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Research Article

A Posteriori Error Estimates with Computable Upper Bound for the Nonconforming Rotated Q_1 Finite Element Approximation of the Eigenvalue Problems

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This paper discusses the nonconforming rotated Q_1 finite element computable upper bound a posteriori error estimate of the boundary value problem established by M. Ainsworth and obtains efficient computable upper bound a posteriori error indicators for the eigenvalue problem associated with the boundary value problem. We extend the a posteriori error estimate to the Steklov eigenvalue problem and also derive efficient computable upper bound a posteriori error indicators. Finally, through numerical experiments, we verify the validity of the a posteriori error estimate of the boundary value problem; meanwhile, the numerical results show that the a posteriori error indicators of the eigenvalue problem and the Steklov eigenvalue problem are effective.

1. Introduction

A posteriori error estimates and adaptive algorithms are the mainstream directions in the study of finite element methods; however, a posteriori error estimates are the theoretical basis of adaptive finite element method. Under these reasons, it is very meaningful to study the a posteriori error estimates. Particularly, it is well known that the residual type a posteriori error estimates usually contain a general constant C, which often affects the validity of the error estimates. Then, it is significant that exploring a computable upper bound a posteriori error estimate does not include constant C.

The residual type a posteriori error estimate of finite element was first proposed by Babushka and Rheinboldt [1] in 1978 and has been studied and applied to many problems. For example, in 2005, Ainsworth [2] gave the a posteriori error estimate of residual type which can provide a computable upper bound for elliptic boundary value problem. In 2007, based on what Ainsworth researched in [2], Carstensen et al. [3] established a framework of a posteriori error estimates of residual type of a class of nonconforming finite element, which includes the nonconforming C-R element, the nonconforming rotated Q_1 element, and Han element, and so forth. In 2010, using the a posteriori error estimates of nonconforming finite element established by Carstensen, Yang [4] founded the a posteriori error indicators for elliptic differential operator eigenvalue problem. Recently, Han and Yang [5] gave a class of a posteriori error estimates of spectral element methods for 2nd-order elliptic eigenvalue problems.

The finite element method is an important approach to solve the Steklov eigenvalue problem (see [6–10]). A posteriori error estimates of finite element for the Steklov eigenvalue problem has attracted attention from mathematical community in recent years. In 2008, Armentano and Padra [11] proposed and analyzed the a posteriori error estimate of the linear finite element approximation for the Steklov eigenvalue problem, and their residual type error estimate can be obtained by the local computation of approximate eigenpairs. In 2011, Ma et al. [12] studied a posteriori error estimate of the nonconforming EQ_1^{rot} element for Steklov eigenvalue problem. For the Steklov eigenvalue problems, Yang and Bi [13] have lately obtained the local a priori/a posteriori error

TABLE 1: The numerical results of Example 1.

$n1 \times n2$	32×32	64×64	128×128	256×256
$\|w_1 - w_{1,h}\|_h$	0.0062	0.0031	0.0015	$7.703e - 004$
$\|w_2 - w_{2,h}\|_h$	0.0224	0.0112	0.0056	0.0028
$\|w_3 - w_{3,h}\|_h$	$8.215e - 004$	$4.110e - 004$	$2.055e - 004$	$1.028e - 004$
$\eta_{\text{com}}/\|w_1 - w_{1,h}\|_h$	1.0038	1.0030	1.0028	1.0027
$\eta_{\text{com}}/\|w_2 - w_{2,h}\|_h$	1.0007	1.0004	1.0003	1.0003
$\eta_{\text{com}}/\|w_3 - w_{3,h}\|_h$	1.0106	1.0071	1.0060	1.0056

TABLE 2: The numerical results of the eigenvalues λ_1, λ_2, and λ_3.

$n1 \times n2$	32×32	64×64	128×128	256×256	512×512
$\lambda_{1,h}$	38.47534	38.52266	38.54373	38.55268	38.55638
$\lambda_{2,h}$	60.83608	60.80069	60.79191	60.78973	60.78919
$\lambda_{3,h}$	78.95635	78.95680	78.95683	78.95683	78.95683
$\lvert\lambda_{1,h} - \lambda_1\rvert$	0.08354	0.03622	0.01515	0.00620	0.00250
$\lvert\lambda_{2,h} - \lambda_2\rvert$	0.04708	0.01169	0.00291	0.00073	0.00019
$\lvert\lambda_{3,h} - \lambda_3\rvert$	0.00050	0.00003	0.00000	0.00000	0.00000
$\eta^2_{\text{com}}/\lvert\lambda_{1,h} - \lambda_1\rvert$	3.55176	02.63591	2.13593	1.84850	1.68088
$\eta^2_{\text{com}}/\lvert\lambda_{2,h} - \lambda_2\rvert$	8.08495	8.12050	8.14971	8.11889	7.89114
$\eta^2_{\text{com}}/\lvert\lambda_{3,h} - \lambda_3\rvert$	1442.526	5026.540	6625.932	2260.119	578.1650

estimates of conforming finite elements approximation and Zhang et al. [14] gave certain results of spectral method.

The nonconforming rotated Q_1 element was proposed by Rannacher and Turek [15]. Based on the existing research results, we discuss further a computable upper bound a posteriori error estimate of the boundary value problem established by Ainsworth and discover that this error estimate does not include a general constant C. So, we use the a posteriori error estimate to establish a computable upper bound a posteriori error indicators for the eigenvalue problem associated with the boundary value problem. In addition, we extend the error estimate to the Steklov eigenvalue problem, and obtain an efficient computable upper bound a posteriori error indicators. Finally, we verify that the computable upper bound a posteriori error estimate of the boundary value problem is effective (see Table 1). Through calculating the validity of the computable upper bound a posteriori error indicators on L-shaped domain, we can ascertain that the indicators of the eigenvalue problem and the Steklov eigenvalue problem are effective (see Tables 2 and 3).

