Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2014, Article ID 972624, 14 pages
http://dx.doi.org/10.1155/2014/972624

Research Article

Gain-Scheduled $\mathscr{H}_2$ Controller Synthesis for Continuous-Time Polytopic LPV Systems

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Received 22 October 2013; Revised 15 January 2014; Accepted 16 January 2014; Published 18 March 2014

Academic Editor: Rongni Yang

This paper is concerned with the problem of gain-scheduled $\mathscr{H}_2$ controller synthesis for continuous-time linear parameter-varying systems. In this problem, the system matrices in the state-space form are polytopic and patameterized and the admissible values of the parameters are assumed to be measurable on-line in a polytope space. By employing a basis-parameter-dependent Lyapunov function and introducing some slack variables to the well-established performance conditions, sufficient conditions for the existence of the desired gain-scheduled $\mathscr{H}_2$ state feedback and dynamic output feedback controllers are established in terms of parameterized linear matrix inequalities. Based on the polytopic characteristic of the dependent parameters and a convexification method, the corresponding controller synthesis problem is then cast into finite-dimensional convex optimization problem which can be efficiently solved by using standard numerical softwares. Numerical examples are given to illustrate the effectiveness and advantage of the proposed methods.

1. Introduction

It is well known that linear parameter-varying (LPV) systems are a class of linear systems whose state-space matrices depend on a set of time-varying parameters which are not known in advance but can be measured or estimated upon operation of the systems. Gain-scheduled control strategies for LPV systems have been studied intensively in the last two decades and significant progress has been made in this area (see, e.g., [1–27]). There are many examples of parameter-dependent systems in practice, such as in aeronautics, aerospace, mechanics, and industrial processes (see, e.g., [2, 4, 12–14, 26]). This has motivated extensive studying of the gain-scheduled analysis and controller synthesis methods for various LPV systems from many aspects, including continuous-time and discrete-time systems, state-space formula and linear fractional transformation representation, affine-type and polytopic systems, state feedback and output feedback controllers, quadratic Lyapunov function-based and parameter-dependent Lyapunov function-based methods, and robust $\mathscr{H}_2$ and $\mathscr{H}_\infty$ control performances.

In most of the existing work, the gain-scheduled LPV control synthesis problems are performed through semidefinite programming and especially linear matrix inequality (LMI) techniques [2, 5–8, 10, 15–17, 19, 26]. This is due mainly to the fact that a number of methods of gain-scheduled control design for LPV systems proposed in the literature are based on small-gain approach (see, e.g., [1–4, 17]) or on the notions of quadratic Lyapunov function (see, e.g., [1, 7, 28]). The advantage of small-gain approach and quadratic Lyapunov function-based methods is that the associated computation is relatively straightforward (e.g., standard numerical software, LMI Control Toolbox [29]). However, the drawback of these methods is in that a single parameter-independent Lyapunov function must be used to guarantee both stability and control performance for all parameter values, and it can produce conservative results (see, e.g., [5, 10, 15, 26]). For the sake of reducing the abovementioned conservativeness, several control methods have been developed in the past decade, such as gain-scheduled $\mathscr{H}_2$ and $\mathscr{H}_\infty$ control based on parameter-dependent Lyapunov function (PDLF) method (see, e.g., [4, 5, 9, 10, 15, 18–21, 26–28, 30]). Until now, some researches have been carried out on $\mathscr{H}_2$ performance synthesis problems for continuous-time LPV systems (see, e.g., [9, 10, 26]). In [9, 10], de Souza et al. presented two novel state feedback $\mathscr{H}_2$ controllers for affine LPV systems based on quadratic PDLF frameworks. Also, Xie recently developed a gain-scheduled $\mathscr{H}_2$ state feedback for polytopic LPV systems with new LMI formulation by introducing additional slack variables [24, 26, 27]. Despite the recent development of gain-scheduled $\mathscr{H}_2$ analysis and controller synthesis for LPV systems, the issue associated with gain-scheduled $\mathscr{H}_2$ control via PDLF is not well documented so far, even in the case of state feedback.

From the above analysis, it is clear that the gain-scheduled $\mathscr{H}_2$ control problem for LPV systems has not been studied thoroughly, and this leads to the first objective of this paper for reducing the conservativeness of the existing state feedback controller. The second objective of this paper is to realize the dynamic output feedback control design for continuous-time polytopic LPV systems. It has to be stressed that the determination of a dynamic output feedback controller for polytopic LPV systems is indeed a difficult problem. The dynamic output feedback control is more flexible than static output feedback since additional dynamics of the controller are introduced [18–24]. Both are new contributions to the existing literature. To begin with, the result of gain-scheduled $\mathscr{H}_2$ analysis in [26] is introduced with some explanations. To reduce the conservativeness of the state feedback controller synthesis, an improved sufficient condition for the existence of desired gain-scheduled $\mathscr{H}_2$ state feedback controller is obtained in terms of parameterized linear matrix inequalities (PLMIs). Furthermore, based on polytopic characteristic of the parameter-dependent system, the corresponding controller synthesis problem is cast into a finite-dimensional convex optimization problem by a convexification method.

Considering the commonly encountered case in practice that the full state variables are unavailable for state feedback control design, we make further research on the gain-scheduled $\mathscr{H}_2$ dynamic output feedback controller synthesis. Inspired by the work of the basis-PDLF approach proposed in [20, 21, 31, 32], some auxiliary slack matrix variables are introduced in the process of expressing the relationships among the terms of the system equation. A sufficient condition for the existence of desired gain-scheduled $\mathscr{H}_2$ dynamic output feedback controller is initially obtained by PLMIs. Such a sufficient condition can guarantee that the closed-loop system is exponentially stable and has a prescribed $\mathscr{H}_2$ disturbance attenuation performance. Similar to the case of state feedback control design, the new convexification method is introduced to derive a finite-dimensional PLMI condition for the dynamic output feedback controller synthesis, which can be efficiently solved by using standard numerical software. Finally, simulation results of an example in a gain-scheduled $\mathscr{H}_2$ state feedback control indicate that our approach can generate less conservativeness than the existing results. The effectiveness of the proposed dynamic output feedback controller is verified by the other numerical example. It should be noticed that $\mathscr{H}_2$ performance in this paper can be used to capture both the response to stationary noise and the transient response of the closed-loop system. Different from the Lyapunov-based robust $\mathscr{H}_2$ performance, other methods lean too heavily on robustness and sacrifice an adequate view of performance. For example, robust $\mathscr{H}_\infty$ method treats disturbances or commands as being the worst in a very broad class which is often unrealistic.

The rest of this paper is organized as follows. The problem formulation and some preliminary results are presented in the next section. Section 3 gives our main results of gain-scheduled $\mathscr{H}_2$ state feedback and dynamic output feedback control design, respectively. Two numerical examples are given in Section 4 and we conclude this paper in Section 5.

