$

where $P \in U\mathbb{C}^{2k\times 2k}$. Then $X \in HH\mathbb{C}^{2k\times 2k}$ if and only if X can be expressed as

$$X = P\begin{pmatrix} 0 & X_{12} \\ X_{12}^* & 0 \end{pmatrix}P^*, \tag{7}$$

where $X_{12} \in \mathbb{C}^{k\times k}$ are arbitrary.

Lemma 9. *Let the eigenvalue decomposition of matrix $J \in ASO\mathbb{R}^{2k\times 2k}$ be (6). Then $X \in AHAH\mathbb{C}^{2k\times 2k}$ if and only if X can be expressed as*

$$X = P\begin{pmatrix} 0 & X_{12} \\ -X_{12}^* & 0 \end{pmatrix}P^*, \tag{8}$$

where $X_{12} \in \mathbb{C}^{k\times k}$ is arbitrary.

Lemma 10. *Let the eigenvalue decomposition of matrix $J \in ASO\mathbb{R}^{2k\times 2k}$ be (6). Then $X \in HAH\mathbb{C}^{2k\times 2k}$ if and only if X can be expressed as*

$$X = P\begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix}P^*, \tag{9}$$

where $X_{11}, X_{22} \in H\mathbb{C}^{k\times k}$ are arbitrary.

Lemma 11. *Let the eigenvalue decomposition of matrix $J \in ASO\mathbb{R}^{2k\times 2k}$ be (6). Then $X \in AHH\mathbb{C}^{2k\times 2k}$ if and only if X can be expressed as*

$$X = P\begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix}P^*, \tag{10}$$

where $X_{11}, X_{22} \in AH\mathbb{C}^{k\times k}$ are arbitrary.

Lemma 12 (see [20]). *Given $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times l}$, $C \in \mathbb{C}^{m\times p}$, and $D \in \mathbb{C}^{n\times l}$, then the system of matrix equations*

$$AX = C, \qquad XB = D \tag{11}$$

has a solution $X \in \mathbb{C}^{n\times p}$ if and only if

$$AA^{\dagger}C = C, \qquad DB^{\dagger}B = D, \qquad AD = CB, \tag{12}$$

in which case the general solutions can be expressed as

$$X = A^{\dagger}C + DB^{\dagger} - A^{\dagger}ADB^{\dagger} + \left(I - A^{\dagger}A\right)W\left(I - BB^{\dagger}\right), \tag{13}$$

where $W \in \mathbb{C}^{n\times p}$ is arbitrary.

By applying the singular value decomposition, similar to the proof of Theorem 1 in [24], the following lemma can be shown.

Lemma 13. *Assume $E, F \in \mathbb{C}^{m\times n}$. Let the singular value decomposition of E be*

$$E = U\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}V^{*}, \tag{14}$$

where

$$\begin{gathered} U \in U\mathbb{C}^{m\times m}, \qquad V \in U\mathbb{C}^{n\times n}, \\ \Sigma = \operatorname{diag}\left(\alpha_1, \ldots, \alpha_r\right), \qquad \alpha_i > 0, \\ i = 1, \ldots, r;\ r = r(E). \end{gathered} \tag{15}$$

Partition

$$\begin{gathered} VXV^{*} = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^{*} & X_{22} \end{pmatrix}, \\ U^{*}FV = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}, \end{gathered} \tag{16}$$

where

$$\begin{gathered} X_{11} \in H\mathbb{C}^{r\times r}, \qquad X_{22} \in H\mathbb{C}^{(n-r)\times(n-r)}, \\ F_{11} \in \mathbb{C}^{r\times r}, \qquad F_{22} \in \mathbb{C}^{(m-r)\times(n-r)}. \end{gathered} \tag{17}$$

Then the matrix equation

$$EX = F \tag{18}$$

has Hermitian solutions if and only if

$$\begin{gathered} EE^{\dagger}F = F, \qquad EF^{*} = FE^{*}, \\ F_{21} = 0, \qquad F_{22} = 0, \end{gathered} \tag{19}$$

in which case the Hermitian solution can be expressed as

$$X = V\begin{pmatrix} \Sigma^{-1}F_{11} & \Sigma^{-1}F_{12} \\ F_{12}^{*}\Sigma^{-1} & X_{22} \end{pmatrix}V^{*}, \tag{20}$$

where $X_{22} \in H\mathbb{C}^{(n-r)\times(n-r)}$ is arbitrary.

By the similar way, the following lemma can also be verified.

Lemma 14. *Assume $M, N \in \mathbb{C}^{m\times n}$. Let the singular value decomposition of M be*

$$M = U\begin{pmatrix} \Pi & 0 \\ 0 & 0 \end{pmatrix}V^{*}, \tag{21}$$

where

$$\begin{gathered} U \in U\mathbb{C}^{m\times m}, \qquad V \in U\mathbb{C}^{n\times n}, \\ \Pi = \operatorname{diag}\left(\beta_1, \ldots, \beta_s\right), \qquad \beta_i > 0, \\ i = 1, \ldots, s;\ s = r(M). \end{gathered} \tag{22}$$

Partition

$$\begin{gathered} VXV^{*} = \begin{pmatrix} X_{11} & X_{12} \\ -X_{12}^{*} & X_{22} \end{pmatrix}, \\ U^{*}NV = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}, \end{gathered} \tag{23}$$

where

$$\begin{gathered} X_{11} \in AH\mathbb{C}^{s\times s}, \qquad X_{22} \in AH\mathbb{C}^{(n-s)\times(n-s)}, \\ N_{11} \in \mathbb{C}^{s\times s}, \qquad N_{22} \in \mathbb{C}^{(m-s)\times(n-s)}. \end{gathered} \tag{24}$$

Then the matrix equation

$$MX = N \tag{25}$$

has an anti-Hermitian solution if and only if

$$\begin{gathered} MM^{\dagger}N = N, \qquad MN^{*} = -NM^{*}, \\ N_{21} = 0, \qquad N_{22} = 0, \end{gathered} \tag{26}$$

in which case the anti-Hermitian solution can be expressed as

$$X = V\begin{pmatrix} \Pi^{-1}N_{11} & \Pi^{-1}N_{12} \\ -N_{12}^{*}\Pi^{-1} & X_{22} \end{pmatrix}V^{*}, \tag{27}$$

where $X_{22} \in AH\mathbb{C}^{(n-s)\times(n-s)}$ is arbitrary.

Lemma 15 (see [31]). *Given $A', B' \in \mathbb{C}^{k\times(m+q)}$, $C', D' \in \mathbb{C}^{k\times(m+q)}$, suppose that the matrices A' and C', respectively, have the following singular value decompositions:*

$$A' = P_1\begin{pmatrix} \Gamma & 0 \\ 0 & 0 \end{pmatrix}Q_1^{*}, \qquad C' = U_1\begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix}V_1^{*}, \tag{28}$$

where

$$\begin{aligned} &P_1 = \begin{pmatrix} P_{11} & P_{12} \end{pmatrix} \in U\mathbb{C}^{k\times k}, \quad P_{11} \in \mathbb{C}^{k\times t_1}; \\ &Q_1 = \begin{pmatrix} Q_{11} & Q_{12} \end{pmatrix} \in U\mathbb{C}^{(m+q)\times(m+q)}, \\ &\qquad Q_{11} \in \mathbb{C}^{(m+q)\times t_1}; \\ &U_1 = \begin{pmatrix} U_{11} & U_{12} \end{pmatrix} \in U\mathbb{C}^{k\times k}, \quad U_{11} \in \mathbb{C}^{k\times t_2}; \\ &V_1 = \begin{pmatrix} V_{11} & V_{12} \end{pmatrix} \in U\mathbb{C}^{(m+q)\times(m+q)}, \\ &\qquad V_{11} \in \mathbb{C}^{(m+q)\times t_2}; \\ &\Gamma = \operatorname{diag}\left(\delta_1, \delta_2, \ldots, \delta_{t_1}\right), \qquad \delta_i > 0, \\ &\qquad 1 \le i \le t_1;\ t_1 = r\left(A'\right); \\ &\Lambda = \operatorname{diag}\left(\gamma_1, \gamma_2, \ldots, \gamma_{t_2}\right), \qquad \gamma_i > 0, \\ &\qquad 1 \le i \le t_2;\ t_2 = r\left(C'\right). \end{aligned} \tag{29}$$

Then the solution set of the problem

$$\begin{aligned} f\left(X_{12}\right) \triangleq {}& \left\|\left(A'\right)^* X_{12} - \left(B'\right)^*\right\|^2 \\ &+ \left\|X_{12}C' - D'\right\|^2 = \min \end{aligned} \tag{30}$$

consists of matrices $X_{12} \in \mathbb{C}^{k\times k}$ with the following form:

$$\begin{aligned} X_{12} &= P_1 \begin{pmatrix} \phi * \left(P_{11}^* D' V_{11}\Lambda + \Gamma Q_{11}^* \left(B'\right)^* U_{12}\right) & \Gamma^{-1} Q_{11}^* \left(B'\right)^* U_{12} \\ P_{12}^* D' V_{11} \Lambda^{-1} & X_{22}' \end{pmatrix} U_1^*, \end{aligned} \tag{31}$$

where

$$\begin{aligned} &\phi = \left(\phi_{ij}\right), \quad \phi_{ij} = \frac{1}{\delta_i^2 + \gamma_j^2}, \\ &\qquad 1 \le i \le t_1,\ 1 \le j \le t_2, \end{aligned} \tag{32}$$

and $X_{22}' \in \mathbb{C}^{(k-t_1)\times(k-t_2)}$ is arbitrary.

