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Research Article

Variational Iteration Method for Singular Perturbation Initial Value Problems with Delays

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The variational iteration method (VIM) is applied to solve singular perturbation initial value problems with delays (SPIVPDs). Some convergence results of VIM for solving SPIVPDs are given. The obtained sequence of iterates is based on the use of general Lagrange multipliers; the multipliers in the functionals can be identified by the variational theory. Moreover, the numerical examples show the efficiency of the method.

1. Introduction

Singular perturbation initial value problems with delays play an important role in the research of various applied sciences, such as control theory, population dynamics, medical science, environment science, biology, and economics [1, 2]. These problems are characterized by a small parameter ϵ multiplying the highest derivatives, and the state variables depend not only on the value of the time, but also on the value prior to the time. Because the classical Lipschitz constant and one-sided Lipschitz constant are generally of size $\mathcal{O}(\epsilon^{-1})$ $(0 < \epsilon \ll 1)$, the classical convergence theory, B-convergence theory, and D-convergence theory cannot be directly applied to SPIVPDs.

Starting from pioneering ideas going back to Inokuti-Sekine-Mura method [3], the variational iteration method was first proposed in later 1990s by He [4–6]. By recent years, this method has been extensively applied to various ODEs, integral equations, delay differential equations and fractional differential equations, two-point boundary value problems, oscillations, and stiff ODEs, notably, He [7, 8], Wazwaz [9, 10], Draganescu et al. [11, 12], Saadatmandi and Dehghan [13], Salkuyeh [14], Lu [15], Xu [16], Rafei et al. [17], Darvishi et al. [18], Tatari and Dehghan [19], Mamode [20], Saadati and Dehghan [21], Yu [22], Marinca et al. [23], Yang and Dumitru [24], and Wu [25] to mention only a few. Recently, Zhao and Xiao [26] have applied this method for solving singular perturbation initial value problems. For more comprehensive survey on this method and its applications, the reader is referred to the review articles [26–28] and the references therein.

In this paper, we apply the VIM to SPIVPDs to obtain the analytical or approximate analytical solutions. The convergence results of VIM for solving SPIVPDs are obtained. Some illustrative examples confirm the theoretical results.

In the remaining parts of the text, we denote $x(t) = x(t,\epsilon)$, $y(t) = y(t,\epsilon)$ for simplicity; here, ϵ is the singular perturbation parameter. The vectors $(a_1, a_2, \ldots, a_n)^T = a \le b = (b_1, b_2, \ldots, b_n)^T$ mean that each component $a_i \le b_i$ $(i = 1, 2, \ldots, n)$. $\|\cdot\|$ denotes the standard Euclidean norm of a vector.

2. Convergence Analysis

2.1. Case 1. Consider the following singular perturbation initial value problem with delays:

$$x'(t) = f(t, x(t), x(t-\tau), y(t), y(t-\tau)),$$
$$0 \le t \le T,$$

$$\epsilon y'(t) = g(t, x(t), x(t-\tau), y(t), x(t-\tau)), \quad 0 < \epsilon \ll 1,$$
$$x(t) = \varphi(t), \quad y(t) = \psi(t), \quad t \le 0, \tag{1}$$

where $x \in R^{n_1}$ and $y \in R^{n_2}$ are the state variables and ϵ is the singular perturbation parameter. $f : [-\tau, T] \times R^{n_1} \times R^{n_1} \times R^{n_2} \times R^{n_2} \to R^{n_1}$, $g : [-\tau, T] \times R^{n_1} \times R^{n_1} \times R^{n_2} \times R^{n_2} \to R^{n_2}$ are given continuous mappings which satisfy the following Lipschitz conditions:

$$\begin{aligned}&\|f(t, x_1, u_1, y_1, v_1) - f(t, x_2, u_2, y_2, v_2)\| \\ &\quad \le l_1(t)\|x_1 - x_2\| + l_2(t)\|u_1 - u_2\| \\ &\quad + l_3(t)\|y_1 - y_2\| + l_4(t)\|v_1 - v_2\|,\end{aligned} \tag{2a}$$

$$\begin{aligned}&\|g(t, x_1, u_1, y_1, v_1) - g(t, x_2, u_2, y_2, v_2)\| \\ &\quad \le k_1(t)\|x_1 - x_2\| + k_2(t)\|u_1 - u_2\| \\ &\quad + k_3(t)\|y_1 - y_2\| + k_4(t)\|v_1 - v_2\|,\end{aligned} \tag{2b}$$

where $l_i(t)$, $k_i(t)$ $(i = 1, \ldots, 4)$ are continuous bounded functions.