2. Model Problem and Preliminaries

2.1. Model Problem. Consider the following eigenvalue problem:

$$\begin{aligned} -\mathbf{div}\,(\alpha\,\mathbf{grad}\,u) &= \lambda u, \quad \text{in } \Omega \\ \mathbf{n}\cdot\alpha\,\mathbf{grad}\,u &= 0, \quad \text{on } \Gamma_N \\ u &= 0, \quad \text{on } \Gamma_D, \end{aligned} \tag{1}$$

where $\Omega \subset R^2$ is a planar polygonal domain with boundary $\Gamma := \partial\Omega$, the disjoint sets Γ_D and Γ_N form a partition of the boundary of Ω, and $\alpha \in L_\infty(\Omega)$ is assumed to be nonnegative. For simplicity, we assume that α is piecewise constant on the finite element mesh.

Then (1) can be written in a weak form: to seek $(\lambda, u) \in \mathbb{R} \times H_E^1(\Omega)$ with $\|u\|_0 = 1$ such that

$$a(u, v) = \lambda b(u, v), \quad \forall v \in H_E^1(\Omega), \tag{2}$$

where $H_E^1(\Omega) = \{v \in H^1(\Omega) : v = 0, \text{ on } \Gamma_D\}$, and $a(u, v) = \int_\Omega \alpha\nabla u\nabla v\,dx\,dy$, $b(u, v) = \int_\Omega uv\,dx\,dy$.

Let $\mathscr{P}_h$ be a partition with mesh diameters h of the domain Ω consisting of disjoint convex quadrilateral elements, and the nonempty intersection of any two distinct elements is either a single common node or a common edge. In addition, the nonempty intersection of an element with the exterior boundary is a portion of either Γ_D or Γ_N. The family of partitions is assumed to be locally quasi-uniform in the sense that the ratio of the diameters of any adjacent elements is bounded above and below uniformly over the whole family of partitions. Define the generalized energy norm $|||v|||$ by

$$|||v|||^2 = (\alpha\,\mathbf{grad}_h v, \mathbf{grad}_h v), \tag{3}$$

where the operator $\mathbf{grad}_h$ satisfies the condition $(\mathbf{grad}_h v)|_\kappa = \mathbf{grad}(v|_\kappa)$, $\forall\kappa \in \mathscr{P}_h$ and the notation $(\cdot,\cdot)_\varpi$ is used to denote the L_2-inner product over a domain ϖ. The subscript ϖ is omitted when it is a physical domain ϖ.

The nonconforming rotated Q_1 finite element space (see [15]) is defined by

$$\begin{aligned} X_h = \Big\{ & v \in L_2(\Omega) : v|_\kappa \in \operatorname{span}\left(1, x, y, x^2 - y^2\right), \\ & \forall\kappa \in \mathscr{P}_h, \int_\gamma [v]\,ds = 0, \forall\gamma \in \partial\mathscr{P}_h \setminus \partial\Omega \Big\}, \end{aligned} \tag{4}$$

Table 3: The numerical results of the Steklov eigenvalues λ_2, λ_3, and λ_4.

$n1 \times n2$	8×8	16×16	32×32	64×64	128×128
$\lambda_{2,h}$	0.88320	0.88930	0.89190	0.89295	0.89337
$\lambda_{3,h}$	1.68213	1.68662	1.68805	1.68846	1.68856
$\lambda_{4,h}$	3.17490	3.20336	3.21370	3.21675	3.21757
$\lvert\lambda_{2,h}-\lambda_2\rvert$	0.01047	0.00436	0.00177	0.00071	0.00030
$\lvert\lambda_{3,h}-\lambda_3\rvert$	0.00647	0.00198	0.00055	0.00015	0.00003
$\lvert\lambda_{4,h}-\lambda_4\rvert$	0.04298	0.01453	0.00418	0.00114	0.00031
$\eta^2_{\text{com}}/\lvert\lambda_{2,h}-\lambda_2\rvert$	1.57977	1.47585	1.41700	1.36649	1.29035
$\eta^2_{\text{com}}/\lvert\lambda_{3,h}-\lambda_3\rvert$	1.56752	1.29357	1.17339	1.10827	1.04199
$\eta^2_{\text{com}}/\lvert\lambda_{4,h}-\lambda_4\rvert$	1.81759	1.35064	1.16560	1.06969	0.97406

where $[v]$ denotes the jump across an interface γ and $\partial\mathcal{P}_h$ the set of element edges. The subspace $X_{h,E}$ of X_h is defined by

$$X_{h,E} = \left\{ v \in X_h : \int_\gamma v\, ds = 0, \text{if } \gamma \subset \Gamma_D \right\}. \tag{5}$$

The nonconforming rotated Q_1 element approximation of (2) is the following: find $(\lambda_h, u_h) \in \mathbb{R} \times X_{h,E}$ such that

$$a_h(u_h, v) = \lambda_h b(u_h, v), \quad \forall v \in X_{h,E}, \tag{6}$$

where $a_h(u_h, v) = \sum_{\kappa\in\mathcal{P}_h} \int_\kappa \alpha_\kappa \nabla u_h \nabla v\, dx\, dy$. Define $\|\cdot\|_h = \sqrt{\sum_{\kappa\in\mathcal{P}_h} \|\|\cdot\|\|^2_\kappa}$. Evidently, $\|\cdot\|_h$ is the norm on $X_{h,E}$.

2.2. A Posteriori Error Estimate of Boundary Value Problem. In this subsection we present the computable upper bound a posteriori error estimate of the boundary value problem established by Ainsworth in [2, 16]. It is the key to establishing a computable upper bound a posteriori error indicator for the eigenvalue problem (1).

Consider the boundary value problem of finding w such that

$$\begin{aligned} -\mathbf{div}(\alpha\, \mathbf{grad}\, w) &= f, \quad \text{in } \Omega \\ \mathbf{n}\cdot\alpha\, \mathbf{grad}\, w &= g, \quad \text{on } \Gamma_N \\ w &= 0, \quad \text{on } \Gamma_D, \end{aligned} \tag{7}$$

where $f \in L_2(\Omega)$, $g \in L_2(\Gamma_N)$.

