

Research Article

Local Fractional Variational Iteration Method for Inhomogeneous Helmholtz Equation within Local Fractional Derivative Operator

Xian-Jin Wang,¹ Yang Zhao,² Carlo Cattani,³ and Xiao-Jun Yang⁴

¹ Foundation College, Ningbo Dahongying University, Ningbo 315175, China

² Electronic and Information Technology Department, Jiangmen Polytechnic, Jiangmen 529090, China

³ Department of Mathematics, University of Salerno, Via Giovanni Paolo II, Fisciano, 84084 Salerno, Italy

⁴ Department of Mathematics and Mechanics, China University of Mining and Technology, Xuzhou, Jiangsu 221008, China

Correspondence should be addressed to Yang Zhao; zhaoyang19781023@gmail.com

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The inhomogeneous Helmholtz equation within the local fractional derivative operator conditions is investigated in this paper. The local fractional variational iteration method is applied to obtain the nondifferentiable solutions and the graphs of the illustrative examples are also shown.

1. Introduction

Helmholtz equation has played an important role in the partial differential equations arising in mathematical physics [1, 2]. In computing the solution of Helmholtz equation, some analytical and numerical methods were presented. For example, Ihlenburg and Babuška used the finite element method to deal with the Helmholtz equation [3]. Momani and Abuasad suggested the variational iteration method to solve the Helmholtz equation [4]. Rafei and Ganji reported the homotopy perturbation method to report the solution to the Helmholtz equation [5]. Bayliss et al. considered the iterative method to discuss the Helmholtz equation [6]. Benamou and Desprès reported the domain decomposition method for solving the Helmholtz equation [7]. Linton presented Green's function method for the Helmholtz equation [8]. Singer and Turkel proposed the finite difference method to solve the Helmholtz equation [9]. Otto and Larsson applied the second-order method to discuss the Helmholtz equation [10].

Recently, the fractional calculus [11, 12] was developed and applied to present some models in the fields, such as the fractional-order digital control systems [13], the fractional-order viscoelasticity [14], the fractional-order quantum mechanics [15], and fractional-order dynamics [16]. More recently, Samuel and Thomas report the fractional Helmholtz equations [17] and some methods for solving the fractional differential equations were reported in [18–23]. However, we are faced with the problem that there must be some calculus to deal with the nondifferentiable solution for Helmholtz equation, which was structured within the local fractional derivative [24–34]. In this paper, we consider the local fractional inhomogeneous Helmholtz equation in two-dimensional case [31, 32]:

$$\frac{\partial^{2\alpha}M(x,y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}M(x,y)}{\partial y^{2\alpha}} + \omega^{2\alpha}M(x,y) = f(x,y), \quad (1)$$

where f(x, y) is a local fractional continuous function and the local fractional partial derivative is defined as follows [24]:

$$\frac{\partial^{\alpha} f(x, y)}{\partial x^{\alpha}} \bigg|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^{\alpha} \left(f(x, y) - f(x_0, y) \right)}{\left(x - x_0 \right)^{\alpha}}, \quad (2)$$

with

$$\Delta^{\alpha} \left(f(x, y) - f(x_0, y) \right)$$

$$\cong \Gamma \left(1 + \alpha \right) \Delta \left(f(x, y) - f(x_0, y) \right).$$
(3)

The local fractional inhomogeneous Helmholtz equation in three-dimensional case was suggested as follows [29, 30]:

$$\frac{\partial^{2\alpha} M(x, y, z)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} M(x, y, z)}{\partial y^{2\alpha}} + \frac{\partial^{2\alpha} M(x, y, z)}{\partial z^{2\alpha}} + \omega^{2\alpha} M(x, y, z) = f(x, y, z),$$
(4)

where f(x, y, z) is a local fractional continuous function. Here, we use the local fractional variational iteration method [30–34] to solve the local fractional inhomogeneous Helmholtz equation in two-dimensional case. The structure of this paper is as follows. In Section 2, we analyze the local fractional variational iteration method. In Section 3, we present some illustrative examples. Finally, the conclusion is given in Section 4.