Lemma 16. *Given $E, F \in \mathbb{C}^{m\times n}$, let the singular value decomposition of E, the partitions of VXV^* and U^*FV be, respectively, as in* (14)–(16). *Then the least squares Hermitian solution to the matrix equation* (18) *can be expressed as*

$$X = V \begin{pmatrix} \Phi * \left(\Sigma F_{11} + F_{11}^* \Sigma\right) & \Sigma^{-1} F_{12} \\ F_{12}^* \Sigma^{-1} & X_{22} \end{pmatrix} V^*, \tag{33}$$

where

$$\Phi = \left(\frac{1}{\alpha_i^2 + \alpha_j^2}\right), \quad 1 \le i, j \le r, \tag{34}$$

and $X_{22} \in H\mathbb{C}^{(n-r)\times(n-r)}$ is arbitrary.

Proof. Combining (14)–(16) and the unitary invariance of the Frobenius norm, it is easy to obtain that

$$\begin{aligned} \|EX - F\|^2 &= \left\|\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^* V \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{pmatrix} - U^* F V\right\|^2 \\ &= \left\|\begin{pmatrix} \Sigma X_{11} & \Sigma X_{12} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}\right\|^2 \\ &= \left\|\Sigma X_{11} - F_{11}\right\|^2 + \left\|\Sigma X_{12} - F_{12}\right\|^2 \\ &\quad + \left\|F_{21}\right\|^2 + \left\|F_{22}\right\|^2. \end{aligned} \tag{35}$$

Then $\|EX - F\|^2$ reaches its minimum if and only if

$$\left\|\Sigma X_{11} - F_{11}\right\|^2, \tag{36}$$

$$\left\|\Sigma X_{12} - F_{12}\right\|^2 \tag{37}$$

reach their minimum. For $X_{11} = (x_{ij}) \in H\mathbb{C}^{r\times r}$, $F_{11} = (f_{ij}) \in \mathbb{C}^{r\times r}$, since $x_{ij} = x_{ij}^*$, $1 \le i, j \le r$, then

$$\begin{aligned} \left\|\Sigma X_{11} - F_{11}\right\|^2 &= \sum_{i=1}^{r}\sum_{j=1}^{r} \left(\alpha_i x_{ij} - f_{ij}\right)^2 \\ &= \sum_{1\le i,j\le r} \left[\left(\alpha_i^2 + \alpha_j^2\right)\left|x_{ij}\right|^2\right. \\ &\qquad \left. + 2\left(\alpha_i f_{ij} + \alpha_j f_{ij}^*\right) x_{ij} + 2\left|f_{ij}\right|^2\right]. \end{aligned} \tag{38}$$

Hence, there exists a unique solution $X_{11} = (\widehat{x}_{ij}) \in H\mathbb{C}^{r\times r}$ for (36) such that

$$\widehat{x}_{ij} = \frac{\alpha_i f_{ij} + \alpha_j f_{ij}^*}{\alpha_i^2 + \alpha_j^2}, \quad 1 \le i, j \le r. \tag{39}$$

That is,

$$X_{11} = \Phi * \left(\Sigma F_{11} + F_{11}^* \Sigma\right), \tag{40}$$

where

$$\Phi = \left(\frac{1}{\alpha_i^2 + \alpha_j^2}\right), \quad 1 \le i, j \le r. \tag{41}$$

When X_{12} can be expressed as

$$X_{12} = \Sigma^{-1} F_{12}, \tag{42}$$

(37) gets its minimum. Therefore, the least squares Hermitian solution to (18) can be described as (33). □

By the similar way, the following result can be obtained.

Lemma 17. *Given $M, N \in \mathbb{C}^{m\times n}$, let the singular value decomposition of M, the partitions of VXV^*, and U^*NV*

be, respectively, as in (21)–(23). Then the least squares anti-Hermitian solution to the matrix equation $MX = N$ can be expressed as

$$X = V\begin{pmatrix} \Psi * (\Pi N_{11} - N_{11}^{*}\Pi) & \Pi^{-1}N_{12} \\ -N_{12}^{*}\Pi^{-1} & X_{22} \end{pmatrix}V^{*}, \quad (43)$$

where

$$\Psi = \left(\frac{1}{\beta_i^2 + \beta_j^2}\right), \quad 1 \le i, j \le s, \quad (44)$$

and $X_{22} \in AH\mathbb{C}^{(n-s)\times(n-s)}$ is arbitrary.

Lemma 18 (see [20]). *Given $F \in \mathbb{C}^{m\times n}$, $G \in \mathbb{C}^{p\times q}$, and $L \in \mathbb{C}^{m\times q}$, then the matrix equation $FXG = L$ has a solution if and only if*

$$FF^{\dagger}LG^{\dagger}G = L, \quad (45)$$

in which case the general solution is

$$X = F^{\dagger}LG^{\dagger} + Y - F^{\dagger}FYGG^{\dagger}, \quad (46)$$

where $Y \in \mathbb{C}^{n\times p}$ is arbitrary.

The following lemma is due to [25, 32] or [29, Lemma 5].

Lemma 19. *Let $M, N \in \mathbb{C}^{m\times n}$. Then there exists a unique matrix $W_1 \in \mathbb{C}^{m\times n}$ such that*

$$\begin{aligned} &\|W_1 - M\|^2 + \|W_1 - N\|^2 \\ &\quad = \min_{W\in\mathbb{C}^{m\times n}} \left(\|W - M\|^2 + \|W - N\|^2\right), \end{aligned} \quad (47)$$

where

$$W_1 = \frac{M + N}{2}. \quad (48)$$

3. The Solvability Conditions and the Expression of the Solution to Problem 5

In this section, our purpose is to derive the necessary and sufficient conditions of and the explicit expression of the solution to Problem 5 by using the results introduced in Section 2.

Theorem 20. *Given $A, B \in \mathbb{C}^{m\times 2k}, C, D \in \mathbb{C}^{2k\times q}$, let the decomposition of $X \in HH\mathbb{C}^{2k\times 2k}$ be (7). Partition*

$$AP = \begin{pmatrix} A_1 & A_2 \end{pmatrix}, \quad A_1 \in \mathbb{C}^{m\times k},\ A_2 \in \mathbb{C}^{m\times k}; \quad (49)$$

$$BP = \begin{pmatrix} B_1 & B_2 \end{pmatrix}, \quad B_1 \in \mathbb{C}^{m\times k},\ B_2 \in \mathbb{C}^{m\times k}; \quad (50)$$

$$P^{*}C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \quad C_1 \in \mathbb{C}^{k\times q},\ C_2 \in \mathbb{C}^{k\times q}; \quad (51)$$

$$P^{*}D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}, \quad D_1 \in \mathbb{C}^{k\times q},\ D_2 \in \mathbb{C}^{k\times q}; \quad (52)$$

$$A' = \begin{pmatrix} A_1 \\ C_1^{*} \end{pmatrix}, \qquad B' = \begin{pmatrix} B_2 \\ D_2^{*} \end{pmatrix}, \quad (53)$$

$$C' = \begin{pmatrix} A_2^{*} & C_2 \end{pmatrix}, \qquad D' = \begin{pmatrix} B_1^{*} & D_1 \end{pmatrix}. \quad (54)$$

Then Problem 5 has a solution $X \in HH\mathbb{C}^{2k\times 2k}$ if and only if

$$\begin{gathered} A'\left(A'\right)^{\dagger}B' = B', \qquad D'\left(C'\right)^{\dagger}C' = D', \\ A'D' = B'C', \end{gathered} \quad (55)$$

in which case the Hermitian generalized Hamiltonian solution to Problem 5 can be expressed as

$$X = P\begin{pmatrix} 0 & X_{12} \\ X_{12}^{*} & 0 \end{pmatrix}P^{*}, \quad (56)$$

where

$$\begin{aligned} X_{12} = {} & \left(A'\right)^{\dagger}B' + D'\left(C'\right)^{\dagger} - \left(A'\right)^{\dagger}A'D'\left(C'\right)^{\dagger} \\ & + L_{A'}WR_{C'} \end{aligned} \quad (57)$$

and $W \in \mathbb{C}^{k\times k}$ is arbitrary.