The variational form of (7) consists of seeking $w \in H^1_E(\Omega)$ such that

$$(a\, \mathbf{grad}\, w, \mathbf{grad}\, v) = (f, v) + \int_{\Gamma_N} g v\, ds, \quad \forall v \in H^1_E(\Omega). \tag{8}$$

The nonconforming rotated Q_1 finite element approximation of (8) is the following: find $w_h \in X_{h,E}$ such that

$$(\alpha\, \mathbf{grad}_h w_h, \mathbf{grad}_h v) = (f, v) + \int_{\Gamma_N} g v\, ds, \quad \forall v \in X_{h,E}. \tag{9}$$

To establish a computable upper bound of nonconforming finite element a posteriori estimate for the error $e = w - w_h$ in the sense of energy norm (3), we use the following Helmholtz decomposition (see [17]) to divide the error e into the conforming part and the nonconforming part.

Lemma 1. *Let*

$$\mathcal{H} = \left\{ w \in H^1(\Omega) : \int_\Omega w\, dx\, dy = 0,\ \frac{\partial w}{\partial \tau} = 0,\ on\ \Gamma_N \right\}, \tag{10}$$

where $\partial w/\partial\tau$ denotes the tangential derivative in direction τ.

Then the error e can be decomposed as the form

$$\alpha\, \mathbf{grad}_h e = \alpha\, \mathbf{grad} \varepsilon + \mathbf{curl}\, \xi, \tag{11}$$

where $\varepsilon \in H^1_E(\Omega)$ satisfies

$$(\alpha\, \mathbf{grad}\, \varepsilon, \mathbf{grad}\, v) = (\alpha\, \mathbf{grad}_h e, \mathbf{grad}\, v), \quad \forall v \in H^1_E(\Omega) \tag{12}$$

and $\xi \in \mathcal{H}$ satisfies

$$(\alpha^{-1} \mathbf{curl}\, \xi, \mathbf{curl} w) = (\alpha\, \mathbf{grad}_h e, \mathbf{grad}\, w), \quad \forall w \in \mathcal{H}, \tag{13}$$

where $\mathbf{curl}$ denotes the operator $\mathbf{curl}\, w = (-\partial_y w, \partial_x w)$. Moreover, it is valid that

$$\|\|e\|\|^2 = \|\|\varepsilon\|\|^2 + (\alpha^{-1} \mathbf{curl}\, \xi, \mathbf{curl}\, \xi). \tag{14}$$

Lemma 1 shows that the error e can be decomposed to the conforming part $\|\|\varepsilon\|\|^2$ and the nonconforming part $(\alpha^{-1} \mathbf{curl}\, \xi, \mathbf{curl}\, \xi)$.

The following Theorem 2 gives the error estimate of the conforming part.

Theorem 2. *Let $r \in L_2(\kappa)$ and $J^\nu \in L_2(\partial\kappa)$ denote the interior residual and the interelement flux jump, respectively. Then*

$$\|\|\varepsilon\|\|^2 \le \sum_{\kappa\in\mathcal{P}_h} \left\{ \left\| \boldsymbol{\sigma}_\kappa + \frac{1}{2} \mathbf{curl} \chi_\kappa \right\|_{\alpha^{-1},\kappa} + \Delta_\kappa \right\}^2, \tag{15}$$

where $\Delta_\kappa = C_{\mathcal{P}} h_\kappa \|r - \pi_\kappa r\|_{\alpha^{-1},\kappa} + (1/2) C_t \sum_{\gamma\subset\partial\kappa} h^{1/2}_\kappa \|J^\nu - \pi_\gamma J^\nu\|_{\alpha^{-1},\gamma}$, $\pi_\gamma r \in L_2(\kappa)$, $\|f\|_{\alpha^{-1},\kappa} = \|\alpha^{1/2} f\|_\kappa, \forall\kappa \in \mathcal{P}_h$, while Δ_κ is a quantity of higher order or even negligible compared with $\|\boldsymbol{\sigma}_\kappa + (1/2)\mathbf{curl}\, \chi_\kappa\|_{\alpha^{-1},\kappa}$. Both the vector-valued function $\boldsymbol{\sigma}_\kappa$ and the scalar-valued function χ_κ contain the interior residual (see [2])

$$r|_\kappa = f + \mathbf{div}_h(\alpha\, \mathbf{grad}_h u_h). \tag{16}$$

Moreover, there exists a positive constant c, independent of mesh-size, such that for each element there holds

$$c\left\|\boldsymbol{\sigma}_\kappa + \frac{1}{2}\mathbf{curl}\,\chi_\kappa\right\|_{\alpha^{-1},\kappa} \le |||\varepsilon|||_{\tilde{\kappa}} + \Delta_\kappa, \tag{17}$$

where $\tilde{\kappa}$ is a block including the element κ and its adjacent elements.

Lemma 3 plays a key role for obtaining the error estimate of the nonconforming part.

Lemma 3. *Let $\xi \in \mathscr{H}$, $\mathscr{H}$ be defined by* (10)*; then*

$$\left(\alpha^{-1}\mathbf{curl}\,\xi, \mathbf{curl}\,\xi\right) = \min_{w^*\in H_E^1(\Omega)} |||w^* - w_h|||. \tag{18}$$

Evidently, (18) gives an upper bound of the nonconforming part. It is important to note that the right hand side of (18) is the minimum value and the interpolation postprocessing function w^* appears in the right hand side of (18). Reference [18] has emphasized that an appropriate selection of w^* is the key to obtaining an effective computable upper bound a posteriori error estimate. And this requires that the function w^* is of a simple form and computable and makes the error of the nonconforming part effective.