Notations. We use the following notations throughout this paper. The superscript "T" stands for matrix transposition, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, and the notation $P > 0$ (≥ 0) means that P is real symmetrical and positive definite (semidefinite). $\mathrm{Tr}(\cdot)$ denotes the matrix trace, and $\mathrm{Her}\{A\}$ stands for $A+A^T$. In symmetric block matrices or long matrix expressions, we use an asterisk $(*)$ to represent a term that is induced by symmetry, and $\mathrm{diag}\{\cdots\}$ stands for a block-diagonal matrix. In addition, $\mathbf{I}$ and $\mathbf{0}$ denote identity matrix and zero matrix, respectively. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2. Problem Description and Preliminaries

2.1. Problem Description. Consider the following class of continuous-time LPV systems:

$$\begin{aligned}\mathcal{S}:\ \dot{x}(t) &= A(\theta(t))\,x(t) \\ &\quad + B_1(\theta(t))\,w(t) + B_2(\theta(t))\,u(t), \\ z(t) &= C_1(\theta(t))\,x(t) + D_1(\theta(t))\,u(t), \\ y(t) &= C_2(\theta(t))\,x(t),\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^m$ is the measured output, $z(t) \in \mathbb{R}^p$ is the controlled output, $u(t) \in \mathbb{R}^q$ is the control input, and $w(t) \in \mathbb{R}^l$ is the disturbance input. All the system matrices have compatible dimensions. The system matrices $A(\theta)$, $B_1(\theta)$, $B_2(\theta)$, $C_1(\theta)$, $D_1(\theta)$, and $C_2(\theta)$ are parameter-dependent matrices with respect to the time-varying scheduling parameter $\theta(t) = [\theta_1(t), \theta_2(t), \ldots, \theta_r(t)]^T \in \mathbb{R}^r$. Similar to the previous gain-scheduled $\mathscr{H}_2$ control problems for LPV systems [10, 26], the following assumptions are given.

Assumption 1. The state-space matrices $A(\theta)$, $B_1(\theta)$, $B_2(\theta)$, $C_1(\theta)$, $D_1(\theta)$, and $C_2(\theta)$ are continuous and bounded functions and depend affinely on $\theta(t)$.

Assumption 2. The parameter values of vector $\theta(t)$ are not known in advance but are measurable in real time. In addition, the parameter $\theta(t)$ is limited to a given convex bounded polyhedral domain $\mathscr{P}$ described by N vertices as

$$\theta(t) \in \mathscr{P} \triangleq \mathrm{Co}\{\omega_1, \omega_2, \ldots, \omega_N\}$$

$$= \left\{ \sum_{i=1}^{N} \alpha_i(t)\,\omega_i : \alpha_i(t) \geqslant 0, \sum_{i=1}^{N} \alpha_i(t) = 1, N = 2^r \right\}, \tag{2}$$

and the rate of variation $\dot{\theta}(t)$ is well defined over the time horizon and varies in a polytope $\mathscr{V}$ as

$$\begin{aligned}\dot{\theta}(t) \in \mathscr{V} &\triangleq \mathrm{Co}\{\nu_1, \nu_2, \ldots, \nu_N\} \\ &= \left\{ \sum_{k=1}^{N} \beta_k(t)\,\nu_k : \beta_k(t) \geqslant 0, \sum_{k=1}^{N} \beta_k(t) = 1, N = 2^r \right\}.\end{aligned} \tag{3}$$

Given the above sets $\mathscr{P}$ and $\mathscr{V}$, we define the parameter set $\mathscr{F}_{\mathscr{P}}^{\mathscr{V}}$ described by N vertices as

$$\mathscr{F}_{\mathscr{P}}^{\mathscr{V}} \triangleq \left\{\theta(t) \in \mathscr{C}^1\left(\mathbb{R}_+, \mathbb{R}^N\right) : \theta(t) \in \mathscr{P}, \dot{\theta}(t) \in \mathscr{V}, \forall t \geqslant 0\right\}. \tag{4}$$

Moreover, the LPV system $\mathscr{S}$ in (1) is called polytopic, when it ranges in a matrix polytope, that is, the LPV system (1) can be expressed as

$$\Omega(\theta) \triangleq \left(A(\theta), B_1(\theta), B_2(\theta), C_1(\theta), D_1(\theta), C_2(\theta)\right) \in \mathscr{R}, \tag{5}$$

where $\theta(t) \in \mathscr{F}_{\mathscr{P}}^{\mathscr{V}}$ and $\mathscr{R}$ is also a given convex bounded polyhedral domain described by N vertices:

$$\mathscr{R} \triangleq \left\{ \sum_{k=1}^{N} \alpha_i(t)\,\Omega_i : \alpha_i(t) \geqslant 0, \sum_{i=1}^{N} \alpha_i(t) = 1, N = 2^r \right\}. \tag{6}$$

Here, we are interested in designing both gain-scheduled $\mathscr{H}_2$ state feedback controller and dynamic output feedback controller for the system $\mathscr{S}$ described by (1). Therefore, two gain-scheduled $\mathscr{H}_2$ control laws are described by $\mathscr{K}_{\mathscr{SF}}$ and $\mathscr{K}_{\mathscr{DOF}}$, respectively, as follows:

$$\mathscr{K}_{\mathscr{SF}} : u(t) = K(\theta)\,x(t), \tag{7}$$

$$\begin{aligned}\mathscr{K}_{\mathscr{DOF}} : \dot{x}_K(t) &= A_K(\theta)\,x_K(t) + B_K(\theta)\,y(t), \\ u(t) &= C_K(\theta)\,x_K(t) + D_K(\theta)\,y(t),\end{aligned} \tag{8}$$

where $x_K(t) \in \mathbb{R}^n$ is controller state vector and $K(\theta)$ and $(A_K(\theta), B_K(\theta), C_K(\theta), D_K(\theta))$ are appropriately dimensioned LPV controller matrices to be determined.

Substituting the state-feedback control law $\mathscr{K}_{\mathscr{SF}}$ into (1), the closed-loop system can be obtained as

$$\begin{aligned}\mathscr{C}_{\mathscr{SF}} : \dot{x}(t) &= A_{\mathrm{cl}}(\theta)\,x(t) + B_{\mathrm{cl}}(\theta)\,w(t) \\ z(t) &= C_{\mathrm{cl}}(\theta)\,x(t),\end{aligned} \tag{9}$$

where

$$\begin{aligned}A_{\mathrm{cl}}(\theta) &= A(\theta) + B_2(\theta)\,K(\theta), \\ B_{\mathrm{cl}}(\theta) &= B_1(\theta), \\ C_{\mathrm{cl}}(\theta) &= C_1(\theta) + D_1(\theta)\,K(\theta).\end{aligned} \tag{10}$$

Augmenting the model of $\mathscr{S}$ to include the state of the gain-scheduled dynamic output feedback control $\mathscr{K}_{\mathscr{DOF}}$, we obtain the closed-loop LPV system $\mathscr{C}_{\mathscr{DOF}}$:

$$\begin{aligned}\mathscr{C}_{\mathscr{DOF}} : \dot{x}_{\mathrm{cl}}(t) &= A_{\mathrm{cl}}(\theta)\,x_{\mathrm{cl}}(t) + B_{\mathrm{cl}}(\theta)\,w(t), \\ z(t) &= C_{\mathrm{cl}}(\theta)\,x_{\mathrm{cl}}(t),\end{aligned} \tag{11}$$

where $x_{\mathrm{cl}}(t) = [x^T(t), x_K^T(t)]^T$ and

$$\begin{aligned}A_{\mathrm{cl}}(\theta) &= \begin{bmatrix} A(\theta) + B_2(\theta)\,D_K(\theta)\,C_2(\theta) & B_2(\theta)\,C_K(\theta) \\ B_K(\theta)\,C_2(\theta) & A_K(\theta) \end{bmatrix}, \\ B_{\mathrm{cl}}(\theta) &= \begin{bmatrix} B_1(\theta) \\ \mathbf{0} \end{bmatrix}, \\ C_{\mathrm{cl}}(\theta) &= \left[C_1(\theta) + D_1(\theta)\,D_K(\theta)\,C_2(\theta)\; D_1(\theta)\,C_K(\theta)\right].\end{aligned} \tag{12}$$

Then, the problems of gain-scheduled $\mathscr{H}_2$ control design for LPV systems can be expressed as follows.