According to VIM, we can construct the correction functionals as follows:

$$\begin{aligned}x_{n+1}(t) &= x_n(t) \\ &+ \int_0^t \lambda_1(s,t) \\ &\times \left(x_n'(s) - \tilde{f}(s, x_n(s), x_n(s-\tau), y_n(s), y_n(s-\tau))\right) ds,\end{aligned} \tag{3a}$$

$$\begin{aligned}y_{n+1}(t) &= y_n(t) \\ &+ \int_0^t \lambda_2(s,t) \\ &\times \left(y_n'(s) - \frac{1}{\epsilon}\tilde{g}(s, x_n(s), x_n(s-\tau), y_n(s), y_n(s-\tau))\right) ds,\end{aligned} \tag{3b}$$

where $\lambda_1(s,t)$, $\lambda_2(s,t)$ are general Lagrange multipliers, which can be defined optimally via variational theory, and $\tilde{f}, \tilde{g}$ denote the restrictive variation; that is, $\delta\tilde{f} = \delta\tilde{g} = \mathbf{0}$. Thus, we have

$$\begin{aligned}\delta x_{n+1}(t) &= \delta x_n(t) + \int_0^t \lambda_1(s,t)\,\delta x_n'(s)\,ds, \\ \delta y_{n+1}(t) &= \delta y_n(t) + \int_0^t \lambda_2(s,t)\,\delta y_n'(s)\,ds,\end{aligned} \tag{4}$$

and the stationary conditions are obtained as

$$\begin{aligned}&1 + \lambda_1(s,t)\big|_{s=t} = 0, \qquad \frac{\partial \lambda_1(s,t)}{\partial s} = 0, \\ &1 + \lambda_2(s,t)\big|_{s=t} = 0, \qquad \frac{\partial \lambda_2(s,t)}{\partial s} = 0.\end{aligned} \tag{5}$$

Moreover, the general Lagrange multiplier can be readily identified by

$$\lambda_1(s,t) = \lambda_2(s,t) = -1. \tag{6}$$

Therefore, the variational iteration formulas can be written as

$$\begin{aligned}x_{n+1}(t) &= x_n(t) - \int_0^t \Big(x_n'(s) \\ &\quad - f(s, x_n(s), x_n(s-\tau), \\ &\quad y_n(s), y_n(s-\tau))\Big) ds,\end{aligned} \tag{7a}$$

$$\begin{aligned}y_{n+1}(t) &= y_n(t) \\ &- \int_0^t \Big(y_n'(s) \\ &\quad - \frac{1}{\epsilon} g(s, x_n(s), \\ &\quad x_n(s-\tau), y_n(s), y_n(s-\tau))\Big) ds.\end{aligned} \tag{7b}$$

Now, we show that the iterative sequences $\{x_n(t)\}_{n=1}^{\infty}$, $\{y_n(t)\}_{n=1}^{\infty}$ defined by (7a) and (7b) with $x_0(t) = \varphi(t)$, $y_0(t) = \psi(t)$ converge to the solution of (1).

Theorem 1. *Let $x(t)$, $x_i(t) \in (C^1[0,T])^{n_1}$, $y(t)$, $y_i(t) \in (C^1[0,T])^{n_2}$, $i = 0, 1, \ldots$. The sequences defined by* (7a) *and* (7b) *with $x_0(t) = \varphi(t)$, $y_0(t) = \psi(t)$ converge to the solution of* (1).

Proof. Obviously from system (1), we have

$$\begin{aligned}x(t) &= x(t) \\ &- \int_0^t \Big(x'(s) \\ &\quad - f(s, x(s), x(s-\tau), y(s), y(s-\tau))\Big) ds,\end{aligned} \tag{8a}$$

$$\begin{aligned}y(t) &= y(t) \\ &- \int_0^t \Big(y'(s) \\ &\quad - \frac{1}{\epsilon} g(s, x(s), x(s-\tau), y(s), y(s-\tau))\Big) ds.\end{aligned} \tag{8b}$$

Introduce $E_i x(t) = x_i(t) - x(t)$, $E_i y(t) = y_i(t) - y(t)$, $E_i x(t-\tau) = x_i(t-\tau) - x(t-\tau)$, $E_i y(t-\tau) = y_i(t-\tau) - y(t-\tau)$, $i = 0, 1, \ldots$, where $E_j x(0) = E_j y(0) = 0$, $j = 0, 1, \ldots$. Now from (7a), (7b)-(8a), and (8b) we obtain

$$\begin{aligned}E_{n+1}x(t) &= E_n x(t) \\ &- \int_0^t \Big(E_n x'(s) - \big(f(s, x_n(s), x_n(s-\tau), \\ &\quad y_n(s), y_n(s-\tau)) \\ &\quad - f(s, x(s), x(s-\tau), \\ &\quad y(s), y(s-\tau))\big)\Big) ds,\end{aligned}$$

$$\begin{aligned} E_{n+1}y(t) = {} & E_n y(t) \\ & - \int_0^t \Big(E_n y'(s) - \frac{1}{\epsilon} \\ & \times \big(g(s, x_n(s), x_n(s-\tau), \\ & y_n(s), y_n(s-\tau)) \\ & - g(s, x(s), x(s-\tau), y(s), \\ & y(s-\tau))\big)\Big) ds. \end{aligned} \tag{9}$$