Considering these factors, [2, 16] made such selection: w^* is taken to be a piecewise (pullback) biquadratic function on each element κ. The interpolation nodes of the function are the element vertices x_n, edge midpoints m_γ, and element centers $\overline{x}_\kappa$. The interpolation conditions are given by

$$\begin{gathered} w^*(x_n) = \begin{cases} 0, & \text{if } x_n \in \Gamma_D \\ \dfrac{1}{|\mathscr{P}_n|}\displaystyle\sum_{\kappa\in\mathscr{P}_n} w_h(x_n)|_\kappa, & \text{otherwise,} \end{cases} \\ w^*\left(m_\gamma\right) = \begin{cases} 0, & \text{if } m_\gamma \in \Gamma_D \\ h_\gamma^{-1}\displaystyle\int_\gamma w_h\,ds, & \text{otherwise,} \end{cases} \\ w^*(\overline{x}_\kappa) = w_h(\overline{x}_\kappa), \end{gathered} \tag{19}$$

where $\mathscr{P}_n \subset \mathscr{P}_h$ denotes the set of elements which share common vertex x_n, $|\mathscr{P}_n| = \text{card}\{\mathscr{P}_n\}$.

It is obvious that the function w^* defined above satisfies $w^* \in H_E^1(\Omega)$ and can be used to obtain an upper bound for the nonconforming part of the a posteriori error estimates.

Theorem 4 gives the reliability and validity of the nonconforming part.

Theorem 4. *Let $w^* \in H_E^1(\Omega)$ be constructed as described above; then*

$$\|\mathbf{curl}\,\boldsymbol{\xi}\|_{\alpha^{-1},\Omega} \le \left|\left\|w_h - w^*\right\|\right|. \tag{20}$$

Moreover, there exists a positive constant C, independent of any mesh-size, such that

$$\left|\left\|w_h - w^*\right\|\right|_\kappa \le C\|\mathbf{curl}\,\boldsymbol{\xi}\|_{\alpha^{-1},\tilde{\kappa}}. \tag{21}$$

Combining (14), (15), and (20), we have the following overall a posteriori error estimate:

$$|||e|||^2 \le \sum_{\kappa\in\mathscr{P}_h}\left(\left\|\boldsymbol{\sigma}_\kappa + \frac{1}{2}\mathbf{curl}\,\chi_\kappa\right\|_{\alpha^{-1},\kappa} + \Delta_\kappa\right)^2 + \left|\left\|w_h - w^*\right\|\right|^2. \tag{22}$$

Note that Δ_κ is a quantity of higher order compared with $\|\boldsymbol{\sigma}_\kappa + (1/2)\mathbf{curl}\,\chi_\kappa\|_{\alpha^{-1},\kappa}$, or even negligible. Let w_h be the approximate solution of (8); we define a computable upper bound a posteriori error indicator by

$$\eta_{\text{com}}^2 = \eta_c^2 + \eta_{\text{nc}}^2 \tag{23}$$

in which $\eta_c^2 = \sum_{\kappa\in\mathscr{P}_h}\|\boldsymbol{\sigma}_\kappa + (1/2)\mathbf{curl}\,\chi_\kappa\|_{\alpha^{-1},\kappa}^2$ denotes the a posteriori error indicator of conforming part and $\eta_{\text{nc}}^2 = |||w_h - w^*|||^2$ the a posteriori error indicator of nonconforming part. Hence, we can use η_{com} as the error estimate indicator of w_h.

Obviously, the error indicator η_{com} does not include a general constant C and is an effective error indicator (see Table 1). So, we are very interested in the error indicator η_{com} and decide to apply the indicator η_{com} to eigenvalue problem (1).

3. A Posteriori Error Estimate of the Eigenvalue Problem

In this section, we apply the error indicator η_{com} to the eigenvalue problem (1) and obtain a computable upper bound a posteriori error indicator η_{com}^2 with $f = \lambda_h u_h$ in (16), where (λ_h, u_h) is the kth eigenpair of (6).

In order to establish the error indicator η_{com}^2, we need the following results, cited from [4, 19, 20], respectively, as our Lemmas 5, 6, and 7.

Lemma 5. *Let (λ_h, u_h) be the kth eigenpair of* (6) *with $\|u_h\|_b = 1$, let λ be the kth eigenvalue of* (2)*, and let $M(\lambda) \subset H^{1+r}(\Omega)$ be the eigenspace corresponding to λ. Then $\lambda_h \to \lambda$, and there exists $u \in M(\lambda)$ with $\|u\|_b = 1$, such that*

$$\begin{gathered} \left|\lambda_h - \lambda\right| \le C\lambda^2 h^{2r}, \\ \left\|u_h - u\right\|_b \le Ch^{2r}, \\ \left\|u_h - u\right\|_h \le C\lambda h^r, \end{gathered} \tag{24}$$

where ω is the largest inner angle of Ω with the edges parallel with axis. If $\omega > \pi$ then $r < \pi/\omega$ and sufficiently close to π/ω, and $\omega < \pi$; then $r = 1$.

Let $\Omega \subset R^2$ be the bounded domain. Define operators $T : L_2(\Omega) \to H_E^1(\Omega)$ satisfies

$$a\left(Tf, v\right) = b\left(f, v\right), \quad \forall f \in L_2(\Omega),\ \forall v \in H_E^1(\Omega), \tag{25}$$

and $T_h : L_2(\Omega) \to X_{h,E}$ satisfies

$$a_h\left(T_h f, v\right) = b\left(f, v\right), \quad \forall f \in L_2(\Omega),\ \forall v \in X_{h,E}. \tag{26}$$

It is easy to know that (2) and (6) have the equivalent operator forms $\lambda Tu = u$ and $\lambda_h T_h u_h = u_h$, respectively. Meanwhile, we have the following estimates.