2. Analysis of the Local Fractional Variational Iteration Method

Here, we give the analysis of the local fractional variational iteration method as follows. We first consider the local fractional linear partial differential equation:

$$L_{\alpha}u + R_{\alpha}u = g(t), \qquad (5)$$

where L_{α} denotes linear local fractional derivative operator of order 2α , R_{α} denotes a lower-order local fractional derivative operator, and g(t) is the nondifferentiable source term.

Let the local fractional operator be defined as [24, 30–34]

$${}_{a}I_{b}^{(\alpha)}f(x) = \frac{1}{\Gamma(1+\alpha)} \int_{a}^{b} f(t) (dt)^{\alpha}$$
$$= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f(t_{j}) (\Delta t_{j})^{\alpha}$$
(6)

with the partitions of the interval [a, b], $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max{\Delta t_1, \Delta t_2, \Delta t_j, \ldots}$, and $j = 0, \ldots, N - 1, t_0 = a, t_N = b$.

We now structure the correctional local fractional functional in the form

$$u_{n+1}(x) = u_n(x) + {}_0I_x^{(\alpha)} \times \{\zeta(s) (L_\alpha u_n(s) + R_\alpha u_n(s) - g(s))\}.$$
(7)

Making the local fractional variation, we have

$$\delta^{\alpha} u_{n+1}(x) = \delta^{\alpha} u_n(x) + {}_0 I_x^{(\alpha)} \delta^{\alpha} \times \{\zeta(s) \left(L_{\alpha} u_n(s) + R_{\alpha} \tilde{u}_n(s) - \tilde{g}(s) \right) \} = 0$$
(8)

such that the following stationary conditions are given as

$$1 - \zeta(s)^{(\alpha)}\Big|_{s=x} = 0, \qquad \zeta(s)\Big|_{s=x} = 0,$$

$$\zeta(s)^{(2\alpha)}\Big|_{s=x} = 0.$$
(9)

In view of (9), we obtain the fractal Lagrange multiplier, which is given by

$$\zeta(s) = \frac{(s-x)^{\alpha}}{\Gamma(1+\alpha)}.$$
(10)

From (7) and (10), we reach at the local fractional variational iteration algorithm

$$u_{n+1}(x) = u_n(x) + {}_0I_x^{(\alpha)} \times \left\{ \frac{(s-x)^{\alpha}}{\Gamma(1+\alpha)} \left(L_{\alpha}u_n(s) + R_{\alpha}u_n(s) - g(s) \right) \right\},$$
(11)

where the nondifferentiable initial value is suggested as

$$u_0(x) = u(0) + \frac{x^{\alpha}}{\Gamma(1+\alpha)} u^{(\alpha)}(0).$$
 (12)

Therefore, from (11), we write the solution of (7) as follows:

$$u = \lim_{n \to \infty} u_n. \tag{13}$$

3. Some Illustrative Examples

In this section, we give some illustrative examples for solving the local fractional inhomogeneous Helmholtz equation in two-dimensional case.

We present the following local fractional inhomogeneous Helmholtz equation:

$$\frac{\partial^{2\alpha}M(x,y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}M(x,y)}{\partial y^{2\alpha}} + M(x,y) = \frac{x^{\alpha}}{\Gamma(1+\alpha)}E_{\alpha}(y^{\alpha}),$$
(14)

subject to the initial-boundary conditions:

$$\frac{\partial^{\alpha} M(0, y)}{\partial x^{\alpha}} = E_{\alpha}(-y^{\alpha}),$$

$$M(0, y) = 0.$$
(15)

Making use of (11), we structure the local fractional variational iteration algorithm as follows:

$$M_{n+1}(x, y) = M_n(x, y) + {}_0I_x^{(\alpha)} \frac{(s-x)^{\alpha}}{\Gamma(1+\alpha)} \times \left\{ \frac{\partial^{2\alpha}M_n(x, y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}M_n(x, y)}{\partial y^{2\alpha}} + M_n(x, y) - \frac{x^{\alpha}}{\Gamma(1+\alpha)} E_{\alpha}(y^{\alpha}) \right\},$$
(16)

where the initial value is presented as

$$M_0(x, y) = \frac{x^{\alpha}}{\Gamma(1+\alpha)} E_{\alpha}(-y^{\alpha}), \qquad (17)$$

whose plot is shown in Figure 1.



