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Research Article

On Blow-Up Structures for a Generalized Periodic Nonlinearly Dispersive Wave Equation

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The local well-posedness for a generalized periodic nonlinearly dispersive wave equation is established. Under suitable assumptions on initial value u_0, a precise blow-up scenario and several sufficient conditions about blow-up results to the equation are presented.

1. Introduction

Recently, Hu and Yin [1] and Yin [2] investigate the following equation:

$$u_t - u_{txx} + 2\omega u_x + 3uu_x = \gamma\left(2u_x u_{xx} + uu_{xxx}\right), \qquad (1)$$

where ω is nonnegative number and γ is arbitrary real number. It is shown from [1, 2] that (1) has solitary wave solutions and blow-up solutions for nonperiodic case and also solutions which blow up in finite time for periodic case.

If $\gamma = 0$, (1) becomes the famous BBM equation modelling the motion of internal gravity waves in shallow channel [3]. Some results related to the equation can be found in [4, 5]. It is worthwhile to mention that the equation does not have integrability and its solitary waves are not solitons [5].

If $\gamma = 1$ in (1), attention is attracted to the well-known Camassa-Holm equation, which models the unidirectional propagation of shallow water waves over a flat bottom. Here, $u(t,x)$ represents the free surface above a flat bottom and ω is a nonnegative parameter related to the critical shallow water speed [6]. As a model to describe the shallow water motion, the Camassa-Holm equation possesses a bi-Hamiltonian structure and infinite conservation laws [7–9] and is completely integrable [10]. It is regarded as a reexpression of geodesic flow on the diffeomorphism group of circle if $\omega = 0$ [11] and on the Bott-Virasoro group if $\omega > 0$ [12]. Recently, some significant results of dynamical behaviors have been obtained for the Cauchy problem of the Camassa-Holm equation. For example, the local well-posedness of corresponding solution for initial data $u_0 \in H^s(\mathbb{R})$ with $s > 3/2$ was given by several authors (see [13–15]). Under certain assumptions on initial data u_0, the equation has global strong solutions and blow-up solutions for periodic and nonperiodic case (see [13, 16–22]). The existence and uniqueness of global weak solutions in $H^1(\mathbb{R})$ for the equation were proved (see [23–25]). It is shown from [6] that the solitary waves of the equation are peakon solitons and are orbitally stable.

If $\omega = 0$ and $\gamma \in \mathbb{R}$, (1) changes into the rod equation derived by Dai [26] recently, which describes finite-length and small amplitude radial deformation waves in thin cylindrical compressible hyperelastic rods (see [26]), and $u(t,x)$ represents the radial stretch relative to a prestressed state in one-dimensional variable. The first investigation of the Cauchy problem of the rod equation on the line was done by Constantin and Strauss [27], the precise blow-up scenario, some blow-up results of strong solution, and the stability of a class of solitary waves to the rod equation are presented. In [28, 29], Zhou found the sufficient conditions to guarantee the finite blow-up of corresponding solution for periodic case. Moreover, Yin [30] discusses the rod equation on the circle and gives some interesting blow-up results.

In this paper, we consider a generalized nonlinearly dispersive wave equation on the circle

$$\begin{aligned} &u_t - u_{txx} + 2\omega u_x + auu_x + \beta(u - u_{xx}) \\ &\quad = \gamma(2u_x u_{xx} + uu_{xxx}), \quad t > 0,\ x \in \mathbb{R}, \\ &u(0,x) = u_0(x), \quad x \in \mathbb{R}, \\ &u(t, x+1) = u(t,x), \quad t > 0,\ x \in \mathbb{R}, \end{aligned} \tag{2}$$

where ω and a are nonnegative fixed constants and β and γ are fixed arbitrary constants. Obviously, (2) reduces to (1) if we define $a = 3$ and $\beta = 0$. Actually, Wu and Yin [31] consider a nonlinearly dissipative Camassa-Holm equation which includes a nonlinearly dissipative term $L(u)$, where L is a differential operator. Thus, we can regard the term $\beta(u-u_{xx})$ as a dissipative term.

Because of the term $\beta(u - u_{xx})$, (2) does not admit conservation laws in previous works [1, 2]:

$$E(u) = \int_{\mathbb{S}} \left(u^2 + u_x^2\right) dx. \tag{3}$$

Several estimates are established to prove several blow-up solutions. More precisely, we establish the local well-posedness of strong solutions for (2) subject to initial value $u_0 \in H^r(\mathbb{S})$, $r > 3/2$, with $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ (the circle of unit length) and give a precise blow-up scenario. Under suitable assumptions on the initial value u_0, relying on the classical mathematical techniques, the several sufficient conditions about blow-up solutions are found.

2. Local Well-Posedness

In this section, we establish the local well-posedness and blow-up scenario for the Cauchy problem (2) in $H^r(\mathbb{S})$, $r > 3/2$.

We denote by $*$ the convolution. Note that if $G(x) := \cosh(x - [x] - 1/2)/2\sinh(1/2)$, where $[x]$ stands for the integer part of $x \in R$, then $(1-\partial_x^2)^{-1} f = G * f$ for all $f \in L^2(\mathbb{R})$ and $G * (u - u_{xx}) = u$. Using this identity, (2) becomes

$$\begin{aligned} &u_t + \gamma u u_x + \beta u + \partial_x G * \left[\frac{a-\gamma}{2}u^2 + \frac{\gamma}{2}(u_x)^2 + 2\omega u\right] = 0, \\ &\qquad t > 0,\ x \in \mathbb{R}, \\ &u(0,x) = u_0(x), \\ &u(t, x+1) = u(t,x), \end{aligned} \tag{4}$$

which is equivalent to

$$\begin{aligned} &y_t + \gamma u y_x + 2\gamma y u_x + 2\omega u_x + \beta y \\ &\quad + (a - 3\gamma) u u_x = 0, \quad t > 0,\ x \in \mathbb{R}, \\ &y(0,x) = u_0(x) - u_{0,xx}(x), \quad x \in \mathbb{R}, \\ &y(t,x) = y(t, x+1), \quad t > 0,\ x \in \mathbb{R}. \end{aligned} \tag{5}$$

Theorem 1. *Given $u_0 \in H^r(\mathbb{S})$ $(r > 3/2)$, there exist a maximal $T = T(a,b,\gamma,\omega,u_0)$ and a unique solution u to problem* (2)*, such that*

$$u = u(\cdot, u_0) \in C([0,T); H^r(\mathbb{S})) \bigcap C^1\left([0,T); H^{r-1}(\mathbb{S})\right). \tag{6}$$

Proof. The proof of Theorem 1 can be finished by using Kato's semigroup theory (see [1] or [2]). Here, we omit the detailed proof. □

3. Blow-Up Solutions

Theorem 2. *Let $u_0 \in H^r(\mathbb{S})$, $r > 3/2$; the solution of $u(\cdot, u_0)$ of problem* (2) *is uniformly bounded. Blow-up in finite time $T < +\infty$ occurs if and only if*

$$\liminf_{t \uparrow T} \left\{\inf_{x \in S} \left[\gamma u_x(t,x)\right]\right\} = -\infty. \tag{7}$$

Before proving the theorem, we give several useful lemmas.

Lemma 3 (Kato and Ponce [32]). *If $r > 0$, then $H^r \cap L^\infty$ is an algebra. Moreover*

$$\|uv\|_{H^r} \le c\left(\|u\|_{L^\infty}\|v\|_{H^r} + \|u\|_{H^r}\|v\|_{L^\infty}\right), \tag{8}$$

where c is a constant depending only on r.