Lemma 6. *Under the assumption of Lemma 5. Moreover, if $\|T - T_h\|_b \to 0$ ($h \to 0$) and there exists a positive constant C independent of mesh-size h and $q_1 < q_2$, such that $\forall f \in L_2(\Omega)$, $\|Tf - T_h f\|_h \leq Ch^{q_1}\|f\|_b$, and $\|Tf - T_h f\|_b \leq Ch^{q_2}\|f\|_b$, then there exists $u \in M(\lambda)$ with $\|u\|_b = 1$, such that $\forall v \in H^1_E(\Omega) \cap X_{h,E}$*

$$\|u_h - u\|_{h,D} = \lambda_h \|Tu_h - T_h u_h\|_{h,D} + R_1, \tag{27}$$

$$\begin{aligned}\lambda_h - \lambda = {} & \lambda_h^2 \|Tu_h - T_h u_h\|_h^2 \\ & + 2\lambda_h a_h\left(Tu_h - T_h u_h, u_h - v\right) + R_2,\end{aligned} \tag{28}$$

where $D \subset \Omega$, R_1, R_2 are infinitesimals of higher order.

Lemma 7. *Let (λ, u) and (λ_h, u_h) be the solutions of problems (2) and (6), respectively. Then $\forall v \in X_{h,E}$*

$$\begin{aligned}\lambda - \lambda_h = {} & \|u - u_h\|_h^2 - \lambda_h \|v - u_h\|_b^2 \\ & - 2\lambda_h b\left(u - v, u_h\right) + 2a_h\left(u - v, u_h\right).\end{aligned} \tag{29}$$

Under the above preparations, we can obtain the following error estimates for the eigenvalue and eigenfunction of problem (1).

Theorem 8. *Let (λ_h, u_h) be the kth nonconforming rotated Q_1 element eigenpair of (6) with $\|u_h\|_b = 1$, and let λ be the kth eigenvalue of (2). Moreover, let Ω be a concave domain and let the eigenfunction u be singular. Then we have*

$$\|u_h - u\|_h \leq \eta_{com} + R_1, \tag{30}$$

$$\left|\lambda - \lambda_h\right| = \|u_h - u\|_h^2 + o\left(h^{2r}\right); \tag{31}$$

Thus,

$$\left|\lambda - \lambda_h\right| \leq \eta_{com}^2 + o\left(h^{2r}\right) + R_2, \tag{32}$$

where $R_2 = R_1^2 + 2R_1\eta_{com}$ is infinitesimal of higher order compared with η_{com}^2.

Proof. Taking $f = \lambda_h u_h$, $g = 0$ in (8), then $w = \lambda_h T u_h$ and $w_h = \lambda_h T_h u_h = u_h$. By (22), we have

$$\|\lambda_h T u_h - \lambda_h T_h u_h\|_h \leq \eta_{\text{com}}. \tag{33}$$

Combining (33) and (27) (taking $D = \Omega$), we get

$$\|u_h - u\|_h = \lambda_h \|Tu_h - T_h u_h\|_h + R_1 \leq \eta_{\text{com}} + R_1, \tag{34}$$

which shows that (30) holds.

In order to prove (31) and (32), we define interpolation operator $I_h : H^1_E(\Omega) \to X_{h,E}$ by

$$\int_\gamma I_h u \, ds = \int_\gamma u \, ds, \quad \forall \gamma \in \partial\kappa, \ \forall \kappa \in \mathcal{P}_h, \ \forall u \in H^1_E(\Omega). \tag{35}$$

Let $v = I_h u$ in Lemma 7; the fourth term on the right-hand side of (29) vanishes (see [21, 22]).

Considering the third term of (29), from the interpolation error estimate, we have

$$\left|-2\lambda_h b\left(u - I_h u, u_h\right)\right| \leq Ch^{1+r}\|u\|_{1+r}, \tag{36}$$

and that, according to Lemma 5, we know that the second and the third terms are infinitesimals of higher order comparing with the first term. Hence, the error $\lambda - \lambda_h$ completely hinges on the first term on the right-hand side of (29); that is, (31) holds. Combining (30) and (31), we obtain (32). □

Remark 9. For the nonconforming rotated Q_1 finite element, only when $\pi < \omega < 2\pi$ would the estimation $| - 2\lambda_h b(u - I_h u, u_h)| \leq Ch^{1+r}\|u\|_{1+r}$ be valid (see [19]). Therefore, it is necessary to assume that Ω be a concave domain.

From (32) and (30), we can obtain the computable upper bound a posteriori error indicators η_{com}^2 and η_{com} for the eigenvalue λ_h and the associated eigenfunction u_h, respectively.

4. Extension and Application

In this section, we extend the error indicator η_{com} to the Steklov eigenvalue problem and also obtain an effective error indicator η_{com}^2 with $f = \lambda_h u_h$ and $\alpha = 1$ in (16), where λ_h and u_h are the approximations of (37).

The Steklov eigenvalue problem reads as follows:

$$-\Delta u + u = 0, \quad \text{in } \Omega, \qquad \mathbf{n} \cdot \mathbf{grad}\, u = \lambda u, \quad \text{on } \partial\Omega, \tag{37}$$

where $\Omega \subset R^2$ is a bounded convex polygonal domain.

We have Steklov eigenvalue problem in its variational formulation: find $(\lambda, u) \in \mathbb{R} \times H^1(\Omega)$ with $\|u\|_b = 1$, so that

$$a(u, v) = \lambda b(u, v), \quad \forall v \in H^1(\Omega), \tag{38}$$

where $a(u, v) = \int_\Omega \nabla u \nabla v + uv \, dx$, $b(u, v) = \int_{\partial\Omega} uv \, ds$, $\|u\|_b = \sqrt{b(u, v)}$. Clearly $a(\cdot, \cdot)$ is a symmetric, continuous, and $H^1(\Omega)$-elliptic bilinear form defined on $H^1(\Omega) \times H^1(\Omega)$.

The nonconforming finite element approximation of (38) is the following: find $(\lambda_h, u_h) \in \mathbb{R} \times X_h$ with $\|u_h\|_b = 1$, such that

$$a_h\left(u_h, v\right) = \lambda_h b\left(u_h, v\right), \quad \forall v \in X_h, \tag{39}$$

where $a_h(u_h, v) = \sum_{\kappa \in \mathcal{P}_h} \int_\kappa \nabla u_h \nabla v + u_h v \, dx$. Define $\|v\|_h = \sqrt{a_h(v, v)}$. Evidently, $\|\cdot\|_h$ is the norm on X_h and $a_h(\cdot, \cdot)$ is uniformly X_h-elliptic. In fact, $a_h(v, v) = \|v\|_h^2, \forall v \in X_h$.