Moreover, we can derive

$$\begin{aligned} E_{n+1}x(t) = {} & \int_0^t \big(f(s, x_n(s), x_n(s-\tau), y_n(s), y_n(s-\tau)) \\ & - f(s, x(s), x(s-\tau), y(s), \\ & y(s-\tau))\big) ds, \end{aligned} \tag{10a}$$

$$\begin{aligned} E_{n+1}y(t) = {} & \frac{1}{\epsilon} \int_0^t \big(g(s, x_n(s), x_n(s-\tau), y_n(s), y_n(s-\tau)) \\ & - g(s, x(s), x(s-\tau), y(s), \\ & y(s-\tau))\big) ds. \end{aligned} \tag{10b}$$

Now, the integration interval is split into two parts

$$\begin{aligned} & E_{n+1}x(t) \\ & = \int_0^\tau \big(f(s, x_n(s), \varphi(s-\tau), y_n(s), \psi(s-\tau)) \\ & \quad - f(s, x(s), \varphi(s-\tau), \\ & \quad y(s), \psi(s-\tau))\big) ds \\ & \quad + \int_\tau^t \big(f(s, x_n(s), x_n(s-\tau), y_n(s), y_n(s-\tau)) \\ & \quad - f(s, x(s), x(s-\tau), \\ & \quad y(s), y(s-\tau))\big) ds, \\ & E_{n+1}y(t) \\ & = \frac{1}{\epsilon} \Big(\int_0^\tau \big(g(s, x_n(s), \varphi(s-\tau), y_n(s), \psi(s-\tau)) \\ & \quad - g(s, x(s), \varphi(s-\tau), y(s), \psi(s-\tau))\big) ds \\ & \quad + \int_\tau^t \big(g(s, x_n(s), x_n(s-\tau), y_n(s), y_n(s-\tau)) \\ & \quad - g(s, x(s), x(s-\tau), y(s), y(s-\tau))\big) ds \Big). \end{aligned} \tag{11}$$

From the Lipschitz conditions (2a) and (2b), we have

$$\begin{aligned} \begin{pmatrix} \|E_{n+1}x(t)\| \\ \|E_{n+1}y(t)\| \end{pmatrix} \le {} & \begin{pmatrix} l_1 & l_3 \\ \frac{k_1}{\epsilon} & \frac{k_3}{\epsilon} \end{pmatrix} \begin{pmatrix} \int_0^t \|E_n x(s)\| ds \\ \int_0^t \|E_n y(s)\| ds \end{pmatrix} \\ & + \begin{pmatrix} l_2 & l_4 \\ \frac{k_2}{\epsilon} & \frac{k_4}{\epsilon} \end{pmatrix} \begin{pmatrix} \int_0^t \|E_n x(s-\tau)\| ds \\ \int_0^t \|E_n y(s-\tau)\| ds \end{pmatrix}, \end{aligned} \tag{12}$$

where $l_i = \max_{0\le s\le T} l_i(s)$, $k_i = \max_{0\le s\le T} k_i(s)$, $i = 1, \ldots, 4$. Therefore,

$$\begin{aligned} \begin{pmatrix} \|E_1x(t)\| \\ \|E_1y(t)\| \end{pmatrix} \le {} & \begin{pmatrix} l_1 & l_3 \\ \frac{k_1}{\epsilon} & \frac{k_3}{\epsilon} \end{pmatrix} \begin{pmatrix} \int_0^t \|E_0 x(s)\| ds \\ \int_0^t \|E_0 y(s)\| ds \end{pmatrix} \\ & + \begin{pmatrix} l_2 & l_4 \\ \frac{k_2}{\epsilon} & \frac{k_4}{\epsilon} \end{pmatrix} \begin{pmatrix} \int_0^t \|E_0 x(s-\tau)\| ds \\ \int_0^t \|E_0 y(s-\tau)\| ds \end{pmatrix} \\ & \le \begin{pmatrix} l_1 + l_2 & l_3 + l_4 \\ \frac{k_1 + k_2}{\epsilon} & \frac{k_3 + k_4}{\epsilon} \end{pmatrix} \begin{pmatrix} \max_{-\tau\le s\le T} \|E_0 x(s)\| t \\ \max_{-\tau\le s\le T} \|E_0 y(s)\| t \end{pmatrix}. \end{aligned} \tag{13}$$

Moreover, we can derive

$$\begin{aligned} \begin{pmatrix} \|E_n x(t)\| \\ \|E_n y(t)\| \end{pmatrix} \le {} & \frac{T^n}{n!} \begin{pmatrix} l_1 + l_2 & l_3 + l_4 \\ \frac{k_1 + k_2}{\epsilon} & \frac{k_3 + k_4}{\epsilon} \end{pmatrix}^n \\ & \times \begin{pmatrix} \max_{-\tau\le s\le T} \|E_0 x(s)\| \\ \max_{-\tau\le s\le T} \|E_0 y(s)\| \end{pmatrix} \\ & \le \frac{(T/\epsilon)^n}{n!} \begin{pmatrix} \mathcal{O}(\epsilon) & \mathcal{O}(\epsilon) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} \begin{pmatrix} \max_{-\tau\le s\le T} \|E_0 x(s)\| \\ \max_{-\tau\le s\le T} \|E_0 y(s)\| \end{pmatrix}. \end{aligned} \tag{14}$$