Lemma 4 (Kato and Ponce [32]). *Let $r > 0$. If $u \in H^r \cap W^{1,\infty}$ and $v \in H^{r-1} \cap L^\infty$, then*

$$\left\|[\Lambda^r, u]v\right\|_{L^2} \le c\left(\left\|\partial_x u\right\|_{L^\infty}\left\|\Lambda^{r-1}v\right\|_{L^2} + \left\|\Lambda^r u\right\|_{L^2}\|v\|_{L^\infty}\right). \tag{9}$$

Lemma 5. *Let $s \ge 3/2$ and $u(t,x)$ is the corresponding solution of* (4) *with initial data $u_0(x) \in H^r(\mathbb{S})$; it holds that if $q \in (0, r-1]$, there is a constant c depending only on q such that*

$$\begin{aligned} \int_{\mathbb{S}} \left(\Lambda^{q+1}u\right)^2 dx &\le \int_{\mathbb{S}} \left(\Lambda^{q+1}u_0\right)^2 dx \\ &\quad + c\int_0^t \left(\|u_x\|_{L^\infty} + 1\right)\|u\|_{H^{q+1}}^2 ds. \end{aligned} \tag{10}$$

Proof. We rewrite (2) in the following equivalent form:

$$\begin{aligned} u_t - u_{txx} &= -auu_x - 2\omega u_x + \frac{\gamma}{2}\partial_x^3 u^2 \\ &\quad - \frac{\gamma}{2}\partial_x\left(u_x^2\right) - \beta u + \beta\partial_x^2 u. \end{aligned} \tag{11}$$

For $q \in (0, s-1]$, applying $(\Lambda^q u)\Lambda^q$ on both sides of (11) and integrating the new equation with respect to x by parts, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}}\left((\Lambda^q u)^2+(\Lambda^q u_x)^2\right)dx &= -a\int_{\mathbb{S}}(\Lambda^q u)\Lambda^q(uu_x)-\gamma\int_{\mathbb{S}}\left(\Lambda^{q+1}u\right)\Lambda^{q+1}(uu_x)\\ &\quad+\gamma\int_{\mathbb{S}}(\Lambda^q u)\Lambda^q(uu_x)+\frac{\gamma}{2}\int_{\mathbb{S}}(\Lambda^q u_x)\Lambda^q\left(u_x^2\right)\\ &\quad-\beta\int_{\mathbb{S}}\left(\Lambda^{q+1}u\right)^2dx,\end{aligned} \tag{12}$$

where the Parseval equality

$$\begin{aligned}\int_{\mathbb{S}}(\Lambda^q u)\Lambda^q\partial_x^3\left(u^2\right)dx &= -2\int_{S}\left(\Lambda^{q+1}u\right)\Lambda^{q+1}(uu_x)\,dx\\ &\quad+2\int_{\mathbb{S}}(\Lambda^q u)\Lambda^q(uu_x)\,dx\end{aligned} \tag{13}$$

is used. We will estimate each of the terms on the right-hand side of (11). For the first and the third terms, using integration by parts, the Cauchy-Schwartz inequality, and Lemmas 3 and 4, we have

$$\begin{aligned}\int_{\mathbb{S}}(\Lambda^q u)\Lambda^q(uu_x)\,dx &= \int_{\mathbb{S}}(\Lambda^q u)\left[\Lambda^q(uu_x)-u\Lambda^q u_x\right]dx\\ &\quad+\int_{\mathbb{S}}(\Lambda^q u)u\Lambda^q(u_x)\,dx\\ &\le c\|u\|_{H^q}^2\|u_x\|_{L^\infty}+\frac{1}{2}\|u_x\|_{L^\infty}\|\Lambda^q u\|_{L^2}^2\\ &\le c\|u\|_{H^q}^2\|u_x\|_{L^\infty},\end{aligned} \tag{14}$$

where c only depends on q. Using the above estimate to the second term yields

$$\int_{\mathbb{S}}\left(\Lambda^{q+1}u\right)\Lambda^{q+1}(uu_x)\,dx \le c\|u\|_{H^{q+1}}^2\|u_x\|_{L^\infty}. \tag{15}$$

For the fourth term, using Lemma 3 gives rise to

$$\begin{aligned}\int_{\mathbb{S}}(\Lambda^q u_x)\Lambda^q\left(u_x^2\right)dx &\le \|\Lambda^q u_x\|_{L^2}\left\|\Lambda^q u_x^2\right\|_{L^2}\\ &\le c\|u\|_{H^{q+1}}\left(\|u_x\|_{L^\infty}\|u_x\|_{H^q}\right)\\ &\le c\|u\|_{H^{q+1}}^2\|u_x\|_{L^\infty}.\end{aligned} \tag{16}$$

It follows from (12)–(16) that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}}\left((\Lambda^q u)^2+(\Lambda^q u_x)^2\right)dx \le c\left(\|\gamma u_x\|_{L^\infty}+1\right)\|u\|_{H^{q+1}}^2, \tag{17}$$

which leads to

$$\begin{aligned}&\int_{\mathbb{S}}\left((\Lambda^q u)^2+(\Lambda^q u_x)^2\right)dx-\int_{\mathbb{S}}\left((\Lambda^q u_0)^2+(\Lambda^q u_{0x})^2\right)dx\\ &\quad\le c\int_0^t\left(\|\gamma u_x\|_{L^\infty}+1\right)\|u\|_{H^{q+1}}^2ds.\end{aligned} \tag{18}$$

It completes the proof of the lemma. □

Proof of Theorem 2. Applying (10) with $q = r-1$, we have

$$\|u\|_{H^r(\mathbb{S})}^2 \le \|u_0\|_{H^r(\mathbb{S})}^2+c\int_0^t\left(\|\gamma u_x\|_{L^\infty(\mathbb{S})}+1\right)\|u\|_{H^r(\mathbb{S})}^2ds. \tag{19}$$

It follows from (19) and Gronwall's inequality that

$$\|u\|_{H^r(\mathbb{S})}^2 \le \|u_0\|_{H^r(\mathbb{S})}^2\exp\left(c\int_0^t\left(\|\gamma u_x\|_{L^\infty(\mathbb{S})}+1\right)ds\right). \tag{20}$$

If there is a constant $M > 0$ such that $\|\gamma u_x\|_{L^\infty} \le M$ on $(0, T]$, then $\|u\|_{H^r(\mathbb{S})}^2$ does not blow up. It completes the proof of Theorem 2. □

Lemma 6 (see [17]). *Let $u(t)$ be the solution to (2) on $[0, T)$ with initial data $u_0 \in H^s(\mathbb{S})$, $s > 3/2$. Then there exists at least one point $\xi(t) \in \mathbb{S}$ with*

$$m(t) = \inf_{x\in\mathbb{S}}\left[u_x(x,t)\right] = u_x(t,\xi(t)). \tag{21}$$

The function $m(t)$ is almost everywhere differentiable on $(0, t)$ with

$$\frac{dm(t)}{dt} = u_{tx}(t,(\xi(t))). \tag{22}$$

Lemma 7 (see [21, 22]). *(i) For every $u \in H^1(\mathbb{S})$, one has*

$$\max_{x\in[0,1]}u^2(x) \le \frac{e+1}{2(e-1)}\|u\|_{H^1(\mathbb{S})}^2, \tag{23}$$

where the constant $(e+1)/2(e-1)$ is sharp.