Gain-Scheduled $\mathscr{H}_2$ Controller Synthesis Problem. Given the polytopic LPV system $\mathscr{S}$ in (1), our concerned problem is to determine the gain-scheduled state feedback controller $\mathscr{K}_{\mathscr{SF}}$ in (7) and dynamic output feedback controller $\mathscr{K}_{\mathscr{DOF}}$ in (8), such that both the closed-loop LPV systems $\mathscr{C}_{\mathscr{SF}}$ and $\mathscr{C}_{\mathscr{DOF}}$ in (9) and (11) are exponentially stable for all $(\theta, \dot{\theta}) \in \mathscr{F}_{\mathscr{P}}^{\mathscr{V}}$, and $z(t)$ reaches the desired controlled output in the sense of the $\mathscr{H}_2$ performance with respect to the disturbance input $w(t)$.

2.2. Preliminaries.

In order to solve the gain-scheduled $\mathscr{H}_2$ control design problem, we present some preliminary results for later use.

First, we introduce the notion of $\mathscr{H}_2$ norm for LPV systems borrowed from linear time-varying (LTV) systems (see [9, 33] for details). In this paper we use the stationary white noise approach and the average output variance to define the $\mathscr{H}_2$ norm. This kind of $\mathscr{H}_2$ norm can be used to capture both the transient response of the system and the response to stationary noise. Furthermore, it has also been proven to be the most appropriate for the Lyapunov-based $\mathscr{H}_2$ performance analysis and control design methods that will be developed in this paper [9, 10].

Definition 3 (see [10]). Let the closed-loop LPV system $\mathscr{C}$ in (9) or (11) be exponentially stable. The $\mathscr{H}_2$ norm of system $\mathscr{C}$ can be defined as

$$\|\mathscr{C}\|_2^2 = \lim_{h \to \infty} E\left\{\frac{1}{h}\int_0^h z^T(t)\,z(t)\,dt\right\} \tag{13}$$

when $x_{\mathrm{cl}}(0) = 0$ and $w(t)$ is a stationary zero-mean white process with an identity power spectrum density matrix, where $E\{\cdot\}$ denotes the mathematical expectation.

From the above definition, $\mathscr{H}_2$ norm performance can be regarded as an index or criterion assessing the elimination of the disturbance or noise. It is clear that the optimal $\mathscr{H}_2$ attenuation levels by the latest approaches are less conservative than that by the approach in the existing literature, and the improvement on conservativeness of the optimal $\mathscr{H}_2$ attenuation level is more apparent [9, 10, 33]. Based on Definition 3, we introduce an important result based on the conclusion in [26] using a PDLF, which is a preliminary result for solving the gain-scheduled $\mathscr{H}_2$ controller synthesis problem in this paper.

Lemma 4 (see [26]). *Given a scalar $\gamma > 0$, the closed-loop system $\mathscr{C}$ in (9) or (11) is exponentially stable with the prescribed performance index $\|\mathscr{C}\|_2 < \gamma$ if there exist matrices $P(\theta) > 0$, $\Pi(\theta) > 0$, and $W(\theta)$ and a sufficiently small positive scalar $\varepsilon > 0$ satisfying*

$$\operatorname{Tr}(\Pi(\theta)) < \gamma, \tag{14}$$

$$\begin{bmatrix} W(\theta) + W^T(\theta) - P(\theta) & W^T(\theta) C_{\mathrm{cl}}^T(\theta) \\ * & \Pi(\theta) \end{bmatrix} > 0, \tag{15}$$

$$\begin{bmatrix} \Theta_1 & \Theta_2 & B_{\mathrm{cl}}(\theta) \\ * & -\varepsilon\left(W(\theta) + W^T(\theta)\right) & \mathbf{0} \\ * & * & -\mathbf{I} \end{bmatrix} < 0, \tag{16}$$

where

$$\begin{aligned} \Theta_1 &\triangleq A_{\mathrm{cl}}(\theta) W(\theta) + W^T(\theta) A_{\mathrm{cl}}^T(\theta) + \frac{dP(\theta)}{dt}, \\ \Theta_2 &\triangleq P(\theta) - W^T(\theta) + \varepsilon A_{\mathrm{cl}}(\theta) W(\theta). \end{aligned} \tag{17}$$

Remark 5. Note that there exists a term $dP(\theta)/dt$ in condition (16) which cannot be implemented since it is not convex in the parameter $\theta(t)$. However, for the polytopic LPV system (1), we can solve this difficulty by using the following method from [11, 26]. Choose the parameter-dependent matrix $P(\theta)$ as

$$P(\theta) = \sum_{i=1}^{N} \theta_i(t) P_i, \quad \left(\theta, \dot{\theta}\right) \in \mathscr{F}_{\mathscr{P}}^{\mathscr{V}}. \tag{18}$$

Based on this expression, its time derivation can be derived as

$$\frac{dP(\theta)}{dt} = \dot{\theta}_1(t) P_1 + \dot{\theta}_2(t) P_2 + \cdots + \dot{\theta}_N(t) P_N = P\left(\dot{\theta}\right). \tag{19}$$

From Assumption 2, we have

$$\frac{dP(\theta)}{dt} = P\left(\dot{\theta}\right) = \sum_{k=1}^{N} \beta_k(t) P_t\left(\nu_k\right), \tag{20}$$

and then $dP(\theta)/dt$ in condition (16) can be substituted by convex parameter-dependent matrix $P_t(\theta)$.

It can be seen from Lemma 4 that, by introducing an extra matrix variable $W(\theta)$, the LMIs (14)–(16) provide a decoupling between Lyapunov function matrix and system matrices and will be useful for a gain-scheduled $\mathscr{H}_2$ controller synthesis for LPV systems. In addition, it has been shown, both theoretically and numerically, that the parameter-dependent approach is less conservative than the results in the quadratic framework, where a common Lyapunov matrix is used for the entire uncertainty domain [26]. However, we have to mention that the result developed in [26] is also conservative due to the imposition of $W(\theta) \equiv W$ when the result is used to synthesise a gain-scheduled $\mathscr{H}_2$ state feedback controller. The reason is that the introduced slack matrix $W(\theta)$ has been involved in the products with system matrices.

Then, there is a natural question: whether the conservativeness could be further reduced if we adopt different approaches other than the imposition of parameter independence as described above in the process of controller synthesis? The answer is affirmative. For state feedback controller synthesis, a possible alternative is the introduction of a new approach in [31] that helps to cast the infinite-dimensional LMI condition into finite-dimensional one. In the case of dynamic output feedback controller synthesis, a possible alternative is the introductions of structural block matrices and basis-parameter-dependent Lyapunov function [20, 21]. To the best of the authors' knowledge, these approaches have not been investigated for gain-scheduled $\mathscr{H}_2$ control problem of LPV systems so far. In the following, we will present some methods to solve both the state feedback and dynamic output feedback controller syntheses for the gain-scheduled $\mathscr{H}_2$ control of LPV systems.

3. Gain-Scheduled $\mathscr{H}_2$ Control Design

To reduce the conservativeness of state feedback controller synthesis mentioned above, we present a new sufficient condition for the existence of desired $\mathscr{H}_2$ state feedback controller in terms of finite-dimensional PLMIs. In order to solve the gain-scheduled $\mathscr{H}_2$ dynamic output feedback controller syntheses, a decoupling technique and a convexification method are introduced to obtain some sufficient conditions for the existence of the desired controller. The following two parts present the main results of gain-scheduled $\mathscr{H}_2$ state feedback and dynamic output feedback controller syntheses, respectively.