To define two useful operators, we need the source problem (40) associated with (38) and the discrete problem (41).

Find $\varphi \in H^1(\Omega)$, satisfies

$$a(\varphi, v) = b(\rho, v), \quad \forall v \in H^1(\Omega), \tag{40}$$

and find $\varphi_h \in X_h$, such that

$$a_h\left(\varphi_h, v\right) = b\left(\rho, v\right), \quad \forall v \in X_h. \tag{41}$$

Using the source problem (40), we define the operators A and T:

$$A : L_2(\partial\Omega) \longrightarrow H^{3/2}(\Omega) \subset H^1(\Omega),$$
$$a(A\rho, v) = b(\rho, v), \quad \forall v \in H^1(\Omega), \tag{42}$$
$$T\rho = (A\rho)', \qquad T : L_2(\partial\Omega) \longrightarrow H^1(\partial\Omega),$$

where the symbol "$\prime$" denotes the restriction to $\partial\Omega$. Bramble and Osborn [8] proved that (38) has the operator form $Tu = (1/\lambda)u$.

Since $a_h(\cdot,\cdot)$ is uniformly elliptic with respect to h, the problem (41) has unique solution. We then define the operators:

$$A_h : L_2(\partial\Omega) \longrightarrow X_h,$$
$$a_h(A_h\rho, v) = b(\rho, v), \quad \forall v \in X_h, \tag{43}$$
$$T_h\rho = (A_h\rho)', \qquad T_h : L_2(\partial\Omega) \longrightarrow \partial X_h \subset L_2(\partial\Omega).$$

From [10], (39) has the operator form $T_h u_h' = (1/\lambda_h)u_h'$. T and T_h are self-adjoint, completely continuous operators and $\|T_h - T\|_b \rightarrow 0, (h \rightarrow 0)$.

For the Steklov eigenvalue problem, we need the following error estimates (see [10]) and expansion (see [21]) which will be used in our subsequent analysis.

Lemma 10. *Let (λ_h, u_h) be the kth nonconforming rotated Q_1 element eigenpair of (39) with $\|u_h\|_b = 1$, and let λ be the kth eigenvalue of (38). Then $\lambda_h \rightarrow \lambda$, and there exists $u \in M(\lambda)$ with $\|u\|_b = 1$, such that*

$$\left|\lambda_h - \lambda\right| \le Ch^2, \tag{44}$$
$$\left\|u_h - u\right\|_b + \left\|A_h u - Au\right\|_b \le Ch^{3/2}, \tag{45}$$
$$\left\|u_h - u\right\|_h + \lambda\left\|A_h u - Au\right\|_h \le Ch, \tag{46}$$

where $M(\lambda)$ is the space spanned by eigenvector corresponding λ.

Lemma 11. *Let $(\lambda, u) \in \mathbb{R}\times H^1(\Omega)$ be an eigenpair of (37) and let $(\lambda_h, u_h) \in \mathbb{R} \times X_h$ be an eigenpair of (39). Then $\forall v \in X_h$*

$$\begin{aligned}\lambda - \lambda_h = {} & \|u - u_h\|_h^2 - \lambda_h\|v - u_h\|_b^2 \\ & + \lambda_h\left(\|v\|_b^2 - \|u\|_b^2\right) + 2a_h(u - v, u_h).\end{aligned} \tag{47}$$

According to the above consequences, we have the following theorem which can be proved with the approach in [4].

Theorem 12. *Under the assumption of Lemma 10. Then there exists $u \in M(\lambda)$ with $\|u\|_b = 1$, such that*

$$\left\|u_h - u\right\|_h = \lambda_h\left\|A_h u_h - Au_h\right\|_h + R, \tag{48}$$

where $|R| \le Ch^{3/2}$.

Proof. From the definitions of A and A_h, we obtain

$$\begin{aligned}u_h - u &= \lambda_h A_h u_h - \lambda A u \\ &= \lambda_h A_h u_h - \lambda_h A u_h + \lambda_h A u_h \\ &\quad - \lambda_h A u + \lambda_h A u - \lambda A u \\ &= \lambda_h(A_h - A)u_h + \lambda_h A(u_h - u) + (\lambda_h - \lambda)Au.\end{aligned} \tag{49}$$

Let $R = \|u_h - u\|_h - \lambda_h\|A_u u_h - Au_h\|_h$, combining the triangle inequality, (49), (44), and (45), we deduce

$$\begin{aligned}|R| &= \left|\left\|u_h - u\right\|_h - \lambda_h\left\|A_u u_h - Au_h\right\|_h\right| \\ &\le \left\|u_h - u - \lambda_h(A_h - A)u_h\right\|_h \\ &\le C\left(\left|\lambda_h - \lambda\right| + \lambda_h\left\|u_h - u\right\|_b\right) \le Ch^{3/2}.\end{aligned} \tag{50}$$

That is, (48) is obtained. □

Based on (48), we have the following computable upper bound a posteriori error estimate of the eigenfunction of (39).