FIGURE 1: The graph of the initial value (17) where $\alpha = \ln 2 / \ln 3$.

In view of (16) and (17), we arrive at the first approximation:

$$M_{1}(x, y) = M_{0}(x, y) + {}_{0}I_{x}^{(\alpha)} \frac{(s-x)^{\alpha}}{\Gamma(1+\alpha)}$$

$$\times \left\{ \frac{\partial^{2\alpha}M_{0}(x, y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}M_{0}(x, y)}{\partial y^{2\alpha}} + M_{0}(x, y) - \frac{x^{\alpha}}{\Gamma(1+\alpha)}E_{\alpha}(-y^{\alpha}) \right\}$$

$$= \frac{x^{\alpha}}{\Gamma(1+\alpha)}E_{\alpha}(-y^{\alpha})$$

$$+ {}_{0}I_{x}^{(\alpha)}\frac{(s-x)^{\alpha}}{\Gamma(1+\alpha)}\left\{ \frac{x^{\alpha}}{\Gamma(1+\alpha)}E_{\alpha}(-y^{\alpha}) \right\}$$

$$= \frac{x^{\alpha}}{\Gamma(1+\alpha)}E_{\alpha}(-y^{\alpha}) - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)}E_{\alpha}(-y^{\alpha})$$

$$= \left(\frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)}\right)E_{\alpha}(-y^{\alpha}).$$
(18)

The second approximation is

$$M_{2}(x, y) = M_{1}(x, y) + {}_{0}I_{x}^{(\alpha)} \frac{(s-x)^{\alpha}}{\Gamma(1+\alpha)}$$

$$\times \left\{ \frac{\partial^{2\alpha}M_{1}(x, y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}M_{1}(x, y)}{\partial y^{2\alpha}} + M_{1}(x, y) - \frac{x^{\alpha}}{\Gamma(1+\alpha)}E_{\alpha}(-y^{\alpha}) \right\}$$

$$= \left(\frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{x^{3\alpha}}{\Gamma(1+\alpha)} \right) E_{\alpha}(-y^{\alpha})$$

$$+ {}_{0}I_{x}^{(\alpha)} \frac{(s-x)^{\alpha}}{\Gamma(1+\alpha)} \left\{ -\frac{2x^{3\alpha}}{\Gamma(1+3\alpha)}E_{\alpha}(-y^{\alpha}) \right\}$$

$$= \frac{x^{\alpha}}{\Gamma(1+\alpha)} E_{\alpha}(-y^{\alpha}) - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} E_{\alpha}(-y^{\alpha})$$
$$+ \frac{2x^{5\alpha}}{\Gamma(1+5\alpha)} E_{\alpha}(-y^{\alpha})$$
$$= \left(\frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{2x^{5\alpha}}{\Gamma(1+5\alpha)}\right)$$
$$\times E_{\alpha}(-y^{\alpha}).$$
(19)

Making best of (16) and (19), the third approximation reads as

$$M_{3}(x, y) = M_{2}(x, y) + {}_{0}I_{x}^{(\alpha)} \frac{(s-x)^{\alpha}}{\Gamma(1+\alpha)}$$

$$\times \left\{ \frac{\partial^{2\alpha}M_{2}(x, y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}M_{2}(x, y)}{\partial y^{2\alpha}} + M_{2}(x, y) - \frac{x^{\alpha}}{\Gamma(1+\alpha)}E_{\alpha}(-y^{\alpha}) \right\}$$

$$= \left(\frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{2x^{5\alpha}}{\Gamma(1+5\alpha)} \right)$$

$$\times E_{\alpha}(-y^{\alpha}) + E_{\alpha}(-y^{\alpha}) {}_{0}I_{x}^{(\alpha)}$$

$$\times \frac{(s-x)^{\alpha}}{\Gamma(1+\alpha)} \left(\frac{4x^{5\alpha}}{\Gamma(1+5\alpha)} \right)$$

$$= \frac{x^{\alpha}}{\Gamma(1+\alpha)}E_{\alpha}(-y^{\alpha}) - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)}E_{\alpha}(-y^{\alpha})$$

$$+ \frac{2x^{5\alpha}}{\Gamma(1+5\alpha)}E_{\alpha}(-y^{\alpha}) - \frac{4x^{7\alpha}}{\Gamma(1+7\alpha)}E_{\alpha}(-y^{\alpha})$$

$$= \left(\frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{2x^{5\alpha}}{\Gamma(1+5\alpha)} \right) E_{\alpha}(-y^{\alpha}).$$
(20)