3.1. State Feedback Control Design

Theorem 6. *Consider the system $\mathscr{S}$ in (1). Given a scalar $\gamma > 0$ and a sufficiently small positive scalar $\varepsilon > 0$, there exists a gain-scheduled $\mathscr{H}_2$ state feedback controller $\mathscr{K}_{\mathscr{S}\mathscr{F}}$ in the form of (7) such that the resulting closed-loop system $\mathscr{C}_{\mathscr{S}\mathscr{F}}$ in (9) is exponentially stable with a prescribed $\mathscr{H}_2$ disturbance attenuation level γ if there exist matrices $\Pi(\theta) > 0$, $P(\theta) > 0$, $P_t(\theta) > 0$, $W(\theta)$, and $M(\theta)$ satisfying*

$$\operatorname{Tr}(\Pi(\theta)) < \gamma, \tag{21}$$

$$\Phi(\theta) \triangleq \begin{bmatrix} \Delta_1 & \Delta_2 \\ * & \Pi(\theta) \end{bmatrix} > 0, \tag{22}$$

$$\Psi(\theta) \triangleq \begin{bmatrix} \Delta_3 & \Delta_4 & B_1(\theta) \\ * & -\varepsilon W(\theta) - \varepsilon W^T(\theta) & \mathbf{0} \\ * & * & -\mathbf{I} \end{bmatrix} < 0, \tag{23}$$

where

$$\begin{aligned}\Delta_1 &\triangleq W(\theta) + W^T(\theta) - P(\theta), \\ \Delta_2 &\triangleq W^T(\theta) C_1^T(\theta) + M^T(\theta) D_1^T(\theta), \\ \Delta_3 &\triangleq \mathrm{Her}\{A(\theta) W(\theta) + B_2(\theta) M(\theta)\} + P_t(\theta), \\ \Delta_4 &\triangleq P(\theta) - W^T(\theta) + \varepsilon A(\theta) W(\theta) + \varepsilon B_2(\theta) M(\theta).\end{aligned} \tag{24}$$

In this case, a desired gain-scheduled $\mathscr{H}_2$ state feedback gain $K(\theta)$ is given by

$$K(\theta) = M(\theta) W^{-1}(\theta). \tag{25}$$

Proof. Define $M(\theta) \triangleq K(\theta)W(\theta)$. With (9) and (10), the results can be derived easily from Lemma 4. □

Note that the matrix variables $W(\theta)$ and $M(\theta)$ in Theorem 6 are dependent on time-varying parameter $\theta(t)$ and are not assumed to be constant matrices, which makes the new state feedback controller design conditions (21)–(23) less conservative than the results in [26]. However, the LMI conditions (21)–(23) in Theorem 6 cannot be implemented since they are not convex in the parameter $\theta(t)$. To solve this problem, we will introduce a new technique that helps convexify the matrix inequalities in Theorem 6 based on the polytopic characteristic of the dependent parameters. Then, we have the main result in the following theorem.

Theorem 7. *Consider the system $\mathcal{S}$ in (1). Given a scalar $\gamma > 0$ and a sufficiently small positive scalar $\varepsilon > 0$, an admissible gain-scheduled $\mathscr{H}_2$ state feedback controller in the form of $\mathscr{K}_{\mathcal{SF}}$ in (7) exists if there exist matrices $\Pi_i > 0$, $P_i > 0$, $P_{tk} > 0$, W_i, and M_i satisfying*

$$\mathrm{Tr}(\Pi_i) < \gamma, \quad i = 1, \ldots, N, \tag{26}$$

$$\Phi_{ij} + \Phi_{ji} - \Lambda_{ij} - \Lambda_{ij}^T \geqslant 0, \quad 1 \leqslant i < j \leqslant N, \tag{27}$$

$$\begin{gathered}\Psi_{ijk} + \Psi_{jik} - \Upsilon_{ijk} - \Upsilon_{ijk}^T \leqslant 0, \\ 1 \leqslant i < j \leqslant N, \quad k = 1, \ldots, N,\end{gathered} \tag{28}$$

$$\Lambda \triangleq \begin{bmatrix} \Phi_{11} & \Lambda_{12} & \cdots & \Lambda_{1N} \\ * & \Phi_{22} & \cdots & \Lambda_{2N} \\ * & * & \ddots & \vdots \\ * & * & * & \Phi_{NN} \end{bmatrix} > 0, \tag{29}$$

$$\Upsilon \triangleq \begin{bmatrix} \Psi_{11k} & \Upsilon_{12k} & \cdots & \Upsilon_{1Nk} \\ * & \Psi_{22k} & \cdots & \Upsilon_{2Nk} \\ * & * & \ddots & \vdots \\ * & * & * & \Psi_{NNk} \end{bmatrix} < 0, \tag{30}$$

where

$$\begin{aligned}\Phi_{ij} &\triangleq \begin{bmatrix} W_i + W_i^T - P_i & W_i^T C_{1j}^T + M_i^T D_{1j}^T \\ * & \Pi_i \end{bmatrix}, \\ \Psi_{ijk} &\triangleq \begin{bmatrix} \mathscr{L}_1 & \mathscr{L}_2 & B_{1j} \\ * & -\varepsilon W_i - \varepsilon W_i^T & \mathbf{0} \\ * & * & -\mathbf{I} \end{bmatrix}, \\ \mathscr{L}_1 &\triangleq \mathrm{Her}\left\{A_j W_i + B_{2j} M_i\right\} + P_{tk}, \\ \mathscr{L}_2 &\triangleq P_i - W_i^T + \varepsilon A_j W_i + \varepsilon B_{2j} M_i.\end{aligned} \tag{31}$$

Moreover, under the above conditions, the admissible gain-scheduled $\mathscr{H}_2$ state feedback gain $K(\theta)$ is given by

$$K(\theta) = \left(\sum_{i=1}^N \alpha_i M_i\right)\left(\sum_{i=1}^N \alpha_i W_i\right)^{-1}. \tag{32}$$

Proof. From Theorem 6, an admissible gain-scheduled $\mathscr{H}_2$ state feedback controller in the form of $\mathscr{K}_{\mathcal{SF}}$ in (7) exists if there exist matrices $\Pi(\theta) > 0$, $P(\theta) > 0$, $P_t(\theta) > 0$, $W(\theta)$, and $M(\theta)$ satisfying (21)–(23). Now, assume that the above matrix functions are of the following forms:

$$\begin{gathered}\Pi(\theta) = \sum_{i=1}^N \alpha_i \Pi_i, \quad W(\theta) = \sum_{i=1}^N \alpha_i W_i, \\ P(\theta) = \sum_{i=1}^N \alpha_i P_i, \quad P_t(\theta) = \sum_{k=1}^N \beta_k P_{tk}, \\ M(\theta) = \sum_{i=1}^N \alpha_i M_i.\end{gathered} \tag{33}$$

Then, with (33), we rewrite $\Phi(\theta)$ and $\Psi(\theta)$ in (22)-(23) as

$$\begin{aligned}\Phi(\theta) &= \sum_{j=1}^N \sum_{i=1}^N \alpha_i \alpha_j \Phi_{ij} \\ &= \sum_{i=1}^N \alpha_i^2 \Phi_{ii} + \sum_{i=1}^{N-1} \sum_{j=i+1}^N \alpha_i \alpha_j \left(\Phi_{ij} + \Phi_{ji}\right), \\ \Psi(\theta) &= \sum_{k=1}^N \sum_{j=1}^N \sum_{i=1}^N \alpha_i \alpha_j \beta_k \Phi_{ijk} \\ &= \sum_{k=1}^N \beta_k \left(\sum_{i=1}^N \alpha_i^2 \Psi_{iik} + \sum_{i=1}^{N-1} \sum_{j=i+1}^N \alpha_i \alpha_j \left(\Psi_{ijk} + \Psi_{jik}\right)\right).\end{aligned} \tag{34}$$

On the other hand, (27)-(28) are equivalent to

$$\begin{gathered}\Phi_{ij} + \Phi_{ji} \geqslant \Lambda_{ij} + \Lambda_{ij}^T, \quad 1 \leqslant i < j \leqslant N, \\ \Psi_{ijk} + \Psi_{jik} \leqslant \Upsilon_{ijk} + \Upsilon_{ijk}^T, \\ 1 \leqslant i < j \leqslant N, \quad k = 1, \ldots, N.\end{gathered} \tag{35}$$