Proof. It follows from (7) and (49)–(52) that the system (3) can be transformed into the following system of matrix equations:

$$\begin{aligned} A_1X_{12} = B_2, \qquad X_{12}A_2^{*} = B_1^{*}, \\ C_1^{*}X_{12} = D_2^{*}, \qquad X_{12}C_2 = D_1. \end{aligned} \quad (58)$$

Then, combining (53) and (54) yields that

$$A'X_{12} = B', \qquad X_{12}C' = D'. \quad (59)$$

Thus, by Lemma 12, the system (59) has a solution $X_{12} \in \mathbb{C}^{k\times k}$ if and only if all equalities in (55) hold, in which case the solution can be written as (57). So the solution to system (3) can be expressed as (56). □

Remark 21. Let C and D vanish in Theorem 20. Partition

$$\begin{aligned} AP = \begin{pmatrix} A_1 & A_2 \end{pmatrix}, \quad A_1 \in \mathbb{C}^{m\times k},\ A_2 \in \mathbb{C}^{m\times k}; \\ BP = \begin{pmatrix} B_1 & B_2 \end{pmatrix}, \quad B_1 \in \mathbb{C}^{m\times k},\ B_2 \in \mathbb{C}^{m\times k}. \end{aligned} \quad (60)$$

Then the matrix equation $AX = B$ has Hermitian generalized Hamiltonian solutions if and only if

$$\begin{gathered} A_1A_1^{\dagger}B_2 = B_2, \qquad A_2A_2^{\dagger}B_1 = B_1, \\ A_1B_1^{*} = B_2A_2^{*}, \end{gathered} \quad (61)$$

in which case its solution can be described as

$$X = P\begin{pmatrix} 0 & X_{12} \\ X_{12}^{*} & 0 \end{pmatrix}P^{*}, \quad (62)$$

where

$$X_{12} = A_1^{\dagger}B_2 + B_1^{*}\left(A_2^{\dagger}\right)^{*} - A_1^{\dagger}A_1B_1^{*}\left(A_2^{\dagger}\right)^{*} + L_{A_1}WL_{A_2} \quad (63)$$

and $W \in \mathbb{C}^{k\times k}$ is arbitrary. It is clear that this result is different from Theorem 3.1 given in [1].

Similarly, by Lemmas 9 and 12, we can get the anti-Hermitian generalized anti-Hamiltonian solution to system (3).

Theorem 22. *Given $A, B \in \mathbb{C}^{m\times 2k}$, $C, D \in \mathbb{C}^{2k\times q}$, let the decomposition of $X \in AHAH\mathbb{C}^{2k\times 2k}$ be* (8)*. AP, BP, P^*C, and P^*D, respectively, have the partitions as in* (49)–(52)*. Put*

$$\begin{aligned} \widetilde{A} &= \begin{pmatrix} A_1 \\ C_1^* \end{pmatrix}, & \widetilde{B} &= \begin{pmatrix} B_2 \\ -D_2^* \end{pmatrix}, \\ \widetilde{C} &= \begin{pmatrix} A_2^* & C_2 \end{pmatrix}, & \widetilde{D} &= \begin{pmatrix} -B_1^* & D_1 \end{pmatrix}. \end{aligned} \tag{64}$$

Then Problem 5 has a solution $X \in AHAH\mathbb{C}^{2k\times 2k}$ if and only if

$$\widetilde{A}\widetilde{A}^{\dagger}\widetilde{B} = \widetilde{B}, \qquad \widetilde{D}\widetilde{C}^{\dagger}\widetilde{C} = \widetilde{D}, \qquad \widetilde{A}\widetilde{D} = \widetilde{B}\widetilde{C}, \tag{65}$$

in which case the anti-Hermitian generalized anti-Hamiltonian solution to Problem 5 can be expressed as

$$X = P\begin{pmatrix} 0 & X_{12} \\ -X_{12}^* & 0 \end{pmatrix}P^*, \tag{66}$$

where

$$X_{12} = \widetilde{A}^{\dagger}\widetilde{B} + \widetilde{D}\widetilde{C}^{\dagger} - \widetilde{A}^{\dagger}\widetilde{A}\widetilde{D}\widetilde{C}^{\dagger} + L_{\widetilde{A}}ZR_{\widetilde{C}} \tag{67}$$

and $Z \in \mathbb{C}^{k\times k}$ is arbitrary.

Now, we investigate the Hermitian generalized anti-Hamiltonian solution to the system (3).

Theorem 23. *Given $A, B \in \mathbb{C}^{m\times 2k}$, $C, D \in \mathbb{C}^{2k\times q}$, let the decomposition of $X \in HAH\mathbb{C}^{2k\times 2k}$ be* (9)*. AP, BP, P^*C, and P^*D, respectively, have the partitions as in* (49)–(52)*. Denote*

$$\overline{A} = \begin{pmatrix} A_1 \\ C_1^* \end{pmatrix}, \qquad \overline{B} = \begin{pmatrix} B_1 \\ D_1^* \end{pmatrix}, \tag{68}$$

$$\overline{C} = \begin{pmatrix} A_2 \\ C_2^* \end{pmatrix}, \qquad \overline{D} = \begin{pmatrix} B_2 \\ D_2^* \end{pmatrix}. \tag{69}$$

Let the singular value decompositions of $\overline{A}$ and $\overline{C}$ be, respectively,

$$\overline{A} = U\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}V^*, \tag{70}$$

$$\overline{C} = Q\begin{pmatrix} \Pi & 0 \\ 0 & 0 \end{pmatrix}R^*, \tag{71}$$

where

$$\begin{aligned} &U \in U\mathbb{C}^{(m+q)\times k}, \qquad V \in U\mathbb{C}^{k\times k}, \\ &\Sigma = \operatorname{diag}(\alpha_1, \ldots, \alpha_r), \qquad \alpha_i > 0, \\ &\qquad i = 1, \ldots, r;\ r = r(\overline{A}), \\ &Q \in U\mathbb{C}^{(m+q)\times k}, \qquad R \in U\mathbb{C}^{k\times k}, \\ &\Pi = \operatorname{diag}(\beta_1, \ldots, \beta_s), \qquad \beta_j > 0, \\ &\qquad j = 1, \ldots, s;\ s = r(\overline{C}). \end{aligned} \tag{72}$$

Set

$$VX_{11}V^* = \begin{pmatrix} \overline{X}_{11} & \overline{X}_{12} \\ \overline{X}_{12}^* & \overline{X}_{22} \end{pmatrix}; \tag{73}$$

$$U^*\overline{B}V = \begin{pmatrix} \overline{B}_{11} & \overline{B}_{12} \\ \overline{B}_{21} & \overline{B}_{22} \end{pmatrix}; \tag{74}$$

$$RX_{22}R^* = \begin{pmatrix} \widehat{X}_{11} & \widehat{X}_{12} \\ \widehat{X}_{12}^* & \widehat{X}_{22} \end{pmatrix}; \tag{75}$$

$$Q^*\overline{D}R = \begin{pmatrix} \overline{D}_{11} & \overline{D}_{12} \\ \overline{D}_{21} & \overline{D}_{22} \end{pmatrix}, \tag{76}$$

where

$$\begin{aligned} \overline{X}_{11} &\in H\mathbb{C}^{r\times r}, & \widehat{X}_{11} &\in H\mathbb{C}^{s\times s}, \\ \overline{X}_{22} &\in H\mathbb{C}^{(k-r)\times(k-r)}, & \widehat{X}_{22} &\in H\mathbb{C}^{(k-s)\times(k-s)}, \\ \overline{B}_{11} &\in \mathbb{C}^{r\times r}, & \overline{D}_{11} &\in \mathbb{C}^{s\times s}, \\ \overline{B}_{22} &\in \mathbb{C}^{(m+q-r)\times(k-r)}, & \overline{D}_{22} &\in \mathbb{C}^{(m+q-s)\times(k-s)}. \end{aligned} \tag{77}$$

Then Problem 5 has a solution $X \in HAH\mathbb{C}^{2k\times 2k}$ if and only if

$$\begin{aligned} &\overline{A}(\overline{A})^{\dagger}\overline{B} = \overline{B}, \qquad \overline{A}(\overline{B})^* = \overline{B}(\overline{A})^*, \\ &\overline{B}_{21} = 0, \qquad \overline{B}_{22} = 0, \end{aligned} \tag{78}$$

$$\begin{aligned} &\overline{C}(\overline{C})^{\dagger}\overline{D} = \overline{D}, \qquad \overline{C}(\overline{D})^* = \overline{D}(\overline{C})^*, \\ &\overline{D}_{21} = 0, \qquad \overline{D}_{22} = 0, \end{aligned} \tag{79}$$

in which case the Hermitian generalized anti-Hamiltonian solution to Problem 5 can be described as

$$X = P\begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix}P^*, \tag{80}$$

where

$$X_{11} = V\begin{pmatrix} \Sigma^{-1}\overline{B}_{11} & \Sigma^{-1}\overline{B}_{12} \\ (\overline{B}_{12})^*\Sigma^{-1} & \overline{X}_{22} \end{pmatrix}V^*, \tag{81}$$

$$X_{22} = R\begin{pmatrix} \Pi^{-1}\overline{D}_{11} & \Pi^{-1}\overline{D}_{12} \\ (\overline{D}_{12})^*\Pi^{-1} & \widehat{X}_{22} \end{pmatrix}R^*, \tag{82}$$

and $\overline{X}_{22} \in H\mathbb{C}^{(k-r)\times(k-r)}$, $\widehat{X}_{22} \in H\mathbb{C}^{(k-s)\times(k-s)}$ are arbitrary.

Proof. It can be derived from (9), (49)–(52), and (68)-(69) that the system (3) is consistent if and only if the following two equations:

$$\overline{A}X_{11} = \overline{B}, \tag{83}$$

$$\overline{C}X_{22} = \overline{D}, \tag{84}$$

are solvable. By (70), (73), and (74), and then combining Lemma 13, we can obtain that there exists Hermitian solution

X_{11} such that (83) holds if and only if all equalities in (78) hold, in which case the solution can be written as (81). By the similar way, there exists Hermitian solution X_{22} such that (84) holds if and only if all equalities in (79) hold, in which case the solution can be described as (82). Therefore, the Hermitian generalized anti-Hamiltonian solution to Problem 5 can be expressed as (80). □

From Lemmas 11 and 14, it is not difficult to obtain the anti-Hermitian generalized Hamiltonian solution to Problem 5, which can be described as follows.