(ii) For every $u \in H^3(\mathbb{S})$, one has

$$\max_{x\in[0,1]}u^2(x) \le c\|u\|_{H^1(\mathbb{S})}^2, \tag{24}$$

with the best possible constant c lying within the range $(1, 13/12]$. Moreover, the best constant c is $(e+1)/2(e-1)$.

Lemma 8 (see [2]). *If $f \in H^3(\mathbb{S})$ is such that $\int_{\mathbb{S}} f(x)dx = a_0/2$, then, for every $\varepsilon > 0$, one has*

$$\max_{x\in[0,1]}f^2(x) \le \frac{\varepsilon+2}{24}\int_{\mathbb{S}}f_x^2dx+\frac{\varepsilon+2}{4\varepsilon}a_0^2. \tag{25}$$

Moreover,

$$\max_{x\in[0,1]}f^2(x) \le \frac{\varepsilon+2}{24}\|f\|_{H^1}^2+\frac{\varepsilon+2}{4\varepsilon}a_0^2. \tag{26}$$

Lemma 9. *Let $u \in H^3(\mathbb{S})$ and $T > 0$ be the maximal existence time of the solution $u(t, x)$ to problem (4). Then it holds that*

(i) $\int_{\mathbb{S}} u(t,x)dx = \int_{\mathbb{S}} u_0(x)dx = \int_{\mathbb{S}} y(t,x)dx = \int_{\mathbb{S}} y_0(x)dx$,

(ii) $\|u\|_{H^1}^2 = \|u\|_{H^1}^2e^{-2\beta t}$.

Proof. The proof of (i) is similar to that of [2, lemma 3.6], so we omit it.

Multiplying u to both sides of (2) and integrating by parts, we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}}\left(u^2+u_x^2\right)dx=-\beta\int_{\mathbb{S}}\left(u^2+u_x^2\right)dx, \tag{27}$$

which yields

$$\|u\|_{H^1(\mathbb{S})}^2=\|u_0\|_{H^1(\mathbb{S})}^2e^{-2\beta t}. \tag{28}$$

It finishes the proof. □

Lemma 10 (see [29]). *Assume that a differential function $y(t)$ satisfies*

$$y'\le -Cy^2(t)+K, \tag{29}$$

with constants $C,K>0$. If the initial datum $y(0)=y_0<-\sqrt{K/C}$, then the solutions to (29) go to $-\infty$ in finite time.

We now present the first blow-up result.

Theorem 11. *Let $\beta<0$ and $u_0\in H^s(\mathbb{S})$, $s\ge 3$.*

(i) *If $0<\gamma<a/3$ and there is a $x_0\in\mathbb{S}$ such that*

$$\gamma u_0'(x_0)<-\beta-\sqrt{2}\left(\frac{\beta^2}{2}+\frac{\gamma(a-2\gamma)(e+1)}{4(e-1)}\|u_0\|_{H^1}^2e^{2\beta T}+4\gamma\omega\sqrt{\frac{e+1}{2(e-1)}}\|u_0\|_{H^1}e^{\beta T}\right)^{1/2}, \tag{30}$$

then the corresponding solution to (2) blows up in finite time.

(ii) *If $a/3<\gamma<a$ and there is a $x_0\in\mathbb{S}$ such that*

$$\gamma u_0'(x_0)<-\beta-\sqrt{2}\left(\frac{\beta^2}{2}+\frac{\gamma(a-\gamma)(e+1)}{8(e-1)}\|u_0\|_{H^1}^2e^{2\beta T}+4\gamma\omega\sqrt{\frac{e+1}{2(e-1)}}\|u_0\|_{H^1}e^{\beta T}\right)^{1/2}, \tag{31}$$

then the corresponding solution to (2) blows up in finite time.

(iii) *If $\gamma<0$ or $\gamma>a$ and there is a $x_0\in\mathbb{S}$ such that*

$$\gamma u_0'(x_0)<-\beta-\sqrt{2}\left(\frac{\beta^2}{2}+\frac{\gamma(\gamma-a)(e+1)}{4(e-1)}\|u_0\|_{H^1}^2e^{2\beta T}+4|\gamma|\omega\sqrt{\frac{e+1}{2(e-1)}}\|u_0\|_{H^1}e^{\beta T}\right)^{1/2}, \tag{32}$$

then the corresponding solution to (2) blows up in finite time.

Proof. Let $T>0$ be the maximal time of existence of the solution u to (2) with the initial data u_0. Note that $\partial_x^2G*f=G*f-f$. Differentiating (4) with respect to x and multiplying the obtained equation by γ, we get

$$\begin{aligned}\gamma u_{tx}=&-\frac{\gamma^2}{2}u_x^2-\gamma^2uu_{xx}+\frac{\gamma(a-\gamma)}{2}u^2+2\gamma\omega u-\gamma\beta u_x\\&-G*\left[\frac{(a-\gamma)\gamma}{2}u^2+\frac{\gamma^2}{2}u_x^2+2\gamma\omega u\right].\end{aligned} \tag{33}$$

Noting that $u_{xx}(t,\xi(t))=0$, for all $t\in[0,T)$, and defining $m(t)=\gamma u_x(t,\xi(t))+\beta=\inf_{x\in\mathbb{S}}(\gamma u_x(t,x)+\beta)$, we obtain

$$\begin{aligned}m'(t)=&-\frac{1}{2}m^2(t)+\frac{\beta^2}{2}+\frac{\gamma(a-\gamma)}{2}u^2+2\gamma\omega u\\&-G*\left[\frac{(a-\gamma)\gamma}{2}u^2+\frac{\gamma^2}{2}u_x^2+2\gamma\omega u\right].\end{aligned} \tag{34}$$

Next, we divide (34) into three cases to prove the theorem.

(i) The first is $0<\gamma<a/3$; note that $(a-3\gamma)\gamma/2\ge 0$ and $G*(u^2+(1/2)u_x^2)\ge(1/2)u^2$. Then, we have

$$\begin{aligned}&G*\left[\frac{(a-\gamma)\gamma}{2}u^2+\frac{\gamma^2}{2}u_x^2\right]\\&\quad=G*\left[\gamma^2u^2+\frac{\gamma^2}{2}u_x^2+\left(\frac{(a-\gamma)\gamma}{2}-\gamma^2\right)u^2\right]\\&\quad\ge\frac{\gamma^2}{2}u^2+G*\left(\frac{(a-3\gamma)\gamma}{2}u^2\right).\end{aligned} \tag{35}$$

Note that $G*(((a-3\gamma)\gamma/2)u^2)\ge 0$. Thus, we obtain

$$\begin{aligned}m'(t)\le&-\frac{1}{2}m^2(t)+\frac{\beta^2}{2}+\frac{\gamma(a-2\gamma)}{2}u^2\\&+2\gamma\omega u-G*(2\gamma\omega u).\end{aligned} \tag{36}$$

From Lemmas 8 and 9, we have

$$\max_{x\in\mathbb{S}}\left\{u^2\right\}\le\frac{e+1}{2(e-1)}\|u\|_{H^1}^2\le\frac{e+1}{2(e-1)}\|u_0\|_{H^1}^2e^{-2\beta T}. \tag{37}$$

Thus,

$$\|u\|_{L^\infty}\le\sqrt{\frac{e+1}{2(e-1)}}\|u_0\|_{H^1}e^{-\beta T}. \tag{38}$$

Note that $\|G\|_{L^1(\mathbb{S})}=1$. Using Young's inequality, we get

$$|G*u|\le\|u\|_{L^\infty}\|G\|_{L^1(\mathbb{S})}\le\sqrt{\frac{e+1}{2(e-1)}}\|u_0\|_{H^1}e^{-\beta T}. \tag{39}$$