Then, from (34)–(35), we have

$$\begin{aligned}\Phi(\theta) &\geqslant \sum_{i=1}^{N}\alpha_i^2\Phi_{ii} + \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\alpha_i\alpha_j\left(\Lambda_{ij}+\Lambda_{ij}^T\right) = \eta^T\Lambda\eta,\\ \Psi(\theta) &\leqslant \sum_{k=1}^{N}\beta_k\left(\sum_{i=1}^{N}\alpha_i^2\Psi_{iik} + \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\alpha_i\alpha_j\left(\Upsilon_{ijk}+\Upsilon_{ijk}^T\right)\right)\\ &= \eta^T\Upsilon\eta,\end{aligned} \tag{36}$$

where $\eta \triangleq [\alpha_1 I, \alpha_2 I, \ldots, \alpha_N I]^T$. Inequalities (29)-(30) guarantee $\Phi(\theta) > 0$ and $\Psi(\theta) < 0$, respectively. As to (26), since $\Pi(\theta) = \sum_{i=1}^{N}\alpha_i\Pi_i$ and $\mathrm{Tr}(\Pi(\theta)) = \sum_{i=1}^{N}\alpha_i \mathrm{Tr}(\Pi_i)$, if (26) is satisfied, we can get (21). By substituting the matrices defined in (33) into (25), we readily obtain (32), and the proof is completed. □

Remark 8. From the proof of Theorems 6 and 7, it can be seen that in the process of solving the gain-scheduled $\mathscr{H}_2$ state feedback control problem, we actually define parameter-dependent Lyapunov function-based matrix for $\mathscr{H}_2$ performance objective. In other words, $P(\theta)$ takes the form of $P(\theta) = \sum_{i=1}^{N}\alpha_i P_i$. The gain-scheduled $\mathscr{H}_2$ state feedback control design for continuous-time LPV systems has been investigated in [26], where parameter-dependent idea is realized at the expense of setting an additional slack variable to be constant for each vertex of the polytope. Notably, in Theorem 7, we do not set any matrix variable to be constant for the whole polytope domain. Therefore, Theorem 7 has the potential to yield less conservative results in the applications of gain-scheduled $\mathscr{H}_2$ state feedback controller synthesis.

Remark 9. The idea behind Theorem 7 is to use convex combinations of vertex matrices in the form of (33) to substitute the matrix functions in Theorem 6. By introducing these matrices and by means of the convexification method used in the proof of Theorem 7, the infinite-dimensional nonlinear matrix inequality conditions in Theorem 6 are cast into finite-dimensional PLMIs conditions, which depend only on the vertex matrices of the polytope $\mathscr{R}$. Therefore, these PLMIs conditions can be readily checked by using standard numerical software [29]. Note that the conditions in Theorem 7 are PLMIs for prescribed scalar ε not only over the matrix variables but also over the scalar γ. This implies that the $\mathscr{H}_2$ performance index γ can be included as optimization variable to obtain a reduction of the attenuation level bound. As in [26, 31], it is usually desired to design $\mathscr{H}_2$ controller with minimized performance γ^*, which can be readily found by solving the following convex optimization problem:

$$\begin{aligned}&\text{Minimize}\quad \gamma\\ &\text{subject to}\quad (26)-(30)\ \text{ for given scalar } \varepsilon.\end{aligned} \tag{37}$$

3.2. Dynamic Output Feedback Control Design

Theorem 10. *Consider the system* $\mathscr{S}$ *in* (1). *Given a scalar* $\gamma > 0$ *and a sufficiently small positive scalar* $\varepsilon > 0$, *there exists a gain-scheduled* $\mathscr{H}_2$ *dynamic output feedback controller* $\mathscr{K}_{\mathscr{DOF}}$ *in the form of* (8) *such that the resulting closed-loop system* $\mathscr{C}_{\mathscr{DOF}}$ *in* (11) *is exponentially stable with a prescribed* $\mathscr{H}_2$ *disturbance attenuation level* γ *if there exist matrices* $\Pi(\theta) > 0$, $\widetilde{P}(\theta) \triangleq \begin{bmatrix}\widetilde{P}_1(\theta) & \widetilde{P}_2(\theta)\\ * & \widetilde{P}_3(\theta)\end{bmatrix} > 0$, $\widetilde{P}_t(\theta) \triangleq \begin{bmatrix}\widetilde{P}_{t1}(\theta) & \widetilde{P}_{t2}(\theta)\\ * & \widetilde{P}_{t3}(\theta)\end{bmatrix} > 0$, $R(\theta)$, $S(\theta)$, $T(\theta)$, $\widetilde{A}_K(\theta)$, $\widetilde{B}_K(\theta)$, $\widetilde{C}_K(\theta)$, *and* $\widetilde{D}_K(\theta)$ *satisfying*

$$\mathrm{Tr}(\Pi(\theta)) < \gamma, \tag{38}$$

$$\Phi(\theta) \triangleq \begin{bmatrix}\Gamma_1 & \Gamma_2 & \Gamma_3\\ * & \Gamma_4 & \Gamma_5\\ * & * & \Pi(\theta)\end{bmatrix} > 0, \tag{39}$$

$$\Psi(\theta) \triangleq \begin{bmatrix}\Xi_1 & \Xi_2 & \Xi_3 & \Xi_4 & B_1(\theta)\\ * & \Xi_5 & \Xi_6 & \Xi_7 & S^T(\theta)B_1(\theta)\\ * & * & \Xi_8 & \Xi_9 & \mathbf{0}\\ * & * & * & \Xi_{10} & \mathbf{0}\\ * & * & * & * & -\mathbf{I}\end{bmatrix} < 0, \tag{40}$$

where

$$\begin{aligned}\Gamma_1 &\triangleq R(\theta) + R^T(\theta) - \widetilde{P}_1(\theta),\\ \Gamma_2 &\triangleq \mathbf{I} + T(\theta) - \widetilde{P}_2(\theta),\\ \Gamma_3 &\triangleq R^T(\theta)C_1^T(\theta) + \widetilde{C}_K^T(\theta)D_1^T(\theta),\\ \Gamma_4 &\triangleq S(\theta) + S^T(\theta) - \widetilde{P}_3(\theta),\\ \Gamma_5 &\triangleq C_1^T(\theta) + C_2^T(\theta)\widetilde{D}_K^T(\theta)D_1^T(\theta),\\ \Xi_1 &\triangleq \mathrm{Her}\left\{A(\theta)R(\theta) + B_2(\theta)\widetilde{C}_K(\theta)\right\} + \widetilde{P}_{t1}(\theta),\\ \Xi_2 &\triangleq A(\theta) + B_2(\theta)\widetilde{D}_K(\theta)C_2(\theta) + \widetilde{A}_K^T(\theta) + \widetilde{P}_{t2}(\theta),\\ \Xi_3 &\triangleq \widetilde{P}_1(\theta) - R^T(\theta) + \varepsilon A(\theta)R(\theta) + \varepsilon B_2(\theta)\widetilde{C}_K(\theta),\\ \Xi_4 &\triangleq \widetilde{P}_2(\theta) - T(\theta) + \varepsilon A(\theta) + \varepsilon B_2(\theta)\widetilde{D}_K(\theta)C_2(\theta),\\ \Xi_5 &\triangleq \mathrm{Her}\left\{S^T(\theta)A(\theta) + \widetilde{B}_K(\theta)C_2(\theta)\right\} + \widetilde{P}_{t3}(\theta),\\ \Xi_6 &\triangleq \widetilde{P}_2^T(\theta) - \mathbf{I} + \varepsilon\widetilde{A}_K(\theta),\\ \Xi_7 &\triangleq \widetilde{P}_3(\theta) - S + \varepsilon S^T(\theta)A(\theta) + \varepsilon\widetilde{B}_K(\theta)C_2(\theta),\\ \Xi_8 &\triangleq -\varepsilon R(\theta) - \varepsilon R^T(\theta),\\ \Xi_9 &\triangleq -\varepsilon\mathbf{I} - \varepsilon T(\theta),\\ \Xi_{10} &\triangleq -\varepsilon S(\theta) - \varepsilon S^T(\theta).\end{aligned} \tag{41}$$