Noting that $\epsilon, T, \max_{-\tau\le s\le T}\|E_0 x(s)\|, \max_{-\tau\le s\le T}\|E_0 y(s)\|$, l_i, k_i, $i = 1, \ldots, 4$ are constants. By using Stirling's formula, we have

$$\begin{aligned} \begin{pmatrix} \|E_n x(t)\| \\ \|E_n y(t)\| \end{pmatrix} \le {} & \frac{((Te/\epsilon)/n)^n}{\sqrt{2\pi n}\,(1 + \mathcal{O}(1/n))} \\ & \times \begin{pmatrix} \mathcal{O}(\epsilon) & \mathcal{O}(\epsilon) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} \begin{pmatrix} \max_{-\tau\le s\le T} \|E_0 x(s)\| \\ \max_{-\tau\le s\le T} \|E_0 y(s)\| \end{pmatrix}; \end{aligned} \tag{15}$$

thus, $(\|E_n x(t)\|, \|E_n y(t)\|)^T \to \mathbf{0}$ as $n \to \infty$. □

2.2. Case 2. Consider the special case of (1):

$$\begin{aligned} x'(t) &= Ax(t) + F(t, x(t), x(t-\tau), y(t), y(t-\tau)), \\ &\qquad 0 \le t \le T, \\ \epsilon y'(t) &= By(t) + G(t, x(t), x(t-\tau), y(t), y(t-\tau)), \\ &\qquad 0 < \epsilon \ll 1, \\ x(t) &= \varphi(t), \quad y(t) = \psi(t), \quad t \le 0, \end{aligned} \tag{16}$$

where $F : [-\tau, T] \times R^{n_1} \times R^{n_1} \times R^{n_2} \times R^{n_2} \to R^{n_1}, G : [-\tau, T] \times R^{n_1} \times R^{n_1} \times R^{n_2} \times R^{n_2} \to R^{n_2}$ are given continuous mappings which satisfy the Lipschitz conditions (2a) and (2b); the matrices $A = (a_{ij}) \in R^{n_1 \times n_1}$, $B = (b_{ij}) \in R^{n_2 \times n_2}$ can be decomposed into $A = A_0 + A_1$, $B = B_0 + B_1$, respectively, where $A_0 = \operatorname{diag}(a_{11}, a_{22}, \ldots, a_{n_1 n_1})$ and $B_0 = \operatorname{diag}(b_{11}, b_{22}, \ldots, b_{n_2 n_2})$:

$$\begin{aligned} &\|F(t, x_1, u_1, y_1, v_1) - F(t, x_2, u_2, y_2, v_2)\| \\ &\quad \le p_1(t)\|x_1 - x_2\| + p_2(t)\|u_1 - u_2\| \\ &\qquad + p_3(t)\|y_1 - y_2\| + p_4(t)\|v_1 - v_2\|, \end{aligned} \tag{17a}$$

$$\begin{aligned} &\|G(t, x_1, u_1, y_1, v_1) - G(t, x_2, u_2, y_2, v_2)\| \\ &\quad \le q_1(t)\|x_1 - x_2\| + q_2(t)\|u_1 - u_2\| \\ &\qquad + q_3(t)\|y_1 - y_2\| + q_4(t)\|v_1 - v_2\|, \end{aligned} \tag{17b}$$

where $p_i(t)$, $q_i(t)$ $(i = 1, \ldots, 4)$ are continuous bounded functions.

It is easy to show that the right hand sides of (16) also satisfy the Lipschitz conditions. If the right hand sides of (16) are considered as nonlinear terms, then we can also use the correction functionals constructed in Case 1 and get similar results to Theorem 1. Now, we construct the following correction functionals:

$$\begin{aligned} x_{n+1}(t) &= x_n(t) \\ &\quad + \int_0^t \Lambda_1(s,t)\Big(x_n'(s) - A_0 x_n(s) - A_1 \widetilde{x}_n(s) \\ &\qquad - \widetilde{F}(s, x_n(s), x_n(s-\tau), \\ &\qquad\quad y_n(s), y_n(s-\tau))\Big)\, ds, \\ y_{n+1}(t) &= y_n(t) \\ &\quad + \int_0^t \Lambda_2(s,t) \\ &\quad \times \Big(y_n'(s) - \frac{1}{\epsilon}(B_0 y_n(s) + B_1 \widetilde{y}_n(s) \\ &\qquad + \widetilde{G}(s, x_n(s), x_n(s-\tau), \\ &\qquad\quad y_n(s), y_n(s-\tau)))\Big)\, ds, \end{aligned} \tag{18}$$