Therefore, we deduce

$$\begin{aligned}m'(t)\le&-\frac{1}{2}m^2(t)+\frac{\beta^2}{2}+\frac{\gamma(a-2\gamma)}{2}\frac{e+1}{2(e-1)}\\&\times\|u_0\|_{H^1}^2e^{-2\beta T}+4\gamma\omega\sqrt{\frac{e+1}{2(e-1)}}\|u_0\|_{H^1}e^{-\beta T},\end{aligned} \tag{40}$$

which results in

$$m'(t) \le -\frac{1}{2}m^2(t) + K, \tag{41}$$

with

$$K = \frac{\beta^2}{2} + \frac{\gamma(a-2\gamma)}{2}\frac{e+1}{2(e-1)}\|u_0\|_{H^1}^2 e^{-2\beta T} + 4\gamma\omega\sqrt{\frac{e+1}{2(e-1)}}\|u_0\|_{H^1} e^{-\beta T}. \tag{42}$$

Note that, from Lemma 10, if $m(0) < -\sqrt{2K}$, then there exists T, such that $\lim_{t\uparrow T} m(t) = -\infty$. Applying Theorem 2, the solution u to (2) does not exist globally in time.

(ii) The second is $a/3 \le \gamma \le a$. Similar to case (i), we have $\gamma(3\gamma - a)/4 \ge 0$ and

$$G * \left[\frac{(a-\gamma)\gamma}{2}u^2 + \frac{\gamma^2}{2}u_x^2\right] \ge \frac{(a-\gamma)\gamma}{4}u^2 + G * \left(\frac{\gamma(3\gamma-a)}{4}u_x^2\right). \tag{43}$$

Therefore, we obtain

$$m'(t) \le -\frac{1}{2}m^2(t) + \frac{\beta^2}{2} + \frac{\gamma(a-\gamma)}{4}u^2 + 2\gamma\omega u - G * (2\gamma\omega u). \tag{44}$$

Using (37)–(39), we get

$$m'(t) \le -\frac{1}{2}m^2(t) + \frac{\beta^2}{2} + \frac{\gamma(a-\gamma)}{4}\frac{e+1}{2(e-1)} \times \|u_0\|_{H^1}^2 e^{-2\beta T} + 4\gamma\omega\sqrt{\frac{e+1}{2(e-1)}}\|u_0\|_{H^1} e^{-\beta T}, \tag{45}$$

which results in

$$m'(t) \le -\frac{1}{2}m^2(t) + K, \tag{46}$$

with

$$K = \frac{\beta^2}{2} + \frac{\gamma(a-\gamma)}{4}\frac{e+1}{2(e-1)}\|u_0\|_{H^1}^2 e^{-2\beta T} + 4\gamma\omega\sqrt{\frac{e+1}{2(e-1)}}\|u_0\|_{H^1} e^{-\beta T}. \tag{47}$$

Note that, from Lemma 10, if $m(0) < -\sqrt{2K}$, then there exists T, such that $\lim_{t\uparrow T} m(t) = -\infty$. Repeating the proof of (i), we conclude that the solution u blows up in finite time.

(iii) The third is $\gamma < 0$ or $\gamma > a$; note that $(a-\gamma)\gamma/2 < 0$. From Young's inequality, we get for all $t \in [0, T)$

$$\left|G * u^2\right| \le \|u\|_{L^\infty}^2 \|G\|_{L^1} \le \frac{e+1}{2(e-1)}\|u_0\|_{H^1}^2 e^{-2\beta T}. \tag{48}$$

Using (37)–(39), we get

$$m'(t) \le -\frac{1}{2}m^2(t) + \frac{\beta^2}{2} + \frac{\gamma(\gamma-a)}{2}\frac{e+1}{2(e-1)}\|u_0\|_{H^1}^2 e^{-2\beta T} + 4|\gamma|\omega\sqrt{\frac{e+1}{2(e-1)}}\|u_0\|_{H^1} e^{-\beta T}, \tag{49}$$

which results in

$$m'(t) \le -\frac{1}{2}m^2(t) + K, \tag{50}$$

with

$$K = \frac{\beta^2}{2} + \frac{\gamma(\gamma-a)}{2}\frac{e+1}{2(e-1)}\|u_0\|_{H^1}^2 e^{-2\beta T} + 4|\gamma|\omega\sqrt{\frac{e+1}{2(e-1)}}\|u_0\|_{H^1} e^{-\beta T}. \tag{51}$$

Note that, from Lemma 10, if $m(0) < -\sqrt{2K}$, then there exists T, such that $\lim_{t\uparrow T} m(t) = -\infty$. Repeating the proof of (i), we deduce that the solution u blows up in finite time. It finishes the proof of the theorem. □

Next, we give the second blow-up result.

Theorem 12. *Let $\beta < 0$ and $u_0 \in H^s(\mathbb{S})$, $s \ge 3$, and $\int_{\mathbb{S}} u_0 dx = a_0/2$.*

(i) *If $0 < \gamma < a/3$ and for all $\epsilon > 0$ there is a $x_0 \in \mathbb{S}$ such that*

$$\gamma u_0'(x_0) < -\beta - \sqrt{2}\left(\frac{\beta^2}{2} + \frac{\gamma(a-2\gamma)}{2} \times \left(\frac{\epsilon+2}{24}\|u_0\|_{H^1}^2 e^{-2\beta T} + \frac{\epsilon+2}{4\epsilon}a_0^2\right) + 4\gamma\omega \times \sqrt{\frac{\epsilon+2}{24}\|u_0\|_{H^1} e^{-2\beta T} + \frac{\epsilon+2}{4\epsilon}a_0^2}\right)^{1/2}, \tag{52}$$

then the corresponding solution to (2) blows up in finite time.

(ii) *If $a/3 < \gamma < a$ and for all $\epsilon > 0$ there is a $x_0 \in \mathbb{S}$ such that*

$$\gamma u_0'(x_0) < -\beta - \sqrt{2}\left(\frac{\beta^2}{2} + \frac{\gamma(a-\gamma)}{4}\left(\frac{\epsilon+2}{24}\|u_0\|_{H^1}^2 e^{-2\beta T} + \frac{\epsilon+2}{4\epsilon}a_0^2\right) + 4\gamma\omega \times \sqrt{\frac{\epsilon+2}{24}\|u_0\|_{H^1} e^{-2\beta T} + \frac{\epsilon+2}{4\epsilon}a_0^2}\right)^{1/2}, \tag{53}$$

then the corresponding solution to (2) blows up in finite time.

(iii) *If $\gamma < 0$ or $\gamma > a$ and for all $\epsilon > 0$ there is a $x_0 \in \mathbb{S}$ such that*

$$\gamma u_0'(x_0) < -\beta - \sqrt{2}\left(\frac{\beta^2}{2} + \frac{\gamma(\gamma-a)}{2}\left(\frac{\epsilon+2}{24}\|u_0\|_{H^1}^2 e^{-2\beta T} + \frac{\epsilon+2}{4\epsilon}a_0^2\right) + 4|\gamma|\omega \times \sqrt{\frac{\epsilon+2}{24}\|u_0\|_{H^1} e^{-2\beta T} + \frac{\epsilon+2}{4\epsilon}a_0^2}\right)^{1/2}, \tag{54}$$

then the corresponding solution to (2) *blows up in finite time.*

Proof. Similar to Theorem 11, we divide (34) into three cases to prove the theorem.

(i) The first is $0 < \gamma < a/3$; note that $(a-3\gamma)\gamma/2 \geq 0$ and $G * (u^2 + (1/2)u_x^2) \geq (1/2)u^2$. Then, we have $G * ((a-3\gamma)\gamma/2)u^2 \geq 0$.