Theorem 24. *Given $A, B \in \mathbb{C}^{m\times 2k}$, $C, D \in \mathbb{C}^{2k\times q}$, let the decomposition of $X \in AHH\mathbb{C}^{2k\times 2k}$ be* (10). *AP, BP, P^*C, and P^*D, respectively, have the partitions as in* (49)–(52). *Denote*

$$\widehat{A} = \begin{pmatrix} A_1 \\ C_1^* \end{pmatrix}, \qquad \widehat{B} = \begin{pmatrix} B_1 \\ -D_1^* \end{pmatrix},$$
$$\widehat{C} = \begin{pmatrix} A_2 \\ C_2^* \end{pmatrix}, \qquad \widehat{D} = \begin{pmatrix} B_2 \\ -D_2^* \end{pmatrix}. \tag{85}$$

Let the singular value decompositions of $\widehat{A}$ and $\widehat{C}$ be, respectively,

$$\widehat{A} = U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^*,$$
$$\widehat{C} = Q \begin{pmatrix} \Pi & 0 \\ 0 & 0 \end{pmatrix} R^*, \tag{86}$$

where

$$U \in U\mathbb{C}^{(m+q)\times k}, \qquad V \in U\mathbb{C}^{k\times k},$$
$$\Sigma = \operatorname{diag}(\alpha_1, \dots, \alpha_r), \qquad \alpha_i > 0,$$
$$i = 1, \dots, r;\ \ r = r(\widehat{A}),$$
$$Q \in U\mathbb{C}^{(m+q)\times k}, \qquad R \in U\mathbb{C}^{k\times k},$$
$$\Pi = \operatorname{diag}(\beta_1, \dots, \beta_s), \qquad \beta_j > 0,$$
$$j = 1, \dots, s;\ \ s = r(\widehat{C}). \tag{87}$$

Set

$$VX_{11}V^* = \begin{pmatrix} \overline{X}_{11} & \overline{X}_{12} \\ -\overline{X}_{12}^* & \overline{X}_{22} \end{pmatrix};$$
$$U^*\widehat{B}V = \begin{pmatrix} \widehat{B}_{11} & \widehat{B}_{12} \\ \widehat{B}_{21} & \widehat{B}_{22} \end{pmatrix};$$
$$RX_{22}R^* = \begin{pmatrix} \widehat{X}_{11} & \widehat{X}_{12} \\ -\widehat{X}_{12}^* & \widehat{X}_{22} \end{pmatrix};$$
$$Q^*\widehat{D}R = \begin{pmatrix} \widehat{D}_{11} & \widehat{D}_{12} \\ \widehat{D}_{21} & \widehat{D}_{22} \end{pmatrix}, \tag{88}$$

where

$$\overline{X}_{11} \in H\mathbb{C}^{r\times r}, \qquad \widehat{X}_{11} \in H\mathbb{C}^{s\times s},$$
$$\overline{X}_{22} \in H\mathbb{C}^{(k-r)\times(k-r)}, \qquad \widehat{X}_{22} \in H\mathbb{C}^{(k-s)\times(k-s)},$$
$$\widehat{B}_{11} \in \mathbb{C}^{r\times r}, \qquad \widehat{D}_{11} \in \mathbb{C}^{s\times s},$$
$$\widehat{B}_{22} \in \mathbb{C}^{(m+q-r)\times(k-r)}, \qquad \widehat{D}_{22} \in \mathbb{C}^{(m+q-s)\times(k-s)}. \tag{89}$$

Then Problem 5 has a solution $X \in AHH\mathbb{C}^{2k\times 2k}$ if and only if

$$\widehat{A}\widehat{A}^\dagger \widehat{B} = \widehat{B}, \qquad \widehat{A}\widehat{B}^* = -\widehat{B}\widehat{A}^*,$$
$$\widehat{B}_{21} = 0, \qquad \widehat{B}_{22} = 0,$$
$$\widehat{C}\widehat{C}^\dagger \widehat{D} = \widehat{D}, \qquad \widehat{C}\widehat{D}^* = -\widehat{D}\widehat{C}^*,$$
$$\widehat{D}_{21} = 0, \qquad \widehat{D}_{22} = 0, \tag{90}$$

in which case the anti-Hermitian generalized Hamiltonian solution to Problem 5 can be described as

$$X = P \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} P^*, \tag{91}$$

where

$$X_{11} = V \begin{pmatrix} \Sigma^{-1}\widehat{B}_{11} & \Sigma^{-1}\widehat{B}_{12} \\ -\widehat{B}_{12}^*\Sigma^{-1} & \overline{X}_{22} \end{pmatrix} V^*,$$
$$X_{22} = R \begin{pmatrix} \Pi^{-1}\widehat{D}_{11} & \Pi^{-1}\widehat{D}_{12} \\ -\widehat{D}_{12}^*\Pi^{-1} & \widehat{X}_{22} \end{pmatrix} R^*, \tag{92}$$

and $\overline{X}_{22} \in AH\mathbb{C}^{(k-r)\times(k-r)}$, $\widehat{X}_{22} \in AH\mathbb{C}^{(k-s)\times(k-s)}$ are arbitrary.

4. The Expression of the Unique Solution to Problem 6

In this section, our aim is to derive the optimal approximation solution to Problem 6.

Theorem 25. *Given $\widehat{X} \in \mathbb{C}^{2k\times 2k}$, under the hypotheses of Theorem 20, let*

$$P^*\widehat{X}P = \begin{pmatrix} \widehat{X}_{11} & \widehat{X}_{12} \\ \widehat{X}_{21} & \widehat{X}_{22} \end{pmatrix}, \quad \widehat{X}_{11} \in \mathbb{C}^{k\times k},\ \widehat{X}_{22} \in \mathbb{C}^{k\times k}. \tag{93}$$

If Problem 5 has Hermitian generalized Hamiltonian solutions, then Problem 6 has a unique solution $\widetilde{X} \in HH\mathbb{C}^{2k\times 2k}$ if and only if

$$L_{A'} \left(\frac{\widehat{X}_{12} + (\widehat{X}_{21})^*}{2} - X_0 \right) R_{C'} = \frac{\widehat{X}_{12} + (\widehat{X}_{21})^*}{2} - X_0, \tag{94}$$

in which case the unique solution $\widetilde{X}$ can be expressed as

$$\widetilde{X} = P \begin{pmatrix} 0 & \overline{X_0} \\ (\overline{X_0})^* & 0 \end{pmatrix} P^*, \tag{95}$$

where

$$\overline{X_0} = \frac{\widehat{X}_{12} + \left(\widehat{X}_{21}\right)^*}{2},$$
$$X_0 = \left(A'\right)^\dagger B' + D'\left(C'\right)^\dagger - \left(A'\right)^\dagger A'D'\left(C'\right)^\dagger. \quad (96)$$

Proof. When the Hermitian generalized Hamiltonian solution set K of Problem 5 is nonempty, it is not difficult to verify that K is a closed convex set. Then by [33], Problem 6 has a unique solution $\widetilde{X} \in HH\mathbb{C}^{2k\times 2k}$. From Theorem 20, for any $X \in K$, X can be expressed as

$$X = P\begin{pmatrix} 0 & X_0 \\ X_0^* & 0 \end{pmatrix} P^* + P\begin{pmatrix} 0 & L_{A'}WR_{C'} \\ R_{C'}W^*L_{A'} & 0 \end{pmatrix} P^*, \quad (97)$$

where

$$X_0 = \left(A'\right)^\dagger B' + D'\left(C'\right)^\dagger - \left(A'\right)^\dagger A'D'\left(C'\right)^\dagger \quad (98)$$

and $W \in \mathbb{C}^{k\times k}$ is arbitrary. Then it follows from the equalities in (93) and (97) and the unitary invariance of the Frobenius norm that

$$\begin{aligned}
&\left\|\widehat{X} - X\right\|^2 \\
&= \left\|P^*\widehat{X}P - P^*XP\right\|^2 \\
&= \left\|\begin{pmatrix} \widehat{X}_{11} & \widehat{X}_{12} - X_0 - L_{A'}WR_{C'} \\ \widehat{X}_{21} - X_0^* - R_{C'}W^*L_{A'} & \widehat{X}_{22} \end{pmatrix}\right\|^2 \\
&= \left\|\widehat{X}_{11}\right\|^2 + \left\|\widehat{X}_{22}\right\|^2 + \left\|\widehat{X}_{12} - X_0 - L_{A'}WR_{C'}\right\|^2 \\
&\quad + \left\|\widehat{X}_{21} - X_0^* - R_{C'}W^*L_{A'}\right\|^2 \\
&= \left\|\widehat{X}_{11}\right\|^2 + \left\|\widehat{X}_{22}\right\|^2 + \left\|L_{A'}WR_{C'} - \left(-X_0 + \widehat{X}_{12}\right)\right\|^2 \\
&\quad + \left\|L_{A'}WR_{C'} - \left(-X_0 + \widehat{X}_{21}^*\right)\right\|^2.
\end{aligned} \quad (99)$$

Thus, Problem 6 has a unique solution $\widetilde{X} \in HH\mathbb{C}^{2k\times 2k}$ if and only if there exists W such that

$$\left\|L_{A'}WR_{C'} - \left(-X_0 + \widehat{X}_{12}\right)\right\|^2 + \left\|L_{A'}WR_{C'} - \left(-X_0 + \left(\widehat{X}_{21}\right)^*\right)\right\|^2 \quad (100)$$

reaches its minimum. Therefore, by Lemma 19, (100) arrives at its minimum if and only if there exists W such that the matrix equation

$$\begin{aligned}
L_{A'}WR_{C'} &= \frac{-X_0 + \widehat{X}_{12} - X_0 + \left(\widehat{X}_{21}\right)^*}{2} \\
&= \frac{\widehat{X}_{12} + \left(\widehat{X}_{21}\right)^*}{2} - X_0
\end{aligned} \quad (101)$$

holds, which, by Lemma 18, has a solution if and only if (94) holds, in which case the solution can be expressed as

$$W = L_{A'}\left(\frac{\widehat{X}_{12} + \left(\widehat{X}_{21}\right)^*}{2} - X_0\right)R_{C'} + Z - L_{A'}ZR_{C'}, \quad (102)$$

where $Z \in \mathbb{C}^{k\times k}$ is arbitrary. Inserting (102) into (97), and then combining (94) yields (95). □

Analogously, the following theorem can be shown.