where $\Lambda_1(s,t) = \operatorname{diag}(\lambda_{11}(s,t), \lambda_{12}(s,t), \ldots, \lambda_{1n_1}(s,t))$, $\Lambda_2(s,t) = \operatorname{diag}(\lambda_{21}(s,t), \lambda_{22}(s,t), \ldots, \lambda_{2n_2}(s,t))$, in which $\lambda_{1i}(s,t)$, $\lambda_{2j}(s,t)$, $i = 1, 2, \ldots, n_1$, $j = 1, 2, \ldots, n_2$ are general Lagrange multipliers and $\widetilde{x}_n, \widetilde{y}_n, \widetilde{F}, \widetilde{G}$ denote the restrictive variations; that is, $\delta\widetilde{x}_n = \delta\widetilde{y}_n = \delta\widetilde{F} = \delta\widetilde{G} = \mathbf{0}$. Thus, we have

$$\begin{aligned} \delta x_{n+1}(t) &= \delta x_n(t) + \int_0^t \Lambda_1(s,t)\left(\delta x_n'(s) - A_0 \delta x_n(s)\right) ds, \\ \delta y_{n+1}(t) &= \delta y_n(t) + \int_0^t \Lambda_2(s,t)\left(\delta y_n'(s) - \frac{B_0}{\epsilon}\delta y_n(s)\right) ds, \end{aligned} \tag{19}$$

and the stationary conditions are obtained as

$$\begin{aligned} &1 + \Lambda_1(s,t)\big|_{s=t} = 0, \qquad \frac{\partial \Lambda_1(s,t)}{\partial s} + A_0 \Lambda_1(s,t) = 0, \\ &1 + \Lambda_2(s,t)\big|_{s=t} = 0, \qquad \frac{\partial \Lambda_1(s,t)}{\partial s} + \frac{B_0}{\epsilon}\Lambda_2(s,t) = 0. \end{aligned} \tag{20}$$

Moreover, the general Lagrange multipliers can be readily identified by

$$\begin{aligned} \Lambda_1(s,t) &= -\exp(-A_0(s-t)), \\ \Lambda_2(s,t) &= -\exp\left(-\frac{B_0}{\epsilon}(s-t)\right). \end{aligned} \tag{21}$$

Therefore, the variational iteration formula can be written as

$$\begin{aligned} x_{n+1}(t) &= x_n(t) \\ &\quad - \int_0^t e^{-A_0(s-t)}\Big(x_n'(s) - Ax_n(s) \\ &\qquad - F(s, x_n(s), x_n(s-\tau), \\ &\qquad\quad y_n(s), y_n(s-\tau))\Big)\, ds, \end{aligned} \tag{22a}$$

$$\begin{aligned} &y_{n+1}(t) \\ &\quad = y_n(t) - \int_0^t e^{((-B_0/\epsilon)(s-t))} \\ &\qquad \times \Big(y_n'(s) - \frac{1}{\epsilon} \\ &\qquad\quad \times (By_n(s) \\ &\qquad\qquad + G(s, x_n(s), x_n(s-\tau), y_n(s), \\ &\qquad\qquad\quad y_n(s-\tau)))\Big)\, ds. \end{aligned} \tag{22b}$$

The following theorem shows that the sequences $\{x_n(t)\}_{n=1}^{\infty}$, $\{y_n(t)\}_{n=1}^{\infty}$ defined by (22a) and (22b) with $x_0(t) = \varphi(t)$, $y_0(t) = \psi(t)$ converge to the solution of (16).

Theorem 2. *Let $x(t)$, $x_i(t) \in (C^1[0,T])^{n1}$, $y(t)$, $y_i(t) \in (C^1[0,T])^{n2}$, $i = 0,1,\ldots$. The sequences defined by* (22a) *and* (22b) *with $x_0(t) = \varphi(t)$, $y_0(t) = \psi(t)$ converge to the solutions of* (16).

Proof. By a similar process to the proof of Theorem 1, we can easily obtain Obviously from system (16) we have

$$\begin{aligned} x(t) = x(t) - \int_0^t \big(x'(s) - Ax(t) \\ - F(s, x(s), x(s-\tau), y(s), \\ y(s-\tau))\big)\, ds, \end{aligned} \tag{23a}$$

$$\begin{aligned} y(t) = y(t) \\ - \int_0^t \Big(y'(s) \\ - \frac{1}{\epsilon}\big(By(t) \\ + G(s, x(s), x(s-\tau), y(s), \\ y(s-\tau))\big)\Big)\, ds. \end{aligned} \tag{23b}$$