From Lemmas 8 and 9, we have

$$\begin{aligned}\max_{x\in S}\{u^2\} &\leq \frac{\epsilon+2}{24}\|u\|_{H^1}^2 + \frac{\epsilon+2}{4\epsilon}a_0^2 \\ &\leq \frac{\epsilon+2}{24}\|u_0\|_{H^1}^2 e^{-2\beta T} + \frac{\epsilon+2}{4\epsilon}a_0^2.\end{aligned} \tag{55}$$

Thus,

$$\|u\|_{L^\infty} \leq \sqrt{\frac{\epsilon+2}{24}\|u_0\|_{H^1} e^{-2\beta T} + \frac{\epsilon+2}{4\epsilon}a_0^2}. \tag{56}$$

Note that $\|G\|_{L^1(\mathbb{S})} = 1$. Using Young's inequality, we get

$$\begin{aligned}|G * u| &\leq \|u\|_{L^\infty}\|G\|_{L^1(\mathbb{S})} \\ &\leq \sqrt{\frac{\epsilon+2}{24}\|u_0\|_{H^1} e^{-2\beta T} + \frac{\epsilon+2}{4\epsilon}a_0^2}.\end{aligned} \tag{57}$$

Therefore, we deduce

$$\begin{aligned}m'(t) \leq -\frac{1}{2}m^2(t) + \frac{\beta^2}{2} + \frac{\gamma(a-2\gamma)}{2} &\times \left(\frac{\epsilon+2}{24}\|u_0\|_{H^1}^2 e^{-2\beta T} + \frac{\epsilon+2}{4\epsilon}a_0^2\right) \\ &+ 4\gamma\omega\sqrt{\frac{\epsilon+2}{24}\|u_0\|_{H^1} e^{-2\beta T} + \frac{\epsilon+2}{4\epsilon}a_0^2},\end{aligned} \tag{58}$$

which results in

$$m'(t) \leq -\frac{1}{2}m^2(t) + K \tag{59}$$

with

$$\begin{aligned}K = \frac{\beta^2}{2} + \frac{\gamma(a-2\gamma)}{2}\left(\frac{\epsilon+2}{24}\|u_0\|_{H^1}^2 e^{-2\beta T} + \frac{\epsilon+2}{4\epsilon}a_0^2\right) &\\ + 4\gamma\omega\sqrt{\frac{\epsilon+2}{24}\|u_0\|_{H^1} e^{-2\beta T} + \frac{\epsilon+2}{4\epsilon}a_0^2}.&\end{aligned} \tag{60}$$

Note that, from the lemma, if $m(0) < -\sqrt{2K}$, then there exists T, such that $\lim_{t\uparrow T} m(t) = -\infty$. Applying Theorem 2, the solution u blows up in finite time.

(ii) The second is $a/3 \leq \gamma \leq a$. Similar to case (i), we have $\gamma(3\gamma-a)/4 \geq 0$ and

$$\begin{aligned}G * \left[\frac{(a-\gamma)\gamma}{2}u^2 + \frac{\gamma^2}{2}u_x^2\right] &\geq \frac{(a-\gamma)\gamma}{4}u^2 \\ &+ G * \left(\frac{\gamma(3\gamma-a)}{4}u_x^2\right).\end{aligned} \tag{61}$$

Using (55)–(57), we get

$$\begin{aligned}m'(t) \leq -\frac{1}{2}m^2(t) + \frac{\beta^2}{2} + \frac{\gamma(a-\gamma)}{4} &\times \left(\frac{\epsilon+2}{24}\|u_0\|_{H^1}^2 e^{-2\beta T} + \frac{\epsilon+2}{4\epsilon}a_0^2\right) \\ &+ 4\gamma\omega\sqrt{\frac{\epsilon+2}{24}\|u_0\|_{H^1} e^{-2\beta T} + \frac{\epsilon+2}{4\epsilon}a_0^2},\end{aligned} \tag{62}$$

which results in

$$m'(t) \leq -\frac{1}{2}m^2(t) + K \tag{63}$$

with

$$\begin{aligned}K = \frac{\beta^2}{2} + \frac{\gamma(a-\gamma)}{4}\frac{\epsilon+2}{24}\|u_0\|_{H^1}^2 e^{-2\beta T} &\\ + \frac{\epsilon+2}{4\epsilon}a_0^2 + 4\gamma\omega\sqrt{\frac{\epsilon+2}{24}\|u_0\|_{H^1} e^{-2\beta T} + \frac{\epsilon+2}{4\epsilon}a_0^2}.&\end{aligned} \tag{64}$$

Following the proof of (i) in Theorem 11, we derive that the solution u blows up in finite time.

(iii) The third is $\gamma < 0$ or $\gamma > a$; note that $(a-\gamma)\gamma/2 < 0$. From Young's inequality, we get for all $t \in [0, T)$

$$\left|G * u^2\right| \leq \|u\|_{L^\infty}^2\|G\|_{L^1} \leq \frac{\epsilon+2}{24}\|u_0\|_{H^1}^2 e^{-2\beta T} + \frac{\epsilon+2}{4\epsilon}a_0^2. \tag{65}$$

Using (55)–(57) and (65), we get

$$\begin{aligned}m'(t) \leq -\frac{1}{2}m^2(t) + \frac{\beta^2}{2} + \frac{\gamma(\gamma-a)}{2} &\times \left(\frac{\epsilon+2}{24}\|u_0\|_{H^1}^2 e^{-2\beta T} + \frac{\epsilon+2}{4\epsilon}a_0^2\right) \\ &+ 4|\gamma|\omega\sqrt{\frac{\epsilon+2}{24}\|u_0\|_{H^1} e^{-2\beta T} + \frac{\epsilon+2}{4\epsilon}a_0^2},\end{aligned} \tag{66}$$

which results in

$$m'(t) \leq -\frac{1}{2}m^2(t) + K \tag{67}$$

with

$$\begin{aligned}K = \frac{\beta^2}{2} + \frac{\gamma(\gamma-a)}{2}\frac{\epsilon+2}{24}\|u_0\|_{H^1}^2 e^{-2\beta T} &\\ + \frac{\epsilon+2}{4\epsilon}a_0^2 + 4|\gamma|\omega\sqrt{\frac{\epsilon+2}{24}\|u_0\|_{H^1} e^{-2\beta T} + \frac{\epsilon+2}{4\epsilon}a_0^2}.&\end{aligned} \tag{68}$$

Following the proof of (i) in Theorem 11, we obtain that the solution u blows up in finite time. It finishes the proof of the theorem. □

Next, we give the third blow-up result.