In this case, a desired gain-scheduled $\mathscr{H}_2$ *dynamic output feedback controller* $\mathscr{K}_{\mathscr{DOF}}$ *in the form of* (8) *can be obtained by solving the following equations:*

$$\begin{aligned}\widetilde{A}_K(\theta) &= S^T(\theta)A(\theta)R(\theta) + G^T(\theta)A_K(\theta)F(\theta)\\ &\quad + G^T(\theta)B_K(\theta)C_2(\theta)R(\theta)\end{aligned}$$

$$
\begin{aligned}
&+ S^T(\theta) B_2(\theta) C_K(\theta) F(\theta) \\
&+ S^T(\theta) B_2(\theta) D_K(\theta) C_2(\theta) R(\theta), \\
\widetilde{B}_K(\theta) &= G^T(\theta) B_K(\theta) + S^T(\theta) B_2(\theta) D_K(\theta), \\
\widetilde{C}_K(\theta) &= C_K(\theta) F(\theta) + D_K(\theta) C_2(\theta) R(\theta), \\
\widetilde{D}_K(\theta) &= D_K(\theta),
\end{aligned} \tag{42}
$$

where $F(\theta)$ and $G(\theta)$ can be obtained by taking any full-rank factorization of $F^T(\theta)G(\theta) = T(\theta) - R^T(\theta)S(\theta)$.

Proof. It can be seen from Lemma 4 that the matrix $W(\theta)$ is nonsingular if (15) holds since $W(\theta) + W^T(\theta) - P(\theta) > 0$ and $P(\theta) > 0$. For notational simplicity, we denote $V(\theta) \triangleq W^{-1}(\theta)$ in the following. Then $W(\theta)$ and $V(\theta)$ can be partitioned as follows:

$$
\begin{aligned}
W(\theta) &\triangleq \begin{bmatrix} W_1(\theta) & W_2(\theta) \\ W_4(\theta) & W_3(\theta) \end{bmatrix}, \\
V(\theta) &\triangleq W^{-1}(\theta) = \begin{bmatrix} V_1(\theta) & V_2(\theta) \\ V_4(\theta) & V_3(\theta) \end{bmatrix}.
\end{aligned} \tag{43}
$$

Without loss of generality, we assume that $W_4(\theta)$ and $V_4(\theta)$ are nonsingular (if not, $W_4(\theta)$ and $V_4(\theta)$ may be perturbed, respectively, by matrices $\Delta W_4(\theta)$ and $\Delta V_4(\theta)$ with sufficiently small norms such that $W_4(\theta) + \Delta W_4(\theta)$ and $V_4(\theta) + \Delta V_4(\theta)$ are nonsingular and satisfy (15)). Then we can define the following nonsingular matrices:

$$
\mathcal{J}_W(\theta) \triangleq \begin{bmatrix} W_1(\theta) & \mathbf{I} \\ W_4(\theta) & \mathbf{0} \end{bmatrix}, \qquad \mathcal{J}_V(\theta) \triangleq \begin{bmatrix} \mathbf{I} & V_1(\theta) \\ \mathbf{0} & V_4(\theta) \end{bmatrix}. \tag{44}
$$

Note that

$$
\begin{gathered}
W(\theta) \mathcal{J}_V(\theta) = \mathcal{J}_W(\theta), \qquad V(\theta) \mathcal{J}_W(\theta) = \mathcal{J}_V(\theta), \\
W_1(\theta) V_1(\theta) + W_2(\theta) V_4(\theta) = \mathbf{I}.
\end{gathered} \tag{45}
$$

Performing congruence transformations to (15) and (16) by matrices $\mathrm{diag}\{\mathcal{J}_V(\theta), \mathbf{I}\}$ and $\mathrm{diag}\{\mathcal{J}_V(\theta), \mathcal{J}_V(\theta), \mathbf{I}\}$, respectively, we have

$$
\begin{bmatrix} \mathrm{Her}\left\{\mathcal{J}_V^T(\theta) \mathcal{J}_W(\theta)\right\} - \widetilde{P}(\theta) & \mathcal{J}_W^T(\theta) C_{\mathrm{cl}}^T(\theta) \\ * & \Pi(\theta) \end{bmatrix} > 0, \tag{46}
$$

$$
\begin{bmatrix} \Delta_1 & \Delta_2 & \mathcal{J}_V^T(\theta) B_{\mathrm{cl}}(\theta) \\ * & \Delta_3 & \mathbf{0} \\ * & * & -\mathbf{I} \end{bmatrix} < 0, \tag{47}
$$

where

$$
\begin{aligned}
\Delta_1 &\triangleq \mathrm{Her}\{\Phi(\theta)\} + \widetilde{P}_t(\theta), \\
\Delta_2 &\triangleq \widetilde{P}(\theta) - \mathcal{J}_W^T(\theta) \mathcal{J}_V(\theta) + \varepsilon \Phi_1(\theta), \\
\Delta_3 &\triangleq -\varepsilon \mathrm{Her}\left\{\mathcal{J}_V^T(\theta) \mathcal{J}_W(\theta)\right\}, \\
\Phi(\theta) &\triangleq \mathcal{J}_V^T(\theta) A_{\mathrm{cl}}(\theta) \mathcal{J}_W(\theta), \\
\widetilde{P}(\theta) &\triangleq \begin{bmatrix} \widetilde{P}_1(\theta) & \widetilde{P}_2(\theta) \\ * & \widetilde{P}_3(\theta) \end{bmatrix} = \mathcal{J}_V^T(\theta) P(\theta) \mathcal{J}_V(\theta) > 0, \\
\widetilde{P}_t(\theta) &\triangleq \begin{bmatrix} \widetilde{P}_{t1}(\theta) & \widetilde{P}_{t2}(\theta) \\ * & \widetilde{P}_{t3}(\theta) \end{bmatrix} = \mathcal{J}_V^T(\theta) P_t(\theta) \mathcal{J}_V(\theta) > 0.
\end{aligned} \tag{48}
$$

Define the following matrices:

$$
\begin{gathered}
R(\theta) \triangleq W_1(\theta), \qquad S(\theta) \triangleq V_1(\theta), \\
T(\theta) \triangleq W_1^T(\theta) V_1(\theta) + W_4^T(\theta) V_4(\theta),
\end{gathered} \tag{49}
$$

$$
\begin{aligned}
\widetilde{A}_K(\theta) &\triangleq V_1^T(\theta) A(\theta) W_1(\theta) + V_4^T(\theta) A_K(\theta) W_4(\theta) \\
&\quad + V_4^T(\theta) B_K(\theta) C_2(\theta) W_1(\theta) \\
&\quad + V_1^T(\theta) B_2(\theta) C_K(\theta) W_4(\theta) \\
&\quad + V_1^T(\theta) B_2(\theta) D_K(\theta) C_2(\theta) W_1(\theta), \\
\widetilde{B}_K(\theta) &\triangleq V_4^T(\theta) B_K(\theta) + V_1^T(\theta) B_2(\theta) D_K(\theta), \\
\widetilde{C}_K(\theta) &\triangleq C_K(\theta) W_4(\theta) + D_K(\theta) C_2(\theta) W_1(\theta), \\
\widetilde{D}_K(\theta) &\triangleq D_K(\theta).
\end{aligned} \tag{50}
$$