Theorem 26. *Given $\widehat{X} \in \mathbb{C}^{2k\times 2k}$, under the hypotheses of Theorem 22, let*

$$P^*\widehat{X}P = \begin{pmatrix} \widehat{X}_{11} & \widehat{X}_{12} \\ \widehat{X}_{21} & \widehat{X}_{22} \end{pmatrix}, \quad \widehat{X}_{11} \in \mathbb{C}^{k\times k}, \ \widehat{X}_{22} \in \mathbb{C}^{k\times k}. \quad (103)$$

If Problem 5 has anti-Hermitian generalized anti-Hamiltonian solutions, then Problem 6 has a unique solution $\widetilde{X} \in AHAH\mathbb{C}^{2k\times 2k}$ if and only if

$$L_{\widetilde{A}}\left(\frac{\widehat{X}_{12} - \left(\widehat{X}_{21}\right)^*}{2} - X_0\right)R_{\widetilde{C}} = \frac{\widehat{X}_{12} - \left(\widehat{X}_{21}\right)^*}{2} - X_0, \quad (104)$$

in which case the unique solution $\widetilde{X}$ can be expressed as

$$\widetilde{X} = P\begin{pmatrix} 0 & \overline{X_0} \\ -\left(\overline{X_0}\right)^* & 0 \end{pmatrix} P^*, \quad (105)$$

where

$$\overline{X_0} = \frac{\widehat{X}_{12} - \left(\widehat{X}_{21}\right)^*}{2},$$
$$X_0 = \left(\widetilde{A}\right)^\dagger \widetilde{B} + \widetilde{D}\left(\widetilde{C}\right)^\dagger - \left(\widetilde{A}\right)^\dagger \widetilde{A}\widetilde{D}\left(\widetilde{C}\right)^\dagger. \quad (106)$$

Now, we give the unique Hermitian generalized anti-Hamiltonian solution to Problem 6.

Theorem 27. *Given $\widehat{X} \in \mathbb{C}^{2k\times 2k}$, under the hypotheses of Theorem 23, let*

$$P^*\frac{\widehat{X} + \widehat{X}^*}{2}P = \begin{pmatrix} \widehat{X}'_{11} & \widehat{X}'_{12} \\ \left(\widehat{X}'_{12}\right)^* & \widehat{X}'_{22} \end{pmatrix},$$
$$\widehat{X}'_{11} \in H\mathbb{C}^{k\times k}, \ \widehat{X}'_{22} \in H\mathbb{C}^{k\times k}; \quad (107)$$

$$V^*\widehat{X}'_{11}V = \begin{pmatrix} \widehat{X}^\circ_{11} & \widehat{X}^\circ_{12} \\ \left(\widehat{X}^\circ_{12}\right)^* & \widehat{X}^\circ_{22} \end{pmatrix},$$
$$\widehat{X}^\circ_{11} \in H\mathbb{C}^{r\times r}, \ \widehat{X}^\circ_{22} \in H\mathbb{C}^{(k-r)\times(k-r)}; \quad (108)$$

$$R^*\widehat{X}'_{22}R = \begin{pmatrix} \widehat{X}'_{11} & \widehat{X}''_{12} \\ \left(\widehat{X}''_{12}\right)^* & \widehat{X}''_{22} \end{pmatrix},$$
$$\widehat{X}''_{11} \in H\mathbb{C}^{s\times s}, \ \widehat{X}''_{22} \in H\mathbb{C}^{(k-s)\times(k-s)}. \quad (109)$$

If Problem 5 has Hermitian generalized anti-Hamiltonian solutions, then the unique solution $\widetilde{X} \in HAH\mathbb{C}^{2k\times 2k}$ to Problem 6 can be expressed as

$$\widetilde{X} = P\begin{pmatrix} X_{11}^{\circ} & 0 \\ 0 & X_{22}^{\circ} \end{pmatrix} P^{*}, \tag{110}$$

where

$$X_{11}^{\circ} = V\begin{pmatrix} \Sigma^{-1}\overline{B}_{11} & \Sigma^{-1}\overline{B}_{12} \\ (\overline{B}_{12})^{*}\Sigma^{-1} & \widehat{X}_{22}^{\circ} \end{pmatrix} V^{*}, \tag{111}$$

$$X_{22}^{\circ} = R\begin{pmatrix} \Pi^{-1}\overline{D}_{11} & \Pi^{-1}\overline{D}_{12} \\ (\overline{D}_{12})^{*}\Pi^{-1} & \widehat{X}_{22}'' \end{pmatrix} R^{*}. \tag{112}$$

Proof. When the Hermitian generalized anti-Hamiltonian solution set K of Problem 5 is nonempty, it is easy to prove that K is a closed convex set. Then, Problem 6 has a unique solution $\widetilde{X} \in HAH\mathbb{C}^{2k\times 2k}$ by the aid of [33]. For any $X \in K$, due to Theorem 23, X can be expressed as

$$X = P\begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} P^{*}, \tag{113}$$

where X_{11} and X_{22} have the expressions as in (81) and (82). Combining the equalities in (80)–(82) and (107) and the unitary invariance of the Frobenius norm yields that

$$\begin{aligned} \left\|X - \widehat{X}\right\|^2 &= \left\|X - \frac{\widehat{X} + \widehat{X}^{*}}{2}\right\|^2 + \left\|\frac{\widehat{X} - \widehat{X}^{*}}{2}\right\|^2 \\ &= \left\|P\begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} P^{*} - \frac{\widehat{X} + \widehat{X}^{*}}{2}\right\|^2 \\ &\quad + \left\|\frac{\widehat{X} - \widehat{X}^{*}}{2}\right\|^2 \\ &= \left\|\begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} - \begin{pmatrix} \widehat{X}_{11}' & \widehat{X}_{12}' \\ (\widehat{X}_{12}')^{*} & \widehat{X}_{22}' \end{pmatrix}\right\|^2 \\ &\quad + \left\|\frac{\widehat{X} - \widehat{X}^{*}}{2}\right\|^2 \\ &= \left\|X_{11} - \widehat{X}_{11}'\right\|^2 + \left\|X_{22} - \widehat{X}_{22}'\right\|^2 \\ &\quad + 2\left\|\widehat{X}_{12}'\right\|^2 + \left\|\frac{\widehat{X} - \widehat{X}^{*}}{2}\right\|^2. \end{aligned} \tag{114}$$

So,

$$\begin{aligned} &\min_{X\in\in HAH\mathbb{C}^{2k\times 2k}} \left\|X - \widehat{X}\right\|^2 \text{ holds} \\ &\Longleftrightarrow \min_{X_{11}\in H\mathbb{C}^{k\times k}} \left\|X_{11} - \widehat{X}_{11}'\right\|^2 \text{ holds} \\ &\quad\text{and } \min_{X_{22}\in H\mathbb{C}^{k\times k}} \left\|X_{22} - \widehat{X}_{22}'\right\|^2 \text{ holds.} \end{aligned} \tag{115}$$

By (81), (108), and the unitary invariance of the Frobenius norm, we obtain

$$\begin{aligned} &\left\|X_{11} - \widehat{X}_{11}'\right\|^2 \\ &\quad = \left\|\begin{pmatrix} \Sigma^{-1}\overline{B}_{11} & \Sigma^{-1}\overline{B}_{12} \\ (\overline{B}_{12})^{*}\Sigma^{-1} & \overline{X}_{22} \end{pmatrix} - \begin{pmatrix} \widehat{X}_{11}^{\circ} & \widehat{X}_{12}^{\circ} \\ (\widehat{X}_{12}^{\circ})^{*} & \widehat{X}_{22}^{\circ} \end{pmatrix}\right\|^2 \\ &\quad = \left\|\Sigma^{-1}\overline{B}_{11} - \widehat{X}_{11}^{\circ}\right\|^2 + \left\|\overline{X}_{22} - \widehat{X}_{22}^{\circ}\right\|^2 \\ &\quad\quad + 2\left\|\Sigma^{-1}\overline{B}_{12} - \widehat{X}_{12}^{\circ}\right\|^2. \end{aligned} \tag{116}$$

Then

$$\begin{aligned} &\min_{X_{11}\in H\mathbb{C}^{k\times k}} \left\|X_{11} - \widehat{X}_{11}'\right\|^2 \textit{ holds} \\ &\Longleftrightarrow \min_{\overline{X}_{22}\in H\mathbb{C}^{(k-r)\times(k-r)}} \left\|\overline{X}_{22} - \widehat{X}_{22}^{\circ}\right\|^2 \textit{ holds.} \end{aligned} \tag{117}$$

Therefore, when $\overline{X}_{22}$ can be expressed as

$$\overline{X}_{22} = \widehat{X}_{22}^{\circ}, \tag{118}$$

$\min_{X_{11}\in H\mathbb{C}^{k\times k}}\|X_{11} - \widehat{X}_{11}'\|^2$ holds. Then combining (81) yields (111). Similarly, we can derive the expression in (112) by (82) and (109). Thus, (110) is the unique solution to Problem 6. □

By the method used in Theorem 27, the following theorem can also be shown.