Theorem 13. *Assume that $\beta < 0$ and $u_0 \in H^s(\mathbb{S})$, $s \geq 3$, and $\| u_0\|_{H^1} \neq 0$.*

(i) *If $0 < \gamma < a/3$ is such that*

$$\int_{\mathbb{S}} \gamma u_{0x}^3 dx < -2\beta \|u_0\|_{H^1}^2 e^{-2\beta T} - \sqrt{\frac{2}{3}} \left(\frac{\beta^2}{2} \|u_0\|_{H^1}^4 e^{-4\beta T} + \frac{3\gamma (a-2\gamma)(e+1)}{4(e-1)} \times \|u_0\|_{H^1}^6 e^{-6\beta T} + 12\omega\gamma \times \sqrt{\frac{e+1}{2(e-1)}} \|u_0\|_{H^1}^5 e^{-5\beta T} \right)^{1/2}, \tag{69}$$

then the corresponding solution to (2) *blows up in finite time.*

(ii) *If $a/3 < \gamma < a$ is such that*

$$\int_{\mathbb{S}} \gamma u_{0x}^3 dx < -2\beta \|u_0\|_{H^1}^2 e^{-2\beta T} - \sqrt{\frac{2}{3}} \left(\frac{\beta^2}{2} \|u_0\|_{H^1}^4 e^{-4\beta T} + \frac{3\gamma (a-\gamma)(e+1)}{8(e-1)} \times \|u_0\|_{H^1}^6 e^{-6\beta T} + 12\omega\gamma \times \sqrt{\frac{e+1}{2(e-1)}} \|u_0\|_{H^1}^5 e^{-5\beta T} \right)^{1/2}, \tag{70}$$

then the corresponding solution to (2) *blows up in finite time.*

(iii) *If $\gamma > a$ or $\gamma < 0$ is such that*

$$\int_{\mathbb{S}} \gamma u_{0x}^3 dx < -2\beta \|u_0\|_{H^1}^2 e^{-2\beta T} - \sqrt{\frac{2}{3}} \left(\frac{\beta^2}{2} \|u_0\|_{H^1}^4 e^{-4\beta T} + \frac{3\gamma (\gamma-a)(e+1)}{8(e-1)} \times \|u_0\|_{H^1}^6 e^{-6\beta T} + 12\omega |\gamma| \times \sqrt{\frac{e+1}{2(e-1)}} \|u_0\|_{H^1}^5 e^{-5\beta T} \right)^{1/2}, \tag{71}$$

then the corresponding solution to (2) *blows up in finite time.*

Proof. Let $T > 0$ be the maximal time of existence of the solution u to (2) with the initial data u_0. Applying $\gamma u_x^2 \partial_x$ to both sides of (2) and integrating by parts, we get

$$\frac{d}{dt} \int_{\mathbb{S}} \gamma u_x^3 dx = -\frac{3\gamma^2}{2} \int_{\mathbb{S}} u_x^4 - 3\beta \int_{\mathbb{S}} \gamma u_x^3 dx + 3 \frac{(a-\gamma)\gamma}{2} \times \int_{\mathbb{S}} u_x^2 u^2 dx + 6\omega\gamma \int_{\mathbb{S}} u_x^2 u\, dx + 3 \times \int_{\mathbb{S}} u_x^2 G * \left(\frac{(a-\gamma)\gamma}{2} u^2 + \frac{\gamma^2}{2} u_x^2 + 2\omega\gamma u \right). \tag{72}$$

Since

$$\left| \int_{\mathbb{S}} u_x^3 dx \right| \leq \left(\int_{\mathbb{S}} u_x^4 dx \right)^{1/2} \left(\int_{\mathbb{S}} u_x^2 dx \right)^{1/2}, \tag{73}$$

thus

$$\int_{\mathbb{S}} u_x^4 dx \geq \frac{\left(\int_{\mathbb{S}} u_x^3 dx\right)^2}{\int_{\mathbb{S}} u_x^2 dx} \geq \frac{\left(\int_{\mathbb{S}} u_x^3 dx\right)^2}{\|u\|_{H^1}^2}. \tag{74}$$

Therefore, we obtain

$$\frac{d}{dt} \int_{\mathbb{S}} \gamma u_x^3 dx \leq -\frac{3}{2\|u_0\|_{H^1}^2 e^{-2\beta T}} \left(\int_{\mathbb{S}} \gamma u_x^3 dx \right)^2 - 3\beta \int_{\mathbb{S}} \gamma u_x^3 dx + 3 \frac{(a-\gamma)\gamma}{2} \int_{\mathbb{S}} u_x^2 u^2 dx + 6\omega\gamma \int_{\mathbb{S}} u_x^2 u\, dx + 3 \int_{\mathbb{S}} u_x^2 G * \left(\frac{(a-\gamma)\gamma}{2} u^2 + \frac{\gamma^2}{2} u_x^2 + 2\omega\gamma u \right) dx. \tag{75}$$

Next, we divide (75) into three cases to prove the theorem.

(i) The first is $0 < \gamma \leq a/3$; from case (i) of Theorem 11, we know that $\gamma(a - 3\gamma) \geq 0$ and $G * (((a - 3\gamma)\gamma/2)u^2) \geq 0$.

From Holder's inequality and Yong's inequality, we have

$$\left| \int_{\mathbb{S}} u_x^2 u^2 dx \right| \leq \|u\|_{L^\infty}^2 \int_{\mathbb{S}} u_x^2 dx \leq \frac{e+1}{2(e-1)} \|u\|_{H^1}^4 \leq \frac{e+1}{2(e-1)} \|u_0\|_{H^1}^4 e^{-4\beta T},$$

$$\left| \int_{\mathbb{S}} u_x^2 u\, dx \right| \leq \|u\|_{L^\infty} \int_{\mathbb{S}} u_x^2 dx \leq \sqrt{\frac{e+1}{2(e-1)}} \|u\|_{H^1}^3 \leq \sqrt{\frac{e+1}{2(e-1)}} \|u_0\|_{H^1}^3 e^{-3\beta T},$$

$$\left|\int_{\mathbb{S}} u_x^2 G * u\,dx\right| \le \|G * u\|_{L^\infty} \int_{\mathbb{S}} u_x^2 dx \le \sqrt{\frac{e+1}{2(e-1)}}\|u\|_{H^1}^3 \le \sqrt{\frac{e+1}{2(e-1)}}\|u_0\|_{H^1}^3 e^{-3\beta T}. \tag{76}$$

Using (76) and (75), it yields

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}} \gamma u_x^3 dx \le{}& -\frac{3}{2\|u_0\|_{H^1}^2 e^{-2\beta T}} \times \left(\int_{\mathbb{S}} \gamma u_x^3 dx + \beta\|u_0\|_{H^1}^2 e^{-2\beta T}\right)^2 + \frac{3\beta^2}{2} \\ &\times \|u_0\|_{H^1}^2 e^{-2\beta T} + 3\frac{(a-2\gamma)\gamma}{2}\frac{e+1}{2(e-1)} \\ &\times \|u_0\|_{H^1}^4 e^{-4\beta T} + 12\omega\gamma\sqrt{\frac{e+1}{2(e-1)}}\|u_0\|_{H^1}^3 e^{-3\beta T}.\end{aligned} \tag{77}$$

Setting

$$m(t) = \int_{\mathbb{S}} \gamma u_x^3 dx + \beta\|u_0\|_{H^1}^2 e^{-2\beta T}, \tag{78}$$

$$K = \frac{3\beta^2}{2}\|u_0\|_{H^1}^2 e^{-2\beta T} + 3\frac{(a-2\gamma)\gamma}{2}\frac{e+1}{2(e-1)}\|u_0\|_{H^1}^4 e^{-4\beta T} + 12\omega\gamma\sqrt{\frac{e+1}{2(e-1)}}\|u_0\|_{H^1}^3 e^{-3\beta T}, \tag{79}$$

we have

$$\frac{d}{dt}m(t) \le -\frac{3}{2\|u_0\|_{H^1}^2 e^{-2\beta T}} m^2(t) + K. \tag{80}$$

Note that, from the lemma, if $m(0) < -\sqrt{2K/3}\|u_0\|_{H^1} e^{-\beta T}$, similar to the proof in case (i) of Theorem 11, we conclude that the corresponding solution will blow up in finite time.