Then, substituting (12) into (46) and (47) and considering (44) and

$$
\begin{aligned}
\mathcal{J}_V^T(\theta) \mathcal{J}_W(\theta) &= \begin{bmatrix} R(\theta) & \mathbf{I} \\ T^T(\theta) & S^T(\theta) \end{bmatrix}, \\
\mathcal{J}_V^T(\theta) A_{\mathrm{cl}}(\theta) \mathcal{J}_W(\theta) &= \begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \widetilde{A}_K(\theta) & \mathcal{A}_3 \end{bmatrix}, \\
\mathcal{J}_V^T(\theta) B_{\mathrm{cl}}(\theta) &= \begin{bmatrix} B_1(\theta) \\ S^T(\theta) B_1(\theta) \end{bmatrix}, \\
\mathcal{J}_W^T(\theta) C_{\mathrm{cl}}^T(\theta) &= \begin{bmatrix} R^T(\theta) C_1^T(\theta) + \widetilde{C}_K^T(\theta) D_1^T(\theta) \\ C_1^T(\theta) + C_2^T(\theta) \widetilde{D}_K^T(\theta) D_1^T(\theta) \end{bmatrix},
\end{aligned} \tag{51}
$$

where

$$
\begin{aligned}
\mathcal{A}_1 &\triangleq A(\theta) R(\theta) + B_2(\theta) \widetilde{C}_K(\theta), \\
\mathcal{A}_2 &\triangleq A(\theta) + B_2(\theta) \widetilde{D}_K(\theta) C_2(\theta), \\
\mathcal{A}_3 &\triangleq S^T(\theta) A(\theta) + \widetilde{B}_K(\theta) C_2(\theta),
\end{aligned} \tag{52}
$$

we obtain that (39) and (40) hold.

Define the following matrices:

$$
F(\theta) \triangleq W_4(\theta), \qquad G(\theta) \triangleq V_4(\theta). \tag{53}
$$

It is noted that

$$F^T(\theta) G(\theta) = T(\theta) - R^T(\theta) S(\theta). \tag{54}$$

Finally, from (50), (47), and (54), (10) holds. This completes the proof. □

Remark 11. Theorem 10 casts the nonlinear matrix inequality condition of dynamic output feedback control design based on Lemma 4 into an LMI condition by using linearization procedures. Based on these procedures, the desired LPV controllers can be constructed by using the obtained matrix functions $\Pi(\theta)$, $\widetilde{P}(\theta)$, $\widetilde{P}_t(\theta)$, $R(\theta)$, $S(\theta)$, $T(\theta)$, $\widetilde{A}_K(\theta)$, $\widetilde{B}_K(\theta)$, $\widetilde{C}_K(\theta)$, and $\widetilde{D}_K(\theta)$. However, these LMI conditions of testing the feasibility in Theorem 10 are infinite-dimensional constraints in terms of the parameter $\theta(t)$. Therefore, these conditions still cannot be implemented since they are not convex in the parameter $\theta(t)$. It is noted that if we set $\Pi(\theta) \equiv \Pi$, $\widetilde{P}(\theta) \equiv \widetilde{P}$, $\widetilde{P}_t(\theta) \equiv \widetilde{P}_t$, $R(\theta) \equiv R$, $S(\theta) \equiv S$, $T(\theta) \equiv T$, $\widetilde{A}_K(\theta) \equiv \widetilde{A}_K$, $\widetilde{B}_K(\theta) \equiv \widetilde{B}_K$, $\widetilde{C}_K(\theta) \equiv \widetilde{C}_K$, and $\widetilde{D}_K(\theta) \equiv \widetilde{D}_K$, we will readily obtain a new gain-scheduled $\mathscr{H}_2$ output feedback control result in a quadratic framework. Then, we can obtain the following corollary based on Theorem 10 immediately.

Corollary 12. *Consider the system $\mathscr{S}$ in* (1). *Given a scalar $\gamma > 0$ and a sufficiently small positive scalar $\varepsilon > 0$, then there exists a gain-scheduled $\mathscr{H}_2$ dynamic output feedback controller $\mathscr{K}_{\mathscr{DOF}}$ in the form of* (8) *such that the resulting closed-loop system $\mathscr{C}_{\mathscr{DOF}}$ in* (11) *is exponentially stable with a prescribed $\mathscr{H}_2$ disturbance attenuation level γ if there exist matrices $\Pi > 0$, $\widetilde{P} \triangleq \begin{bmatrix} \widetilde{P}_1 & \widetilde{P}_2 \\ * & \widetilde{P}_3 \end{bmatrix} > 0$, $\widetilde{P}_t \triangleq \begin{bmatrix} \widetilde{P}_{t1} & \widetilde{P}_{t2} \\ * & \widetilde{P}_{t3} \end{bmatrix} > 0$, R, S, T, $\widetilde{A}_K$, $\widetilde{B}_K$, $\widetilde{C}_K$, and $\widetilde{D}_K$ satisfying*

$$\begin{gathered} \operatorname{Tr}(\Pi) < \gamma, \\ \Phi_i \triangleq \begin{bmatrix} \Gamma_1 & \Gamma_2 & \Gamma_3 \\ * & \Gamma_4 & \Gamma_5 \\ * & * & \Pi \end{bmatrix} > 0, \quad i = 1, \ldots, N, \\ \Psi_i \triangleq \begin{bmatrix} \Xi_1 & \Xi_2 & \Xi_3 & \Xi_4 & B_{1i} \\ * & \Xi_5 & \Xi_6 & \Xi_7 & S^T B_{1i} \\ * & * & \Xi_8 & \Xi_9 & \mathbf{0} \\ * & * & * & \Xi_{10} & \mathbf{0} \\ * & * & * & * & -\mathbf{I} \end{bmatrix} < 0, \quad i = 1, \ldots, N, \end{gathered} \tag{55}$$

where

$$\begin{aligned} \Gamma_1 &\triangleq R + R^T - \widetilde{P}_1, \\ \Gamma_2 &\triangleq \mathbf{I} + T - \widetilde{P}_2, \\ \Gamma_3 &\triangleq R^T C_{1i}^T + \widetilde{C}_K^T D_{1i}^T, \\ \Gamma_4 &\triangleq S + S^T - \widetilde{P}_3, \\ \Gamma_5 &\triangleq C_{1i}^T + C_{2i}^T \widetilde{D}_K^T D_{1i}^T, \\ \Xi_1 &\triangleq \operatorname{Her}\left\{A_i R + B_{2i} \widetilde{C}_K\right\} + \widetilde{P}_{t1}, \\ \Xi_2 &\triangleq A_i + B_{2i} \widetilde{D}_K C_{2i} + \widetilde{A}_K^T + \widetilde{P}_{t2}, \\ \Xi_3 &\triangleq \widetilde{P}_1 - R^T + \varepsilon A_i R + \varepsilon B_{2i} \widetilde{C}_K, \\ \Xi_4 &\triangleq \widetilde{P}_2 - T + \varepsilon A_i + \varepsilon B_{2i} \widetilde{D}_K C_{2i}, \\ \Xi_5 &\triangleq \operatorname{Her}\left\{S^T A_i + \widetilde{B}_K C_{2i}\right\} + \widetilde{P}_{t3}, \\ \Xi_6 &\triangleq \widetilde{P}_2^T - \mathbf{I} + \varepsilon \widetilde{A}_K, \\ \Xi_7 &\triangleq \widetilde{P}_3 - S + \varepsilon S^T A_i + \varepsilon \widetilde{B}_K C_{2i}, \\ \Xi_8 &\triangleq -\varepsilon R - \varepsilon R^T, \\ \Xi_9 &\triangleq -\varepsilon \mathbf{I} - \varepsilon T, \\ \Xi_{10} &\triangleq -\varepsilon S - \varepsilon S^T. \end{aligned} \tag{56}$$

In this case, a desired gain-scheduled $\mathscr{H}_2$ dynamic output feedback controller $\mathscr{K}_{\mathscr{DOF}}$ in the form of (8) *can be obtained by solving the following equations:*

$$\begin{aligned} \widetilde{A}_K &= S^T A(\theta) R + G^T A_K(\theta) F \\ &\quad + G^T B_K(\theta) C_2(\theta) R + S^T B_2(\theta) C_K(\theta) F \\ &\quad + S^T B_2(\theta) D_K(\theta) C_2(\theta) R, \\ \widetilde{B}_K &= G^T B_K(\theta) + S^T B_2(\theta) D_K(\theta), \\ \widetilde{C}_K &= C_K(\theta) F + D_K(\theta) C_2(\theta) R, \\ \widetilde{D}_K &= D_K(\theta), \end{aligned} \tag{57}$$

where F and G can be obtained by taking any full-rank factorization of $F^T G = T - R^T S$.