From (16) and (20), we obtain the fourth approximation of (14) given as

$$\begin{split} M_4\left(x,y\right) &= M_3\left(x,y\right) + {}_0I_x^{(\alpha)} \, \frac{(s-x)^{\alpha}}{\Gamma\left(1+\alpha\right)} \\ &\times \left\{ \frac{\partial^{2\alpha}M_3\left(x,y\right)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}M_3\left(x,y\right)}{\partial y^{2\alpha}} + M_3\left(x,y\right) \right. \\ &\left. - \frac{x^{\alpha}}{\Gamma\left(1+\alpha\right)} E_{\alpha}\left(-y^{\alpha}\right) \right\} \end{split}$$

$$= \left(\frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{2x^{5\alpha}}{\Gamma(1+5\alpha)}\right)$$
$$- \frac{4x^{7\alpha}}{\Gamma(1+7\alpha)} E_{\alpha}(-y^{\alpha}) + E_{\alpha}(-y^{\alpha}) e_{\alpha}^{(\alpha)}$$
$$\times \frac{(s-x)^{\alpha}}{\Gamma(1+\alpha)} \left(-\frac{8x^{7\alpha}}{\Gamma(1+7\alpha)}\right)$$
$$= \frac{x^{\alpha}}{\Gamma(1+\alpha)} E_{\alpha}(-y^{\alpha}) - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} E_{\alpha}(-y^{\alpha})$$
$$+ \frac{2x^{5\alpha}}{\Gamma(1+5\alpha)} E_{\alpha}(-y^{\alpha}) - \frac{4x^{7\alpha}}{\Gamma(1+7\alpha)} E_{\alpha}(-y^{\alpha})$$
$$+ \frac{8x^{9\alpha}}{\Gamma(1+9\alpha)} E_{\alpha}(-y^{\alpha})$$
$$= \left(\frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{2x^{5\alpha}}{\Gamma(1+5\alpha)}\right)$$
$$- \frac{4x^{7\alpha}}{\Gamma(1+7\alpha)} + \frac{8x^{9\alpha}}{\Gamma(1+9\alpha)}\right) E_{\alpha}(-y^{\alpha}).$$
(21)

As similar manner, from (21), we arrive at the fifth approximate formula:

$$\begin{split} M_{5}\left(x,y\right) &= M_{4}\left(x,y\right) + {}_{0}I_{x}^{(\alpha)} \frac{\left(s-x\right)^{\alpha}}{\Gamma\left(1+\alpha\right)} \\ &\times \left\{ \frac{\partial^{2\alpha}M_{4}\left(x,y\right)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}M_{4}\left(x,y\right)}{\partial y^{2\alpha}} + M_{4}\left(x,y\right) \right. \\ &\left. - \frac{x^{\alpha}}{\Gamma\left(1+\alpha\right)} E_{\alpha}\left(-y^{\alpha}\right) \right\} \\ &= \left(\frac{x^{\alpha}}{\Gamma\left(1+\alpha\right)} - \frac{x^{3\alpha}}{\Gamma\left(1+3\alpha\right)} + \frac{2x^{5\alpha}}{\Gamma\left(1+5\alpha\right)} \right. \\ &\left. - \frac{4x^{7\alpha}}{\Gamma\left(1+7\alpha\right)} + \frac{8x^{9\alpha}}{\Gamma\left(1+9\alpha\right)} \right) E_{\alpha}\left(-y^{\alpha}\right) \right. \\ &\left. + E_{\alpha}\left(-y^{\alpha}\right) {}_{0}I_{x}^{(\alpha)} \frac{\left(s-x\right)^{\alpha}}{\Gamma\left(1+\alpha\right)} \left(\frac{16x^{9\alpha}}{\Gamma\left(1+9\alpha\right)} \right) \right. \\ &= \frac{x^{\alpha}}{\Gamma\left(1+\alpha\right)} E_{\alpha}\left(-y^{\alpha}\right) - \frac{x^{3\alpha}}{\Gamma\left(1+3\alpha\right)} E_{\alpha}\left(-y^{\alpha}\right) \\ &\left. + \frac{2x^{5\alpha}}{\Gamma\left(1+5\alpha\right)} E_{\alpha}\left(-y^{\alpha}\right) - \frac{4x^{7\alpha}}{\Gamma\left(1+7\alpha\right)} E_{\alpha}\left(-y^{\alpha}\right) \right. \\ &\left. + \frac{8x^{9\alpha}}{\Gamma\left(1+9\alpha\right)} E_{\alpha}\left(-y^{\alpha}\right) - \frac{16x^{11\alpha}}{\Gamma\left(1+11\alpha\right)} \right. \end{split}$$