(ii) The second is $a/3 < \gamma \le a$; from case (ii) of the theorem, we have $\gamma(3\gamma - a) \ge 0$ and $G * ((\gamma(3\gamma - a)/4)u_x^2) \ge 0$.

Using (76) and (75), it yields

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}} \gamma u_x^3 dx \le{}& -\frac{3}{2\|u_0\|_{H^1}^2 e^{-2\beta T}} \times \left(\int_{\mathbb{S}} \gamma u_x^3 dx + \beta\|u_0\|_{H^1}^2 e^{-2\beta T}\right)^2 + \frac{3\beta^2}{2} \\ &\times \|u_0\|_{H^1}^2 e^{-2\beta T} + 3\frac{(a-\gamma)\gamma}{4}\frac{e+1}{2(e-1)}\|u_0\|_{H^1}^4 \\ &\times e^{-4\beta T} + 12\omega\gamma\sqrt{\frac{e+1}{2(e-1)}}\|u_0\|_{H^1}^3 e^{-3\beta T}.\end{aligned} \tag{81}$$

Setting

$$m(t) = \int_{\mathbb{S}} \gamma u_x^3 dx + \beta\|u_0\|_{H^1}^2 e^{-2\beta T},$$

$$K = \frac{3\beta^2}{2}\|u_0\|_{H^1}^2 e^{-2\beta T} + 3\frac{(a-\gamma)\gamma}{4}\frac{e+1}{2(e-1)}\|u_0\|_{H^1}^4 e^{-4\beta T} + 12\omega\gamma\sqrt{\frac{e+1}{2(e-1)}}\|u_0\|_{H^1}^3 e^{-3\beta T}, \tag{82}$$

we have

$$\frac{d}{dt}m(t) \le -\frac{3}{2\|u_0\|_{H^1}^2 e^{-2\beta T}} m^2(t) + K. \tag{83}$$

Note that, from the lemma, if $m(0) < -\sqrt{2K/3}\|u_0\|_{H^1} e^{-\beta T}$, similar to the proof in case (i) of Theorem 11, we derive that the corresponding solution will blow up in finite time.

(iii) The third is $\gamma > a$ or $\gamma < 0$; note that $(a-\gamma)\gamma/2 < 0$.

Using (76) and (75), it yields

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}} \gamma u_x^3 dx \le{}& -\frac{3}{2\|u_0\|_{H^1}^2 e^{-2\beta T}} \times \left(\int_{\mathbb{S}} \gamma u_x^3 dx + \beta\|u_0\|_{H^1}^2 e^{-2\beta T}\right)^2 + \frac{3\beta^2}{2} \\ &\times \|u_0\|_{H^1}^2 e^{-2\beta T} + 3\frac{(\gamma-a)\gamma}{4}\frac{e+1}{2(e-1)}\|u_0\|_{H^1}^4 \\ &\times e^{-4\beta T} + 12\omega|\gamma|\sqrt{\frac{e+1}{2(e-1)}}\|u_0\|_{H^1}^3 e^{-3\beta T}.\end{aligned} \tag{84}$$

Setting

$$\begin{aligned}m(t) &= \int_{\mathbb{S}} \gamma u_x^3 dx + \beta\|u_0\|_{H^1}^2 e^{-2\beta T}, \\ K &= \frac{3\beta^2}{2}\|u_0\|_{H^1}^2 e^{-2\beta T} + 3\frac{(\gamma-a)\gamma}{4}\frac{e+1}{2(e-1)} \times \|u_0\|_{H^1}^4 e^{-4\beta T} + 12\omega|\gamma| \times \sqrt{\frac{e+1}{2(e-1)}}\|u_0\|_{H^1}^3 e^{-3\beta T},\end{aligned} \tag{85}$$

we have

$$\frac{d}{dt}m(t) \le -\frac{3}{2\|u_0\|_{H^1}^2 e^{-2\beta T}} m^2(t) + K. \tag{86}$$

Note that, from the lemma, if $m(0) < -\sqrt{2K/3}\|u_0\|_{H^1} e^{-\beta T}$, similar to the proof in case (i) of Theorem 11, we deduce that the corresponding solution will blow up in finite time. It completes the proof of Theorem 13. □

Finally, we give the fourth blow-up result.

Theorem 14. *Let $\beta \geq 0$ and $u_0 \in H^s$, $s \geq 3$. Assume that $u_0 \neq 0$ and $\int_{\mathbb{S}} u_0 dx = 0$. If γ and ω satisfy one of the following conditions:*

(i) $a\sinh(1/2)/(6+\sinh(1/2)) < \gamma \leq a$ *and* $\omega < (6\sqrt{3}/169)(3\gamma/\sinh(1/2) - (a-\gamma)/2)\|u_0\|_{H^1} e^{-4\beta T}$,

(ii) $\gamma > a$ *and* $\omega < (6\sqrt{3}/169)(3\gamma/\sinh(1/2) + (a-\gamma)/2)\|u_0\|_{H^1} e^{-4\beta T}$,

(iii) $\gamma < a\sinh(1/2)/(\sinh(1/2) - 6)$ *and* $\omega < (6\sqrt{3}/169|\gamma|)(3\gamma^2/\sinh(1/2) + \gamma(a-\gamma)/2)\|u_0\|_{H^1} e^{-4\beta T}$,

then the corresponding solution of (2) *blows up in finite time.*

Proof. Supposing that the statement is not correct, then the solution of (2) exists globally in time.

Thus, applying $\gamma u_x^2 \partial_x$ to both sides of (4) and integrating by parts, we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}} \gamma u_x^3 dx = &-\frac{3\gamma^2}{2}\int_{\mathbb{S}} u_x^4 - 3\beta\int_{\mathbb{S}} \gamma u_x^3 dx + 3\frac{(a-\gamma)\gamma}{2} \\ &\times \int_{\mathbb{S}} u_x^2 u^2 dx + 6\omega\gamma\int_{\mathbb{S}} u_x^2 u\, dx \\ &- 3\int_{\mathbb{S}} u_x^2 G * \frac{(a-\gamma)\gamma}{2} u^2 dx \\ &- 3\int_{\mathbb{S}} u_x^2 G * \frac{\gamma^2}{2} u_x^2 dx - 6\omega\gamma\int_{\mathbb{S}} u_x^2 G * u\, dx.\end{aligned} \tag{87}$$

Due to $\|u\|_{L^\infty}^2 \leq (1/12)\int_S u_x^2 dx \leq (1/12)\|u\|_{H^1}^2$ and $\int_S u_x^2 dx \geq (12/13)\|u\|_{H^1}^2$ (see [29]), we get

$$\begin{aligned}&\int_{\mathbb{S}} u^2 u_x^2 dx \leq \max\{u^2\}\int_{\mathbb{S}} u_x^2 dx \leq \frac{1}{12}\left(\int_{\mathbb{S}} u_x^2 dx\right)^2, \\ &\int_{\mathbb{S}} u u_x^2 dx \leq \max\{|u|\}\int_{\mathbb{S}} u_x^2 dx \leq \frac{\sqrt{3}}{6}\left(\int_{\mathbb{S}} u_x^2 dx\right)^{3/2}, \\ &\int_{\mathbb{S}} u_x^2 G * \left(u^2\right) dx \leq \|G\|_{L^1}\max\{u^2\} \\ &\qquad \times \int_{\mathbb{S}} u_x^2 dx \leq \frac{1}{12}\left(\int_{\mathbb{S}} u_x^2 dx\right)^2, \\ &\int_{\mathbb{S}} u_x^2 G * u\, dx \leq \|G\|_{L^1}\max\{u\}\int_{\mathbb{S}} u_x^2 dx \leq \frac{\sqrt{3}}{6}\left(\int_{\mathbb{S}} u_x^2 dx\right)^{3/2}.\end{aligned} \tag{88}$$