Theorem 13. *Under the assumption of Lemma 10. Then exists $u \in M(\lambda)$ with $\|u\|_b = 1$, such that*

$$\left\|u_h - u\right\|_h \le \eta_{com} + o\left(h^{3/2}\right) + o\left(\left\|u_h - w\right\|_{-1}\right). \tag{51}$$

Proof. Consider the auxiliary problem

$$-\Delta w = -u_h, \quad \text{in } \Omega, \qquad \mathbf{n}\cdot\mathbf{grad}\, w = \lambda_h u_h, \quad \text{on } \partial\Omega; \tag{52}$$

under the condition of $\int_\Omega w\,dx = 0$, the auxiliary problem exists a unique solution only up to additive constant. Let w be the exact solution and let w_h be the approximate solution of (52) and (λ_h, u_h) be a rotated Q_1 element eigenpair of (39) obviously, $w_h = u_h$. Taking $f = -u_h$, $\alpha = 1$, and $g = \lambda_h u_h$ in (7), from the a posteriori error estimate (22) and the definition of η_{com}, we have

$$\left\|w_h - w\right\|_h \le \eta_{\text{com}}. \tag{53}$$

For the source problem (40) and (41), taking $\rho = \lambda_h u_h$, then $\varphi = \lambda_h A u_h$, $\varphi_h = \lambda_h A_h u_h = u_h$. We deduce

$$\begin{aligned}a(\varphi - w, v) &= a(\varphi - u_h + u_h - w, v) \\ &= a(\varphi, v) - \int_\Omega \nabla u_h \nabla v - \int_\Omega u_h v \\ &\quad + \int_\Omega \nabla(u_h - w)\nabla v + (u_h - w, v) \\ &= a(\varphi, v) - \int_\Omega u_h v + \int_\Omega \Delta w v \\ &\quad - \int_{\partial\Omega}(\mathbf{n}\cdot\mathbf{grad}\, w)\, v + (u_h - w, v) \\ &= a(\varphi, v) - b(\rho, v) + (u_h - w, v) \\ &= (u_h - w, v).\end{aligned} \tag{54}$$

Let $H^{-1}(\Omega)$ denote the dual space of $H^1(\Omega)$ with norm

$$\|f\|_{-1} = \sup_{v\in H^1(\Omega)} \frac{(f, v)}{\|v\|_1}, \quad \forall f \in H^{-1}(\Omega). \tag{55}$$

Setting $v = \varphi - w$, it follows that

$$\|\varphi - w\|_1^2 \le \sup_{v\in H^1(\Omega)} \frac{(u_h - w, v)}{\|v\|_1}\|v\|_1 = \|u_h - w\|_{-1}\|\varphi - w\|_1; \tag{56}$$

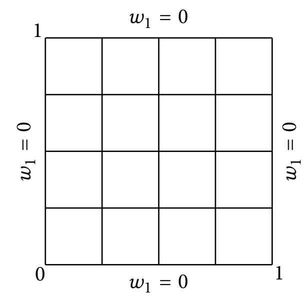

Figure 1

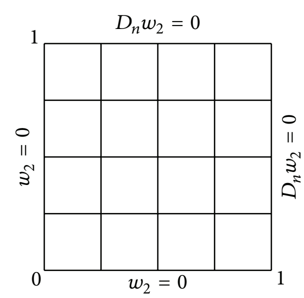

Figure 2

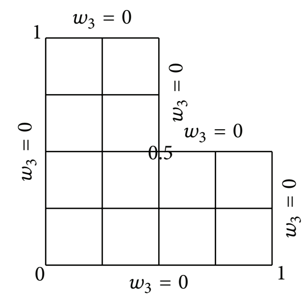

Figure 3

thus,

$$\|\varphi - w\|_1 \leq \|u_h - w\|_{-1}. \tag{57}$$

Further,

$$\begin{aligned} \|\varphi - u_h\|_h &= \|\varphi - w + w - u_h\|_h \\ &= \|u_h - w\|_h + o\left(\|u_h - w\|_{-1}\right). \end{aligned} \tag{58}$$

From (53) and (58), we find

$$\begin{aligned} \|\lambda_h A u_u - \lambda_h A_h u_h\|_h &= \|\varphi - \varphi_h\|_h = \|\varphi - u_h\|_h \\ &= \|w_h - w\|_h + o\left(\|u_h - w\|_{-1}\right) \\ &\leq \eta_{\text{com}} + o\left(\|u_h - w\|_{-1}\right). \end{aligned} \tag{59}$$

Substituting (59) into (48), we obtain (51). □

In Theorem 13, $o(h^{3/2})$ and $o(\|u_h - w\|_{-1})$ are generally infinitesimals of higher order comparing with η_{com}. Therefore, we can use η_{com} as a computable upper bound a posteriori error indicator for the eigenfunction u_h of (39).

The next corollary gives a relation between the error in the eigenvalue and eigenfunction approximations.

Corollary 14. *Under the assumption of Lemma 10, we have*

$$\lambda - \lambda_h \leq \eta_{com}^2 + 2\int_\Omega (u - I_h u)\, u_h + O\left(h^{5/2}\right) + R, \tag{60}$$

where R is infinitesimal of higher order comparing with η_{com}^2.

Proof. We define interpolation operator $I_h : H^1(\Omega) \to X_h$, such that

$$\int_\gamma I_h u\, ds = \int_\gamma u\, ds, \quad \forall \gamma \subset \partial\kappa,\ \forall \kappa \in \mathcal{P}_h,\ \forall u \in H^1(\Omega). \tag{61}$$

Taking $v = I_h u$ in Lemma 11, for the fourth term and the third term on the right-hand side of (47), we have (see [10], pp: 2397–2398)

$$\begin{aligned} a_h(u - I_h u, u_h) &= \int_\Omega (u - I_h u)\, u_h dx, \\ \left| \|I_h u\|_b^2 - \|u\|_b^2 \right| &\leq C h^{5/2}. \end{aligned} \tag{62}$$

For the second term on the right-hand side of (47), by (45), we get

$$\left| -\lambda_h \|I_h u - u_h\|_b^2 \right| \leq C h^3. \tag{63}$$

Thus,

$$\lambda - \lambda_h = \|u - u_h\|_h^2 + 2\int_\Omega (u - I_h u)\, u_h + O\left(h^{5/2}\right). \tag{64}$$

Combining (51) and (64), we obtain (60). □

Remark 15. Under certain conditions, we can prove that $2\int_\Omega (u - I_h u) u_h$ is a infinitesimal of higher order than $\|u - u_h\|_h^2$. From Lemma 11 of Yang et al. [10], we have $|\int_\Omega (u - I_h u) u_h| \leq C h^2$ and $\|u - u_h\|_h \leq Ch$ provided that $u \in H^2(\Omega)$. Similarly, if $u \in H^{1+r}(0 < r < 1)$, Ω is a concave domain and the eigenfunction u_h is singular. Then, $|\int_\Omega (u - I_h u) u_h|$ may reach $O(h^{1+r})$ convergence rate and $\|u - u_h\|_h$ may reach $O(h^r)$. So, the error $\lambda - \lambda_h$ hinges on $\|u - u_h\|_h^2$. By (60) and (64), we can obtain a computable upper a posteriori error indicator η_{com}^2 for the eigenvalue λ_h. The numerical results (see Table 3) show that this hypothesis is appropriate.