Theorem 28. *Given $\widehat{X} \in \mathbb{C}^{2k\times 2k}$, under the hypotheses of Theorem 24, let*

$$\begin{aligned} &P^{*}\frac{\widehat{X} - \widehat{X}^{*}}{2}P = \begin{pmatrix} \widehat{X}_{11}' & \widehat{X}_{12}' \\ -(\widehat{X}_{12}')^{*} & \widehat{X}_{22}' \end{pmatrix}, \\ &\widehat{X}_{11}' \in AH\mathbb{C}^{k\times k},\ \widehat{X}_{22}' \in AH\mathbb{C}^{k\times k}; \\ &V^{*}\widehat{X}_{11}'V = \begin{pmatrix} \widehat{X}_{11}^{\circ} & \widehat{X}_{12}^{\circ} \\ -(\widehat{X}_{12}^{\circ})^{*} & \widehat{X}_{22}^{\circ} \end{pmatrix}, \\ &\widehat{X}_{11}^{\circ} \in AH\mathbb{C}^{r\times r},\ \widehat{X}_{22}^{\circ} \in AH\mathbb{C}^{(k-r)\times(k-r)}; \\ &R^{*}\widehat{X}_{22}'R = \begin{pmatrix} \widehat{X}_{11}'' & \widehat{X}_{12}'' \\ -(\widehat{X}_{12}'')^{*} & \widehat{X}_{22}'' \end{pmatrix}, \\ &\widehat{X}_{11}'' \in AH\mathbb{C}^{s\times s},\ \widehat{X}_{22}'' \in AH\mathbb{C}^{(k-s)\times(k-s)}. \end{aligned} \tag{119}$$

If Problem 5 has anti-Hermitian generalized Hamiltonian solutions, then the unique solution $\widetilde{X} \in AHH\mathbb{C}^{2k\times 2k}$ to Problem 6 can be expressed as

$$\widetilde{X} = P\begin{pmatrix} X_{11}^{\circ} & 0 \\ 0 & X_{22}^{\circ} \end{pmatrix} P^{*}, \tag{120}$$

where

$$\begin{aligned} X_{11}^{\circ} &= V\begin{pmatrix} \Sigma^{-1}\widehat{B}_{11} & \Sigma^{-1}\widehat{B}_{12} \\ -\widehat{B}_{12}^{*}\Sigma^{-1} & \widehat{X}_{22}^{\circ} \end{pmatrix} V^{*}, \\ X_{22}^{\circ} &= R\begin{pmatrix} \Pi^{-1}\widehat{D}_{11} & \Pi^{-1}\widehat{D}_{12} \\ -\widehat{D}_{12}^{*}\Pi^{-1} & \widehat{X}_{22}'' \end{pmatrix} R^{*}. \end{aligned} \tag{121}$$

5. The Expression of the Solution to Problem 7

If the solvability conditions of linear matrix equations are not satisfied, the least squares solution is usually considered. So, in this section, the solution to Problem 7 is constructed.

Theorem 29. *Given $A, B \in \mathbb{C}^{m\times 2k}$, $C, D \in \mathbb{C}^{2k\times q}$, let the decomposition of $X \in HH\mathbb{C}^{2k\times 2k}$ be* (7). *AP, BP, P^*C, P^*D, C', and D', respectively, have the partitions as in* (49)–(52) *and* (54). *Denote*

$$A' = \begin{pmatrix} A_1^* & C_1 \end{pmatrix}, \qquad B' = \begin{pmatrix} B_2^* & D_2 \end{pmatrix}. \tag{122}$$

Let the singular value decompositions of A' and C' be as given in (28). *Then the least squares Hermitian generalized Hamiltonian solution to Problem 7 can be described as* (7), *where X_{12} has the expression as in* (31).

Proof. Combining (7), (49)–(52), (54), (122), and the unitary invariance of the Frobenius norm yields that

$$\begin{aligned} &\|AX - B\|^2 + \|XC - D\|^2 \\ &\quad = \left\| \left(A'\right)^* X_{12} - \left(B'\right)^* \right\|^2 + \left\| X_{12}C' - D' \right\|^2. \end{aligned} \tag{123}$$

Therefore, by Lemma 15, if X_{12} has the expression as in (31), then (123) reaches its minimum. Then, substituting (31) into (7), we obtain the least squares Hermitian generalized Hamiltonian solution to Problem 7. □

Corollary 30. *Given $A, B \in \mathbb{C}^{m\times 2k}$, $C, D \in \mathbb{C}^{2k\times q}$, under the conditions of Theorem 29, the least squares Hermitian generalized Hamiltonian solution with minimum norm to Problem 7 can be described as* (7), *where X_{12} has the expression as in* (31) *with $X'_{22} = 0$.*

By the same way, we can also derive the least squares anti-Hermitian generalized anti-Hamiltonian solution to Problem 7.

Theorem 31. *Given $A, B \in \mathbb{C}^{m\times 2k}$, $C, D \in \mathbb{C}^{2k\times q}$, let the decomposition of $X \in AHAH\mathbb{C}^{2k\times 2k}$ be* (8). *AP, BP, P^*C, and P^*D, respectively, have the partitions as in* (49)–(52). *Denote*

$$\begin{aligned} A' &= \begin{pmatrix} A_1^* & C_1 \end{pmatrix}, \qquad B' = \begin{pmatrix} B_2^* & -D_2 \end{pmatrix}, \\ C' &= \begin{pmatrix} A_2^* & C_2 \end{pmatrix}, \qquad D' = \begin{pmatrix} -B_1^* & D_1 \end{pmatrix}. \end{aligned} \tag{124}$$

Let the singular value decompositions of A' and C' be as in (28). *Then the least squares anti-Hermitian generalized anti-Hamiltonian solution to Problem 7 can be described as* (8), *where X_{12} has the expression as in* (31).

Corollary 32. *Given $A, B \in \mathbb{C}^{m\times 2k}$, $C, D \in \mathbb{C}^{2k\times q}$, under the conditions of Theorem 31, the least squares anti-Hermitian generalized anti-Hamiltonian solution with minimum norm to Problem 7 can be described as* (8), *where X_{12} has the expression as in* (31) *with $X'_{22} = 0$.*

At present, we give the least squares Hermitian generalized anti-Hamiltonian solution to Problem 7.

Theorem 33. *Assume $A, B \in \mathbb{C}^{m\times 2k}$, $C, D \in \mathbb{C}^{2k\times q}$. Let the decomposition of $X \in HAH\mathbb{C}^{2k\times 2k}$ be* (9). *$AP, BP, P^*C, P^*D, \overline{A}, \overline{B}, \overline{C}$, and $\overline{D}$, respectively, have the partitions as in* (49)–(52), (68), *and* (69). *Let the singular value decompositions of $\overline{A}$ and $\overline{C}$ be, respectively,* (70) *and* (71), *$VX_{11}V^*, U^*\overline{B}V, RX_{22}R^*$, and $Q^*\overline{D}R$ have the partitions as in* (73)–(76). *Then the least squares Hermitian generalized anti-Hamiltonian solution to Problem 7 can be expressed as* (9) *with*

$$X_{11} = V \begin{pmatrix} \Phi_1 * \left(\Sigma\overline{B}_{11} + \overline{B}_{11}^*\Sigma\right) & \Sigma^{-1}\overline{B}_{12} \\ \overline{B}_{12}^*\Sigma^{-1} & \overline{X}_{22} \end{pmatrix} V^*, \tag{125}$$

$$X_{22} = R \begin{pmatrix} \Phi_2 * \left(\Pi\overline{D}_{11} + \overline{D}_{11}^*\Pi\right) & \Pi^{-1}\overline{D}_{12} \\ \overline{D}_{12}^*\Pi^{-1} & \widehat{X}_{22} \end{pmatrix} R^*, \tag{126}$$

where

$$\begin{aligned} \Phi_1 &= \left(\frac{1}{\alpha_i^2 + \alpha_j^2}\right), \quad 1 \le i,\ j \le r; \\ \Phi_2 &= \left(\frac{1}{\beta_i^2 + \beta_j^2}\right), \quad 1 \le i,\ j \le s, \end{aligned} \tag{127}$$

and $\overline{X}_{22} \in H\mathbb{C}^{(k-r)\times(k-r)}$, $\widehat{X}_{22} \in H\mathbb{C}^{(k-s)\times(k-s)}$ are arbitrary.