Since $1/2\sinh(1/2) \leq G(x) \leq \cosh(1/2)/2\sinh(1/2)$, we have

$$\begin{aligned}\int_{\mathbb{S}} u_x^2 G * \left(u_x^2\right) dx &\geq \frac{1}{2\sinh(1/2)}\left(\int_{\mathbb{S}} u_x^2 dx\right)^2 \\ &\geq \frac{72}{169\sinh(1/2)}\|u\|_{H^1}^4.\end{aligned} \tag{89}$$

Therefore, we deduce

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}} \gamma u_x^3 dx = &-\frac{3\gamma^2}{2}\int_{\mathbb{S}} u_x^4 - 3\beta\int_{\mathbb{S}} \gamma u_x^3 dx + 2\sqrt{3}\omega|\gamma| \\ &\times\left(\int_{\mathbb{S}} u_x^2 dx\right)^{3/2} \\ &+\left(\frac{(a-\gamma)\gamma}{8} - \frac{3\gamma^2}{4\sinh(1/2)}\right)\left(\int_{\mathbb{S}} u_x^2 dx\right)^2.\end{aligned} \tag{90}$$

Next, we divide (90) three cases to prove the theorem.

(i) The first is $a\sinh(1/2)/(6+\sinh(1/2)) < \gamma \leq a$. Note that $\gamma(a-\gamma)/8 > 0$ and $\gamma(a-\gamma)/8 - 3\gamma^2/4\sinh(1/2) < 0$. From the condition of Theorem 14, we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}} \gamma u_x^3 dx \leq &-\frac{3\gamma^2}{2}\int_{\mathbb{S}} u_x^4 - 3\beta\int_{\mathbb{S}} \gamma u_x^3 dx + 2\sqrt{3}\omega\gamma \\ &\times\|u_0\|_{H^1}^3 + \frac{36}{169}\left(\frac{(a-\gamma)\gamma}{2} - \frac{3\gamma^2}{\sinh(1/2)}\right) \\ &\times\|u_0\|_{H^1}^4 e^{-4\beta T} \\ \leq &-\frac{3\gamma^2}{2}\int_{\mathbb{S}} u_x^4 - 3\beta\int_{\mathbb{S}} \gamma u_x^3 dx - \delta,\end{aligned} \tag{91}$$

where $\delta = 2\sqrt{3}\omega\gamma\|u_0\|_{H^1}^3 + (36/169)((a-\gamma)\gamma/2 - 3\gamma^2/\sinh(1/2))\|u_0\|_{H^1}^4 e^{-4\beta T} > 0$. From Holder's inequality, we obtain

$$\int_{\mathbb{S}} \gamma u_x^3 dx \leq \gamma\left(\int_{\mathbb{S}} u_x^3 dx\right)^{3/4}. \tag{92}$$

Thus, we have

$$\frac{d}{dt}\int_{\mathbb{S}} \gamma u_x^3 dx \leq -\frac{3\gamma^{2/3}}{2}\left(\int_{\mathbb{S}} \gamma u_x^3 dx\right)^{4/3} - 3\beta\int_{\mathbb{S}} \gamma u_x^3 dx - \delta. \tag{93}$$

From (93), we have

$$\frac{d}{dt}\int_{\mathbb{S}} \gamma u_x^3 dx \leq -3\beta\int_{\mathbb{S}} \gamma u_x^3 dx - \delta. \tag{94}$$

Solving (94), we obtain

$$\int_{\mathbb{S}} \gamma u_x^3 dx \leq -\frac{\delta}{3\beta} + Ce^{-3\beta t}, \tag{95}$$

where $C < 0$ is constant which implies that there is a $t_0 \geq 0$ such that

$$\int_{\mathbb{S}} \gamma u_x^3 dx < 0, \quad t \geq t_0. \tag{96}$$

On the other hand,

$$\frac{d}{dt}\int_{\mathbb{S}} \gamma u_x^3 dx \leq -\frac{3\gamma^{2/3}}{2}\left(\int_{\mathbb{S}} \gamma u_x^3 dx\right)^{4/3} - 3\beta\int_{\mathbb{S}} \gamma u_x^3 dx. \tag{97}$$

Solving (97), we obtain

$$0 > \left(\int_{\mathbb{S}} \gamma u_x^3 dx\right)^{-1} \geq \left(-\frac{\gamma^{2/3}}{2\beta} + C_1 e^{\beta t}\right)^3, \tag{98}$$

where $C_1 \geq 0$ is constant, which leads to a contradiction as t becomes large enough. Therefore, case (i) is proved.

(ii) The second is $\gamma \geq a$. Note that $\gamma(a-\gamma)/8 < 0$ and $-\gamma(a-\gamma)/8 - 3\gamma^2/4\sinh(1/2) < 0$. From the condition of the theorem, we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}} \gamma u_x^3 dx &\leq -\frac{3\gamma^2}{2}\int_{\mathbb{S}} u_x^4 - 3\beta\int_{\mathbb{S}} \gamma u_x^3 dx + 2\sqrt{3}\omega\gamma \\ &\quad \times \|u_0\|_{H^1}^3 + \frac{36}{169}\left(-\frac{(a-\gamma)\gamma}{2} - \frac{3\gamma^2}{\sinh(1/2)}\right) \\ &\quad \times \|u_0\|_{H^1}^4 e^{-4\beta T} \\ &\leq -\frac{3\gamma^2}{2}\int_{\mathbb{S}} u_x^4 - 3\beta\int_{\mathbb{S}} \gamma u_x^3 dx - \delta,\end{aligned} \tag{99}$$

where $\delta = 2\sqrt{3}\omega\gamma\|u_0\|_{H^1}^3 + (36/169)(-(a-\gamma)\gamma/2 - 3\gamma^2/\sinh(1/2))\|u_0\|_{H^1}^4 e^{-4\beta T} > 0$. Repeating the arguments in the proof of case (i), we deduce that the corresponding solution blows up.

(iii) The third is $\gamma < a\sinh(1/2)/(\sinh(1/2) - 6)$. Note that $\gamma(a-\gamma)/8 < 0$ and $-\gamma(a-\gamma)/8 - 3\gamma^2/4\sinh(1/2) < 0$. From the condition of the theorem, we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}} \gamma u_x^3 dx &\leq -\frac{3\gamma^2}{2}\int_{S} u_x^4 - 3\beta\int_{\mathbb{S}} \gamma u_x^3 dx + 2\sqrt{3}\omega|\gamma| \\ &\quad \times \|u_0\|_{H^1}^3 + \frac{36}{169}\left(-\frac{(a-\gamma)\gamma}{2} - \frac{3\gamma^2}{\sinh(1/2)}\right) \\ &\quad \times \|u_0\|_{H^1}^4 e^{-4\beta T} \\ &\leq -\frac{3\gamma^2}{2}\int_{\mathbb{S}} u_x^4 - 3\beta\int_{\mathbb{S}} \gamma u_x^3 dx - \delta,\end{aligned} \tag{100}$$

where $\delta = 2\sqrt{3}\omega|\gamma|\|u_0\|_{H^1}^3 + (36/169)(-(a-\gamma)\gamma/2 - 3\gamma^2/\sinh(1/2))\|u_0\|_{H^1}^4 e^{-4\beta T} > 0$. Repeating the arguments in the proof of case (i), we derive that the corresponding solution blows up. This finishes the proof of the theorem. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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