5. Numerical Examples

This section will report some computational results for the computable upper bound a posteriori error indicators η_{com} and η_{com}^2. For the sake of simplicity, we take $\alpha = 1$, and the partition $\mathcal{P}_h$ is uniform square meshes in problems (1) and (7).

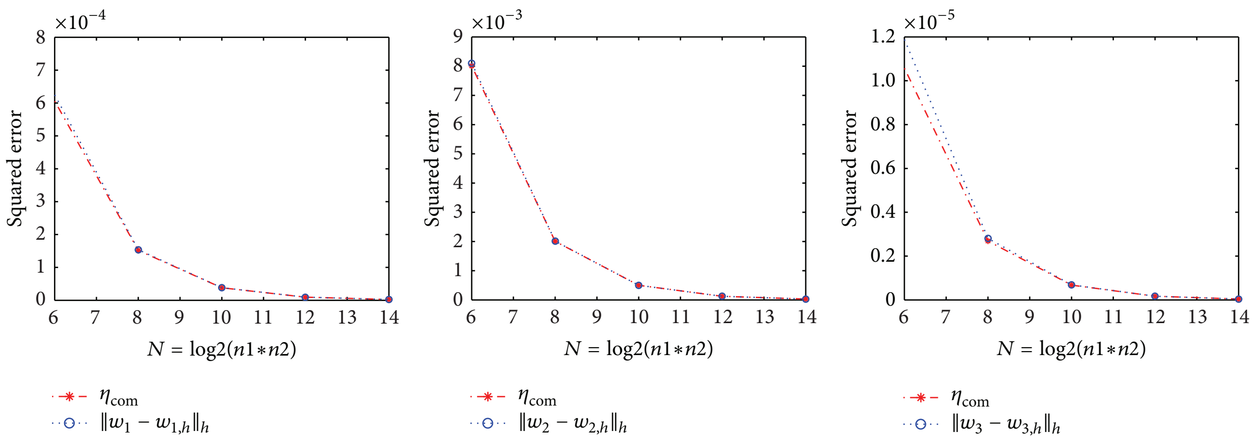

Figure 4: The error for test functions 1, 2, and 3.

We now verify that the error indicator η_{com} is effective for the boundary value problem (7) by the following three different types of test functions. The corresponding boundary conditions are shown in Figures 1, 2, and 3. The numerical results are listed in Table 1.

Example 1. Consider the equation $-\Delta w = f$ on the square domain $\Omega = [0,1]^2$ and the L-shaped domain $\Omega = [0,1]^2 \setminus [0.5,1]^2$, respectively.

Test function 1: $w_1 = x(1-x)y(1-y)$, corresponding $f = 2x(1-x) + 2y(1-y)$.

Test function 2: $w_2 = \sin((\pi/2)x)(e^y - 1 - ey)$, corresponding $f = \sin((\pi/2)x)[-(\pi^2/4)(e^y - 1 - ey) + e^y]$.

Test function 3: $w_3 = (x-0.5)(y-0.5)x(x-1)y(y-1)$, corresponding $f = (3/2)(2x-1)(2y-1)(-x^2 + x - y^2 + y)$, where $w_{i,h}$ $(i = 1,2,3)$ denotes the nonconforming rotate Q_1 element approximations. From Table 1 we find out the ratio $\eta_{com}/\|w_i - w_{i,h}\|_h$ converges that to 1 rapidly, when the number $n1 \times n2$ of the elements increases gradually. Namely, the a posteriori error indicator η_{com} is effective (see Figure 4).

Next we will compute the validity of the error indicator η^2_{com} of the eigenvalue problem (1). The numerical results are listed in Table 2.

Example 2. Consider the eigenvalue problem $-\Delta u = \lambda u$ on the L-shaped domain $\Omega = [0,1]^2 \setminus [0.5,1]^2$ (see Figure 3). Here we take $\lambda_1 \approx 4 * 9.63972$, $\lambda_2 \approx 4 * 15.19725$, and $\lambda_3 \approx 4 * 19.73921$, respectively. $\lambda_{i,h}$ denotes the ith approximate eigenvalue.

In Table 2, we can see that the indicators η^2_{com} for the first eigenvalue and second eigenvalue are effective and reliable, respectively. But the indicator η^2_{com} for the third eigenvalue is distortion, obviously, for which reason is that the eigenfunction is smooth corresponding to the eigenvalue λ_3. So, in Theorem 8, the assumptions, in which Ω is a concave domain and the eigenfunction is singular, are necessary.

According to the explanation in Remark 15, we compute the validity of the error indicator η^2_{com} of the Steklov eigenvalue problem on the L-shaped domain $\Omega = [0,1]^2 \setminus [0.5,1]^2$ (see Figure 3).

Example 3. Consider the problems (37), where λ_2, λ_3, and λ_4 were taken at the midpoint value of the interval [0.893602779, 0.893736398], [1.688598128, 1.688606742], and [3.21786760, 3.217900202], respectively. Taking into account the possibility that $M(\lambda_1) \subset H^2(\Omega)$, λ_1 can be omitted.

We can see that the ratio $\eta^2_{com}/|\lambda_{i,h} - \lambda_i| (i = 2,3,4)$ converges to 1 from Table 3, which proves that our supposition is rational in Remark 15.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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