Introduce $E_i x(t) = x_i(t) - x(t)$, $E_i y(t) = y_i(t) - y(t)$, $i = 0,1,\ldots$, where $E_j x(0) = E_j y(0) = 0$, $j = 0,1,\ldots$. Now from (22a), (22b)-(23a), and (23b) we obtain

$$\begin{aligned} &E_{n+1}x(t) \\ &= \int_0^t e^{-A_0(s-t)} \\ &\quad \times \big(A_1 E_n x(s) \\ &\qquad + \big(F(s, x_n(s), x_n(s-\tau), y_n(s), y_n(s-\tau)) \\ &\qquad - F(s, x(s), x(s-\tau), y(s), y(s-\tau))\big)\big)\, ds, \\ &E_{n+1}y(t) \\ &= \int_0^t e^{((-B_0/\epsilon)(s-t))} \\ &\quad \times \Big(\frac{1}{\epsilon}\big(B_1 E_n y(s) \\ &\qquad + G(s, x_n(s), x_n(s-\tau), y_n(s), \\ &\qquad y_n(s-\tau)) \\ &\qquad - G(s, x(s), x(s-\tau), y(s), \\ &\qquad y(s-\tau))\big)\Big)\, ds. \end{aligned} \tag{24}$$

From the Lipschitz conditions (17a) and (17b), we have

$$\begin{aligned} \begin{pmatrix} \|E_{n+1}x(t)\| \\ \|E_{n+1}y(t)\| \end{pmatrix} &\le \begin{pmatrix} e^{-\|A_0\|(s-t)} & 0 \\ 0 & e^{-(\|B_0\|/\epsilon)(s-t)} \end{pmatrix} \\ &\quad \times \begin{pmatrix} \|A_1\| + p_1 & p_3 \\ \dfrac{q_1}{\epsilon} & \dfrac{\|B_1\| + q_3}{\epsilon} \end{pmatrix} \\ &\quad \times \begin{pmatrix} \int_0^t \|E_n x(s)\|\, ds \\ \int_0^t \|E_n y(s)\|\, ds \end{pmatrix} \\ &\quad + \begin{pmatrix} e^{-\|A_0\|(s-t)} & 0 \\ 0 & e^{-(\|B_0\|/\epsilon)(s-t)} \end{pmatrix} \\ &\quad \times \begin{pmatrix} p_2 & p_4 \\ \dfrac{q_2}{\epsilon} & \dfrac{q_4}{\epsilon} \end{pmatrix} \\ &\quad \times \begin{pmatrix} \int_0^t \|E_n x(s-\tau)\|\, ds \\ \int_0^t \|E_n y(s-\tau)\|\, ds \end{pmatrix}; \end{aligned} \tag{25}$$

similarly, we can derive

$$\begin{aligned} \begin{pmatrix} \|E_n x(t)\| \\ \|E_n y(t)\| \end{pmatrix} &\le e(\epsilon)\, \frac{T^n}{n!} \begin{pmatrix} \|A_1\| + p_1 + p_2 & p_3 + p_4 \\ \dfrac{q_1 + q_2}{\epsilon} & \dfrac{\|B_1\| + q_3 + q_4}{\epsilon} \end{pmatrix}^n \\ &\quad \times \begin{pmatrix} \max\limits_{-\tau \le s \le T} \|E_0 x(s)\| \\ \max\limits_{-\tau \le s \le T} \|E_0 y(s)\| \end{pmatrix} \le \frac{e(\epsilon)\,((Te/\epsilon)/n)^n}{\sqrt{2\pi n}\,(1 + \mathcal{O}(1/n))} \\ &\quad \times \begin{pmatrix} \mathcal{O}(\epsilon) & \mathcal{O}(\epsilon) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} \begin{pmatrix} \max\limits_{-\tau \le s \le T} \|E_0 x(s)\| \\ \max\limits_{-\tau \le s \le T} \|E_0 y(s)\| \end{pmatrix}, \end{aligned} \tag{26}$$

where $e(\epsilon) = \max_{0 \le s \le t, 0 \le t \le T}(e^{-\|A_0\|(s-t)}, e^{-(\|B_0\|/\epsilon)(s-t)})$, $p_i = \max_{0 \le s \le T} p_i(s)$, $q_i = \max_{0 \le s \le T} q_i(s)$, $i = 1,\ldots,4$. Noting that ϵ, T, $\|A_1\|$, $\|B_1\|$, $e(\epsilon)$, p_i, q_i, $i = 1,\ldots,4$, $\max_{-\tau \le s \le T}\|E_0 x(s)\|$ and $\max_{-\tau \le s \le T}\|E_0 y(s)\|$ are constants, we can derive from (26) that $(\|E_n x(t)\|, \|E_n y(t)\|)^T \to \mathbf{0}$ as $n \to \infty$. □

3. Numerical Examples

In this section, some numerical examples are given to show the efficiency of the VIM for solving SPIVPs.