Proof. It follows from (9), (49)–(52), (68), (69), and the unitary invariance of the Frobenius norm that

$$\begin{aligned} &\|AX - B\|^2 + \|XC - D\|^2 \\ &\quad = \left\|\overline{A}X_{11} - \overline{B}\right\|^2 + \left\|\overline{C}X_{22} - \overline{D}\right\|^2. \end{aligned} \tag{128}$$

Then

$$\|AX - B\|^2 + \|XC - D\|^2 \tag{129}$$

gains its minimum value if and only if

$$\min = \left\|\overline{A}X_{11} - \overline{B}\right\|^2 \text{ holds}, \tag{130}$$

$$\min = \left\|\overline{C}X_{22} - \overline{D}\right\|^2 \text{ holds}. \tag{131}$$

So, by (68), (70), (73), and (74) and then combining Lemma 16, we get that if X_{11} has the expression as in (125), then (130) holds. Similarly, if X_{22} has the expression as in (126), then (131) holds. Thus, the least squares Hermitian generalized anti-Hamiltonian solution to Problem 7 can be expressed as (9), where X_{11} and X_{22} have the expressions as in (125) and (126). □

Corollary 34. *Given $A, B \in \mathbb{C}^{m\times 2k}$, $C, D \in \mathbb{C}^{2k\times q}$, under the conditions of Theorem 33, the least squares Hermitian generalized anti-Hamiltonian solution with minimum norm to Problem 7 can be expressed as* (9) *with X_{11} and X_{22} having the expressions as in* (125) *and* (126), *where $\overline{X}_{22} = 0, \widehat{X}_{22} = 0$.*

At last, on the basis of Lemma 17, we can obtain the least squares anti-Hermitian generalized Hamiltonian solution to Problem 7, the proof of which is analogous to the proof of Theorem 33.

Theorem 35. *Given $A, B \in \mathbb{C}^{m\times 2k}$, $C, D \in \mathbb{C}^{2k\times q}$, let the decomposition of $X \in AHH\mathbb{C}^{2k\times 2k}$ be* (10). *$AP, BP, P^*C, P^*D, \widehat{A}, \widehat{B}, \widehat{C}$, and $\widehat{D}$, respectively, have the partitions as in* (49)–(52), (85). *Assume that the singular value decompositions of $\widehat{A}$ and $\widehat{C}$ are, respectively, expressed as in* (86) *and $VX_{11}V^*, U^*\widehat{B}V$, $RX_{22}R^*, Q^*\widehat{D}R$ have the partitions as in* (88). *Then the least squares anti-Hermitian generalized Hamiltonian solution to Problem 7 can be expressed as* (10) *with*

$$X_{11} = V\begin{pmatrix} \Psi_1 * \left(\Sigma\widehat{B}_{11} - \widehat{B}_{11}^*\Sigma\right) & \Sigma^{-1}\widehat{B}_{12} \\ -\widehat{B}_{12}^*\Sigma^{-1} & \overline{X}_{22} \end{pmatrix}V^*,$$
$$X_{22} = R\begin{pmatrix} \Psi_2 * \left(\Pi\widehat{D}_{11} - \widehat{D}_{11}^*\Pi\right) & \Pi^{-1}\widehat{D}_{12} \\ -\widehat{D}_{12}^*\Pi^{-1} & \widehat{X}_{22} \end{pmatrix}R^*, \tag{132}$$

where

$$\Psi_1 = \left(\frac{1}{\alpha_i^2 + \alpha_j^2}\right), \quad 1 \le i,\ j \le r;$$
$$\Psi_2 = \left(\frac{1}{\beta_i^2 + \beta_j^2}\right), \quad 1 \le i,\ j \le s, \tag{133}$$

and $\overline{X}_{22} \in AH\mathbb{C}^{(k-r)\times(k-r)}$, $\widehat{X}_{22} \in AH\mathbb{C}^{(k-s)\times(k-s)}$ are arbitrary.

Corollary 36. *Given $A, B \in \mathbb{C}^{m\times 2k}$, $C, D \in \mathbb{C}^{2k\times q}$, under the conditions of Theorem 35, the least squares anti-Hermitian generalized Hamiltonian solution with minimum norm to Problem 7 can be expressed as* (10) *with X_{11} and X_{22} having the expressions as in* (132), *where $\overline{X}_{22} = 0, \widehat{X}_{22} = 0$.*

6. Algorithms and Numerical Examples

In this section, algorithms are given to compute the solution to Problem 7, and meanwhile some numerical examples are presented to show that the algorithms provided are feasible. Note that all the tests are performed by MATLAB 7.6.

An algorithm is firstly presented to compute the least squares Hermitian generalized Hamiltonian solution to Problem 7.

Algorithm 37. *Step 1.* Input A, B, C, D, J.

Step 2. Compute the eigenvalue decomposition of J according to (6).

Step 3. Compute AP, BP, P^*C, P^*D according to (49)–(52).

Step 4. Compute A', B', C', D' according to (53) and (54). If the conditions in (55) hold, then compute the Hermitian generalized Hamiltonian solution to Problem 5 according to (56) and (57). Otherwise, turn to Step 5.

Step 5. Compute A', B', C', D' according to (53) and (122).

Step 6. Compute the singular value decompositions of A' and C' according to (28).

Step 7. Compute X_{12} according to (31).

Step 8. Compute X according to (7), and output X.

Example 38. Given

$$A = \begin{pmatrix} 3+6i & 2+i & 7-2i & 8+3i \\ 2-3i & 5-4i & 1+4i & 9+3i \end{pmatrix},$$
$$B = \begin{pmatrix} 2-4i & 3+2i & 5+i & 4+i \\ 6+i & 2-5i & 1+6i & 5+3i \end{pmatrix},$$
$$C = \begin{pmatrix} 4+7i & 10+3i & 7+i \\ 8+7i & 3+9i & 1-6i \\ 2-5i & 5+6i & 2+7i \\ 2 & 3i & 3+7i \end{pmatrix},$$
$$D = \begin{pmatrix} 7+3i & 5 & 2i \\ 5+2i & 2-3i & 6-i \\ 3+i & 9-i & 4 \\ 4-2i & 5+2i & 1+4i \end{pmatrix},$$
$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{134}$$

it can be easily verified that the conditions in (55) are not satisfied. Then, according to Algorithm 37, the least squares Hermitian generalized Hamiltonian solution X to Problem 7 can be expressed as

$$X = \begin{pmatrix} 0.0865 - i & 0.0141 - 0.0359i & 0.0317 - 0.0312i & 0.1174 + 0.0824i \\ 0.0141 + 0.0359i & -0.0148 & -0.0201 - 0.0682i & 0.1189 + 0.0954i \\ 0.0317 + 0.0302i & -0.0201 + 0.0682i & -0.0724 & 0.1114 + 0.0359i \\ 0.1173 - 0.0824i & 0.1189 - 0.0954i & 0.1114 - 0.0359i & 0.0006 \end{pmatrix},$$
$$\min_{X\in HH\mathbb{C}^{2k\times 2k}} \left\|X - X^*\right\| = 0.0000,$$
$$\min_{X\in HH\mathbb{C}^{2k\times 2k}} \left\|X^* - JXJ\right\| = 0.6000. \tag{135}$$

Remark 39. (1) There exists a unique least squares Hermitian generalized Hamiltonian solution to Problem 7 if and only if both A' and C' in Theorem 29 have full row ranks. Example 38 just illustrates it.

(2) Similarly, the algorithm about computing the least squares anti-Hermitian generalized anti-Hamiltonian solution to Problem 7 can be shown. We omit it here.

Now, we provide another algorithm to compute the least squares Hermitian generalized anti-Hamiltonian solution to Problem 7.

Algorithm 40. Step 1. Input A, B, C, D, J.

Step 2. Compute the eigenvalue decomposition of J according to (6).

Step 3. Compute AP, BP, P^*C, P^*D according to (49)–(52).

Step 4. Compute $\overline{A}, \overline{B}, \overline{C}, \overline{D}$ according to (68) and (69).

Step 5. Compute the singular value decompositions of $\overline{A}$ and $\overline{C}$ according to (70)-(71).

Step 6. Compute the partitions of $U^*\overline{B}V, Q^*\overline{D}R$ according to (74) and (76). If the conditions in (78) and (79) are all satisfied, then compute the Hermitian generalized anti-Hamiltonian solution to Problem 5 according to (80)–(82). Otherwise, turn to Step 7.

Step 7. Compute X_{11} and X_{22} according to (125) and (126).

Step 8. Compute X according to (9), and output X.

Example 41. Let A, B, C, D, J be as given in Example 38.

It is not difficult to prove that the conditions in (78) and (79) do not hold. So, according to Algorithm 40, the least squares Hermitian generalized anti-Hamiltonian solution to Problem 7 can be written as

$$X = \begin{pmatrix} 0.3671 & 0.1579 - 0.1804i & 0.1822 + 0.1400i & 0.0179 - 0.0012i \\ 0.1579 + 0.1804i & 0.2324 & 0.0179 + 0.1149i & -0.0662 - 0.1400i \\ 0.1822 - 0.1400i & 0.0179 - 0.1149i & 0.2143 & 0.1221 - 0.0849i \\ 0.0179 + 0.0012i & -0.0662 + 0.1400i & 0.1221 + 0.0849i & 0.5764 \end{pmatrix},$$
$$\min_{X\in HAH\mathbb{C}^{2k\times 2k}} \left\|X - X^*\right\| = 0.0000,$$
$$\min_{X\in HAH\mathbb{C}^{2k\times 2k}} \left\|X^* + JXJ\right\| = 0.8309. \tag{136}$$

Remark 42. (1) There exists a unique least squares Hermitian generalized anti-Hamiltonian solution to Problem 7 if and only if both $\overline{A}$ and $\overline{C}$ in Theorem 33 have full column ranks. Example 41 is just the case.