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$$= \left(\frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{2x^{5\alpha}}{\Gamma(1+5\alpha)} - \frac{4x^{7\alpha}}{\Gamma(1+7\alpha)} + \frac{8x^{9\alpha}}{\Gamma(1+9\alpha)} - \frac{16x^{11\alpha}}{\Gamma(1+11\alpha)}\right) \times E_{\alpha}\left(-y^{\alpha}\right).$$
(22)

Hence, we have the local fractional series solution of (14):

$$M_{n}(x, y) = \left(\frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{2x^{5\alpha}}{\Gamma(1+5\alpha)} - \frac{4x^{7\alpha}}{\Gamma(1+7\alpha)} + \frac{8x^{9\alpha}}{\Gamma(1+9\alpha)} - \frac{16x^{11\alpha}}{\Gamma(1+11\alpha)} + \cdots\right) \times E_{\alpha}(-y^{\alpha}) = \left(\frac{1}{2}\frac{x^{\alpha}}{\Gamma(1+\alpha)} + \frac{1}{2}\sum_{i=0}^{\infty}(-1)^{i}\frac{2^{i}x^{(2i+1)\alpha}}{\Gamma(1+(2i+1)\alpha)}\right) \times E_{\alpha}(-y^{\alpha}).$$

$$(23)$$

From (13), we get the exact solution of (14) given as

$$M = \lim_{n \to \infty} M_n(x, y)$$

=
$$\lim_{n \to \infty} \left(\frac{1}{2} \frac{x^{\alpha}}{\Gamma(1+\alpha)} + \frac{1}{2} \sum_{i=0}^{\infty} (-1)^i \frac{2^i x^{(2i+1)\alpha}}{\Gamma(1+(2i+1)\alpha)} \right)$$

×
$$E_{\alpha}(-y^{\alpha})$$

=
$$\left[\frac{1}{2} \frac{x^{\alpha}}{\Gamma(1+\alpha)} + \frac{1}{2} \sin_{\alpha}(2x^{\alpha}) \right] E_{\alpha}(-y^{\alpha})$$

(24)

and its plot is illustrated in Figure 2.

Example 1. We suggest the following local fractional inhomogeneous Helmholtz equation:

$$\frac{\partial^{2\alpha} M(x, y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} M(x, y)}{\partial y^{2\alpha}} + M(x, y)$$

$$= \frac{x^{\alpha}}{\Gamma(1+\alpha)} \frac{y^{\alpha}}{\Gamma(1+\alpha)},$$
(25)

and the initial-boundary conditions read as

$$\frac{\partial^{\alpha} M(0, y)}{\partial x^{\alpha}} = \frac{y^{\alpha}}{\Gamma(1 + \alpha)},$$

$$M(0, y) = 0.$$
(26)



FIGURE 2: The graph of exact solution of (14) where $\alpha = \ln 2 / \ln 3$.