Example 3. Consider SPIVPD (cf. [2]):

$$x'(t) = x(t-1)\, y(t-1) - 1000x(t) + 2y^2(t) + R_x(t), \quad t > 0,$$

$$\epsilon y'(t) = x(t-1) - y(t-1) - (1 + x(t))\, y(t) + R_y(t), \quad 0 < \epsilon \ll 1,$$
$$x(t) = e^{-0.5t} + e^{-0.2t}, \quad y(t) = -e^{-0.5t} + e^{-0.2t}, \quad t \le 0, \tag{27}$$

where

$$\begin{aligned} R_x(t) &= 999.5e^{-0.5t} + 999.8e^{-0.2t} + e^{-(t-1)} \\ &\quad - e^{-0.4(t-1)} - 2e^{-t} - 2e^{-0.4t} + 4e^{-0.7t}, \\ R_y(t) &= (0.5\epsilon - 1)\, e^{-0.5t} + (1 - 0.2\epsilon)\, e^{-0.2t} \\ &\quad - e^{-0.5(t-1)} - e^{-t} + e^{-0.4t}. \end{aligned} \tag{28}$$

By using the VIM in Case 1, we construct the following iteration formula:

$$\begin{aligned} x_{n+1}(t) &= x_n(t) \\ &\quad - \int_0^t \Big(x_n'(s) - x_n(s-1)\, y_n(s-1) \\ &\qquad + 1000 x_n(s) - 2y_n^2(s) - R_x(s) \Big)\, ds, \end{aligned} \tag{29a}$$

$$\begin{aligned} y_{n+1}(t) &= y_n(t) \\ &\quad - \int_0^t \Big(y_n'(s) - \frac{1}{\epsilon} \\ &\qquad \times \big(x_n(s-1) - y_n(s-1) - (1 + x_n(s)) \\ &\qquad \times\ y_n(s) + R_y(s) \big) \Big)\, ds. \end{aligned} \tag{29b}$$

To get iterate sequence, we start with an initial approximation $x_0(t) = e^{-0.5t} + e^{-0.2t}$, $y_0(t) = -e^{-0.5t} + e^{-0.2t}$ and let $\epsilon = 10^{-6}$. By means of formulas (29a) and (29b), we have

$$x_1(t) = e^{-0.5t} + e^{-0.2t}, \quad y_1(t) = -e^{-0.5t} + e^{-0.2t}. \tag{30}$$

Figure 1 shows the efficiency of VIM for SPIVPDs.

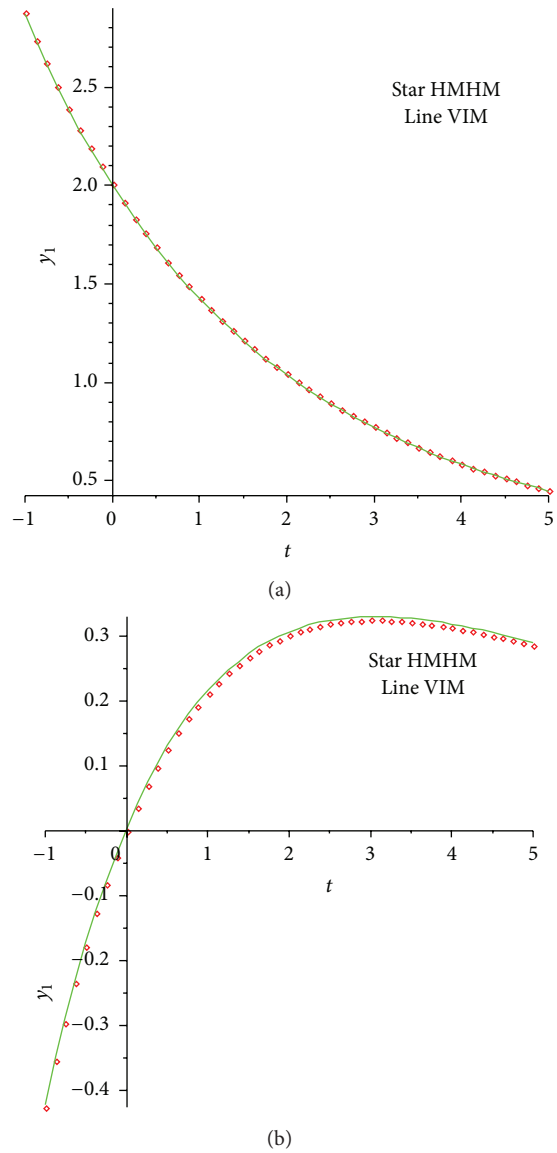


Figure 1: Results for problem 1.