(2) Similarly, the algorithm about computing the least squares anti-Hermitian generalized Hamiltonian solution to Problem 7 can be obtained. We also omit it here.

7. Conclusions

In the previous sections, using the decomposition of the (anti-)Hermitian generalized (anti-)Hamiltonian matrices, the necessary and sufficient conditions for the existence of and the expression for the solution to Problem 5 have been firstly derived, respectively. Then the solutions to Problems 6 and 7 have been individually given. Finally, algorithms have been given to compute the least squares Hermitian generalized Hamiltonian solution and the least squares Hermitian generalized anti-Hamiltonian solution to Problem 7, and the corresponding examples have also been presented to show that the algorithms are reasonable.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This research was supported by the Grants from the National Natural Science Foundation of China (11171205), the National Natural Science Foundation of China (11301330), the Key Project of Scientific Research Innovation Foundation of Shanghai Municipal Education Commission (13ZZ080), the Natural Science Foundation of Shanghai (11ZR1412500), the Youth Funds of Natural Science Foundation of Hebei province (A2012403013), and the Natural Science Foundation of Hebei province (A2012205028).

References

[1] Z.-Z. Zhang, X.-Y. Hu, and L. Zhang, "On the Hermitian-generalized Hamiltonian solutions of linear matrix equations," *SIAM Journal on Matrix Analysis and Applications*, vol. 27, no. 1, pp. 294–303, 2005.

[2] Z.-J. Bai, "The solvability conditions for the inverse eigenvalue problem of Hermitian and generalized skew-Hamiltonian matrices and its approximation," *Inverse Problems*, vol. 19, no. 5, pp. 1185–1194, 2003.

[3] M. Jamshidi, "An overview on the solutions of the algebraic matrix Riccati equation and related problems," *Large Scale Systems*, vol. 1, no. 3, pp. 167–192, 1980.

[4] H. K. Wimmer, "Decomposition and parametrization of semidefinite solutions of the continuous-time algebraic Riccati

equation," *SIAM Journal on Control and Optimization*, vol. 32, no. 4, pp. 995–1007, 1994.

[5] Z.-Z. Zhang, X.-Y. Hu, and L. Zhang, "The solvability conditions for the inverse eigenvalue problem of Hermitian-generalized Hamiltonian matrices," *Inverse Problems*, vol. 18, no. 5, pp. 1369–1376, 2002.

[6] L. Guan and Y. Jiang, "Least-squares solutions for the inverse problem of anti-Hermitian generalized anti-Hamiltonian matrices on the linear manifold and their optimal approximation problem," *Natural Science Journal of Xiangtan University*, vol. 28, no. 2, pp. 13–18, 2006.

[7] Z.-Z. Zhang, *On the Hermitan and generalized Hamiltonian constrained matrix equations [Ph.D. thesis]*, Hunan University, 2002.

[8] C. G. Khatri and S. K. Mitra, "Hermitian and nonnegative definite solutions of linear matrix equations," *SIAM Journal on Applied Mathematics*, vol. 31, no. 4, pp. 579–585, 1976.

[9] S. K. Mitra, "The matrix equations $AX = B$, $XC = D$," *Linear Algebra and its Applications*, vol. 59, pp. 171–181, 1984.

[10] Q.-W. Wang, "Bisymmetric and centrosymmetric solutions to systems of real quaternion matrix equations," *Computers & Mathematics with Applications*, vol. 49, no. 5-6, pp. 641–650, 2005.

[11] Q.-W. Wang, X. Liu, and S.-W. Yu, "The common bisymmetric nonnegative definite solutions with extreme ranks and inertias to a pair of matrix equations," *Applied Mathematics and Computation*, vol. 218, no. 6, pp. 2761–2771, 2011.

[12] Q.-X. Xu, "Common Hermitian and positive solutions to the adjointable operator equations $AX = C$, $XB = D$," *Linear Algebra and Its Applications*, vol. 429, no. 1, pp. 1–11, 2008.

[13] Y.-X. Yuan, "Least-squares solutions to the matrix equations $AX = B$ and $XC = D$," *Applied Mathematics and Computation*, vol. 216, no. 10, pp. 3120–3125, 2010.

[14] H.-X. Chang, Q.-W. Wang, and G.-J. Song, "(R, S)-conjugate solution to a pair of linear matrix equations," *Applied Mathematics and Computation*, vol. 217, no. 1, pp. 73–82, 2010.

[15] A. Dajić and J. J. Koliha, "Positive solutions to the equations $AX = C$ and $XB = D$ for Hilbert space operators," *Journal of Mathematical Analysis and Applications*, vol. 333, no. 2, pp. 567–576, 2007.

[16] C.-Z. Dong, Q.-W. Wang, and Y.-P. Zhang, "On the Hermitian R-conjugate solution of a system of matrix equations," *Journal of Applied Mathematics*, Article ID 398085, 14 pages, 2012.

[17] F.-L. Li, X.-Y. Hu, and L. Zhang, "The generalized reflexive solution for a class of matrix equations $AX = B$, $XC = D$," *Acta Mathematica Scientia B*, vol. 28, no. 1, pp. 185–193, 2008.

[18] F.-L. Li, X.-Y. Hu, and L. Zhang, "Least-squares mirrorsymmetric solution for matrix equations $AX = B$, $XC = D$," *Numerical Mathematics*, vol. 15, no. 3, pp. 217–226, 2006.

[19] Y.-Y. Qiu, Z.-Y. Zhang, and J.-F. Lu, "The matrix equations $AX = B$, $XC = D$ with $PX = sXP$ constraint," *Applied Mathematics and Computation*, vol. 189, no. 2, pp. 1428–1434, 2007.

[20] C. R. Rao and S. K. Mitra, *Generalized Inverse of Matrices and Its Applications*, John Wiley & Sons, New York, NY, USA, 1971.

[21] Q.-W. Wang, "A system of four matrix equations over von Neumann regular rings and its applications," *Acta Mathematica Sinica*, vol. 21, no. 2, pp. 323–334, 2005.

[22] Q.-W. Wang and J. Yu, "Constrained solutions of a system of matrix equations," *Journal of Applied Mathematics*, vol. 2012, Article ID 471573, 19 pages, 2012.

[23] Q. Zhang and Q.-W. Wang, "The (P, Q)-(skew)symmetric extremal rank solutions to a system of quaternion matrix equations," *Applied Mathematics and Computation*, vol. 217, no. 22, pp. 9286–9296, 2011.

[24] H. Dai, "On the symmetric solutions of linear matrix equations," *Linear Algebra and its Applications*, vol. 131, pp. 1–7, 1990.

[25] Z.-Y. Peng, X.-Y. Hu, and L. Zhang, "One kind of inverse problems for symmetric and skew anti-symmetric matrices," *Numerical Mathematics*, vol. 25, no. 2, pp. 144–152, 2003.

[26] J. Qian and R. C. E. Tan, "On some inverse eigenvalue problems for Hermitian and generalized Hamiltonian/skew-Hamiltonian matrices," *Journal of Computational and Applied Mathematics*, vol. 250, pp. 28–38, 2013.

[27] Q.-W. Wang and J. Yu, "On the generalized bi (skew-) symmetric solutions of a linear matrix equation and its procrust problems," *Applied Mathematics and Computation*, vol. 219, no. 19, pp. 9872–9884, 2013.

[28] S.-F. Yuan, Q.-W. Wang, and X.-F. Duan, "On solutions of the quaternion matrix equation and their applications in color image restoration," *Applied Mathematics and Computation*, vol. 221, pp. 10–20, 2013.

[29] Z.-Z. Zhang, X.-Y. Hu, and L. Zhang, "Least-squares solutions of inverse problem for Hermitian generalized Hamiltonian matrices," *Applied Mathematics Letters*, vol. 17, no. 3, pp. 303–308, 2004.

[30] Z.-Z. Zhang, X.-Y. Hu, and L. Zhang, "Least-squares problem and optimal approximate problem for anti-Hermitian generalized Hamiltonian matrices on linear manifolds," *Acta Mathematica Scientia A*, vol. 26, no. 6, pp. 978–986, 2006.

[31] L. Zhang and D.-X. Xie, "A class of inverse eigenvalue problems," *Acta Mathematica Scientia A*, vol. 13, no. 1, pp. 94–99, 1993 (Chinese).

[32] L. Yu, K.-Y. Zhang, and Z.-K. Shi, "The solvability conditions for inverse problems of anti-symmetric ortho-symmetric matrices," *Journal of Applied Mathematics and Computing*, vol. 28, no. 1-2, pp. 225–233, 2008.

[33] L. Zhang, "The approximation on the closed convex cone and its numerical application," *Hunan Annals of Mathematics*, vol. 6, no. 2, pp. 43–48, 1986.