From (11), we set up the local fractional variational iteration algorithm as follows:

$$M_{n+1}(x, y) = M_n(x, y) + {}_0I_x^{(\alpha)} \frac{(s-x)^{\alpha}}{\Gamma(1+\alpha)} \times \left\{ \frac{\partial^{2\alpha}M_n(x, y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}M_n(x, y)}{\partial y^{2\alpha}} + M_n(x, y) - \frac{x^{\alpha}}{\Gamma(1+\alpha)} \frac{y^{\alpha}}{\Gamma(1+\alpha)} \right\},$$
(27)

where the initial value is suggested as

$$M_0(x, y) = \frac{x^{\alpha}}{\Gamma(1+\alpha)} \frac{y^{\alpha}}{\Gamma(1+\alpha)}.$$
 (28)

Appling (27) and (28) gives the first approximate solution:

$$M_{1}(x, y) = M_{0}(x, y) + {}_{0}I_{x}^{(\alpha)} \frac{(s-x)^{\alpha}}{\Gamma(1+\alpha)} \times \left\{ \frac{\partial^{2\alpha}M_{0}(x, y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}M_{0}(x, y)}{\partial y^{2\alpha}} + M_{0}(x, y) - \frac{x^{\alpha}}{\Gamma(1+\alpha)} \frac{y^{\alpha}}{\Gamma(1+\alpha)} \right\}$$

$$= \frac{x^{\alpha}}{\Gamma(1+\alpha)} \frac{y^{\alpha}}{\Gamma(1+\alpha)}.$$
(29)

Using (27) and (29), we obtain the second approximate term, which is expressed as follows:

$$M_{2}(x, y) = M_{1}(x, y) + {}_{0}I_{x}^{(\alpha)} \frac{(s-x)^{\alpha}}{\Gamma(1+\alpha)} \times \left\{ \frac{\partial^{2\alpha}M_{1}(x, y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}M_{1}(x, y)}{\partial y^{2\alpha}} + M_{1}(x, y) - \frac{x^{\alpha}}{\Gamma(1+\alpha)} \frac{y^{\alpha}}{\Gamma(1+\alpha)} \right\}$$

$$= \frac{x^{\alpha}}{\Gamma(1+\alpha)} \frac{y^{\alpha}}{\Gamma(1+\alpha)}.$$
(30)

In view of (27) and (30), we obtain the third approximation, which reads as follows:

$$M_{3}(x, y) = M_{2}(x, y) + {}_{0}I_{x}^{(\alpha)} \frac{(s-x)^{\alpha}}{\Gamma(1+\alpha)} \times \left\{ \frac{\partial^{2\alpha}M_{2}(x, y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}M_{2}(x, y)}{\partial y^{2\alpha}} + M_{2}(x, y) - \frac{x^{\alpha}}{\Gamma(1+\alpha)} \frac{y^{\alpha}}{\Gamma(1+\alpha)} \right\}$$

$$= \frac{x^{\alpha}}{\Gamma(1+\alpha)} \frac{y^{\alpha}}{\Gamma(1+\alpha)}.$$
(31)

Therefore, we arrive at the approximate term

$$M_n(x, y) = \frac{x^{\alpha}}{\Gamma(1+\alpha)} \frac{y^{\alpha}}{\Gamma(1+\alpha)},$$
(32)

which leads to the exact solution of (25) given as

$$M(x, y)$$

$$= \lim_{n \to \infty} M_n(x, y)$$

$$= \lim_{n \to \infty} \frac{x^{\alpha}}{\Gamma(1 + \alpha)} \frac{y^{\alpha}}{\Gamma(1 + \alpha)}$$

$$= \frac{x^{\alpha}}{\Gamma(1 + \alpha)} \frac{y^{\alpha}}{\Gamma(1 + \alpha)}$$
(33)

and its plot is illustrated in Figure 3.

4. Conclusions

In this work, the boundary value problems for the inhomogeneous Helmholtz equation within local fractional derivative operator were discussed by using the local fractional variational iteration method. Their nondifferentiable solutions are obtained and the graphs of the solutions with fractal dimension $\alpha = \ln 2 / \ln 3$ are also given.



FIGURE 3: The plot of exact solution of (25) where $\alpha = \ln 2 / \ln 3$.

Conflict of Interests

The authors declare that they have no competing interests in this paper.

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