Example 4. Consider SPIVPD (cf. [2]):

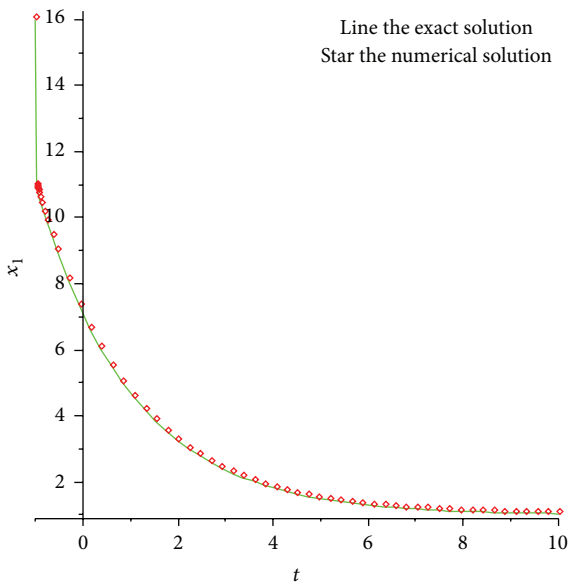
$$x'(t) = 2x(t-1) + y(t-1) - 1000x(t) + y(t) + R_x(t), \quad t > 0,$$
$$\epsilon y'(t) = x(t-1) - y(t-1) + 3x(t) - y(t) + R_y(t), \quad 0 < \epsilon \ll 1,$$
$$x(t) = 1 + 10e^{-(t+1)/2} + 5e^{-(t+1)/\epsilon}, \quad t \le 0,$$
$$y(t) = -1 - 9e^{-(t+1)/2} + 4e^{-(t+1)/\epsilon}, \quad t \le 0, \tag{31}$$

where

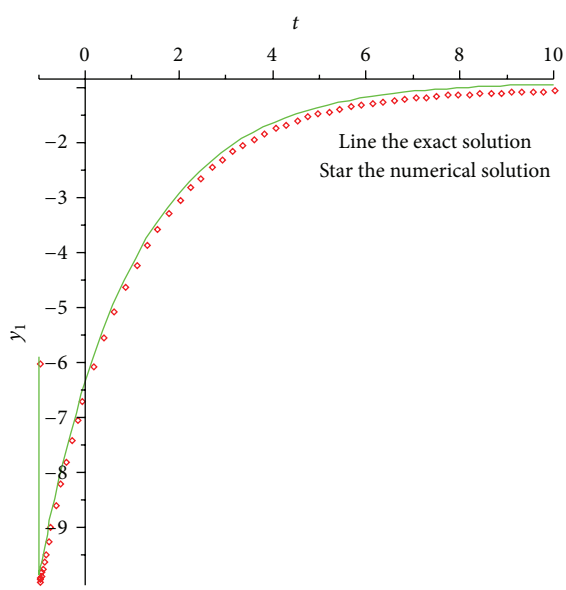
$$\begin{aligned} R_x(t) &= 10004e^{-(t+1)/2} + \left(4996 - \frac{5}{\epsilon}\right) e^{-(t+1)/\epsilon} \\ &\quad - 11e^{-t/2} - 14e^{-t/\epsilon} + 1000, \\ R_y(t) &= \left(\frac{9\epsilon}{2} - 39\right) e^{-(t+1)/2} + 15e^{-(t+1)/\epsilon} \\ &\quad - 19e^{-t/2} - e^{-t/\epsilon} - 6. \end{aligned} \tag{32}$$

By using the VIM in Case 2, we construct the following iteration formula:

$$\begin{aligned} x_{n+1}(t) &= x_n(t) \\ &\quad - \int_0^t e^{1000(s-t)} \big(x_n'(s) - 2x_n(s-1) - y_n(s-1) \\ &\qquad + 1000 x_n(s) - y_n(s) - R_x(s) \big)\, ds, \end{aligned} \tag{33}$$



(a)



(b)

Figure 2: Results for problem 2.

$$\begin{aligned} y_{n+1}(t) &= y_n(t) - \int_0^t e^{(s-t)/\epsilon} \\ &\quad \times \Big(y_n'(s) - \frac{1}{\epsilon} \\ &\quad \times \big(x_n(s-1) - y_n(s-1) + 3x_n(s) \\ &\quad - y_n(s) + R_y(s) \big) \Big) ds. \end{aligned} \tag{34}$$

To get iterate sequence, we start with an initial approximation $x_0(t) = 1 + 10e^{-(t+1)/2} + 5e^{-(t+1)/\epsilon}$, $y_0(t) = -1 - 9e^{-(t+1)/2} + 4e^{-(t+1)/\epsilon}$ and let $\epsilon = 10^{-3}$. By means of formulas (33) and (34), we have

$$\begin{aligned} x_1(t) &= 1 + 10e^{-(t+1)/2} + 5e^{-(t+1)/\epsilon}, \\ y_1(t) &= -1 - 9e^{-(t+1)/2} + 4e^{-(t+1)/\epsilon}. \end{aligned} \tag{35}$$

Figure 2 shows the efficiency of VIM for SPIVPDs.

4. Conclusion

The VIM used in this paper is the variational iteration algorithm I; there are also variational iteration algorithms II and III [29]. In this paper, we apply the VIM to obtain the analytical or approximate analytical solutions of SPIVPDs. The convergence results of VIM for solving SPIVPDs are given. The illustrative examples show the efficiency of the method. When considering the system (16), the choice of correction functionals of Case 1 or Case 2 relies on the practical problems and this choice will result in the difference of the speed of convergence.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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