Although Corollary 12 gives a finite-dimensional LMI approach to design a gain-scheduled $\mathscr{H}_2$ dynamic output feedback controller $\mathscr{K}_{\mathscr{DOF}}$ in the form of (8), all vertices of the polytope share a common Lyapunov function, which may lead to the conservative result. In order to reduce the conservativeness and derive a solvable condition, we develop a new PLMI condition in the following. Similar to the introduction of the convexification method used in Theorem 7, a finite-dimensional PLMI condition that depends on the vertices of the polytope $\mathscr{R}$ is presented in the following theorem, which can be efficiently solved by using standard numerical software.

Theorem 13. *Consider the system $\mathscr{S}$ in* (1). *Given a scalar $\gamma > 0$ and a sufficiently small positive scalar $\varepsilon > 0$, then an admissible gain-scheduled $\mathscr{H}_2$ dynamic output feedback controller in the form of $\mathscr{K}_{\mathscr{DOF}}$ in* (8) *exists if there exist matrices $\Pi_i > 0$, $\widetilde{P}_i \triangleq \begin{bmatrix} \widetilde{P}_{1i} & \widetilde{P}_{2i} \\ * & \widetilde{P}_{3i} \end{bmatrix} > 0$, $\widetilde{P}_{tk} \triangleq \begin{bmatrix} \widetilde{P}_{t1k} & \widetilde{P}_{t2k} \\ * & \widetilde{P}_{t3k} \end{bmatrix} > 0$, R_i, S_i, T_i, $\widetilde{A}_{Ki}$, $\widetilde{B}_{Ki}$, $\widetilde{C}_{Ki}$, $\widetilde{D}_{Ki}$, Λ_{ij}, and Υ_{ijk} satisfying*

$$\operatorname{Tr}(\Pi_i) < \gamma, \quad i = 1, \ldots, N, \tag{58}$$

$$\Phi_{ij} + \Phi_{ji} - \Lambda_{ij} - \Lambda_{ij}^T \geqslant 0, \quad 1 \leqslant i < j \leqslant N, \tag{59}$$

$$\Psi_{ijk} + \Psi_{jik} - \Upsilon_{ijk} - \Upsilon_{ijk}^T \leqslant 0, \quad 1 \leqslant i < j \leqslant N, \quad k = 1, \ldots, N, \tag{60}$$

$$\Lambda \triangleq \begin{bmatrix} \Phi_{11} & \Lambda_{12} & \cdots & \Lambda_{1N} \\ * & \Phi_{22} & \cdots & \Lambda_{2N} \\ * & * & \ddots & \vdots \\ * & * & * & \Phi_{NN} \end{bmatrix} > 0, \tag{61}$$

$$\Upsilon \triangleq \begin{bmatrix} \Psi_{11k} & \Upsilon_{12k} & \cdots & \Upsilon_{1Nk} \\ * & \Psi_{22k} & \cdots & \Upsilon_{2Nk} \\ * & * & \ddots & \vdots \\ * & * & * & \Psi_{NNk} \end{bmatrix} < 0, \tag{62}$$

where

$$\begin{aligned}
\Phi_{ij} &\triangleq \begin{bmatrix} R_i + R_i^T - \widetilde{P}_{1i} & \mathbf{I} + T_i - \widetilde{P}_{2i} & \Theta_1 \\ * & S_i + S_i^T - \widetilde{P}_{3i} & \Theta_2 \\ * & * & \Pi_i \end{bmatrix}, \\
\Psi_{ijk} &\triangleq \begin{bmatrix} \Sigma_1 & \Sigma_2 & \Sigma_3 & \Sigma_4 & B_{1j} \\ * & \Sigma_5 & \Sigma_6 & \Sigma_7 & S_i^T B_{1j} \\ * & * & \Sigma_8 & \Sigma_9 & \mathbf{0} \\ * & * & * & \Sigma_{10} & \mathbf{0} \\ * & * & * & * & -\mathbf{I} \end{bmatrix}, \\
\Theta_1 &\triangleq R_i^T C_{1j}^T + \widetilde{C}_{Ki}^T D_{1j}^T, \\
\Theta_2 &\triangleq C_{1j}^T + C_{2j}^T \widetilde{D}_{Ki}^T D_{1j}^T, \\
\Sigma_1 &\triangleq \operatorname{Her}\left\{A_j R_i + B_{2j} \widetilde{C}_{Ki}\right\} + \widetilde{P}_{t1k}, \\
\Sigma_2 &\triangleq A_j + B_{2j} \widetilde{D}_{Ki} C_{2j} + \widetilde{A}_{Ki}^T + \widetilde{P}_{t2k}, \\
\Sigma_3 &\triangleq \widetilde{P}_{1i} - R_i^T + \varepsilon A_j R_i + \varepsilon B_{2j} \widetilde{C}_{Ki}, \\
\Sigma_4 &\triangleq \widetilde{P}_{2i} - T_i + \varepsilon A_j + \varepsilon B_{2j} \widetilde{D}_{Ki} C_{2j}, \\
\Sigma_5 &\triangleq \operatorname{Her}\left\{S_i^T A_j + \widetilde{B}_{Ki} C_{2j}\right\} + \widetilde{P}_{t3k}, \\
\Sigma_6 &\triangleq \widetilde{P}_{2i}^T - \mathbf{I} + \varepsilon \widetilde{A}_{Ki}, \\
\Sigma_7 &\triangleq \widetilde{P}_{3i} - S_i^T + \varepsilon S_i^T A_j + \varepsilon \widetilde{B}_{Ki} C_{2j}, \\
\Sigma_8 &\triangleq -\varepsilon R_i - \varepsilon R_i^T, \\
\Sigma_9 &\triangleq -\varepsilon \mathbf{I} - \varepsilon T_i, \\
\Sigma_{10} &\triangleq -\varepsilon S_i - \varepsilon S_i^T.
\end{aligned} \tag{63}$$

Moreover, under the above conditions, the matrix functions for an admissible gain-scheduled $\mathscr{H}_2$ dynamic output feedback controller $\mathscr{K}_{\mathscr{DOF}}$ in (8) *are given by solving the following equations:*

$$\begin{aligned}
\widetilde{A}_K(\theta) &= S^T(\theta) A(\theta) R(\theta) + G^T(\theta) A_K(\theta) F(\theta) \\
&\quad + G^T(\theta) B_K(\theta) C_2(\theta) R(\theta) \\
&\quad + S^T(\theta) B_2(\theta) C_K(\theta) F(\theta) \\
&\quad + S^T(\theta) B_2(\theta) D_K(\theta) C_2(\theta) R(\theta), \\
\widetilde{B}_K(\theta) &= G^T(\theta) B_K(\theta) + S^T(\theta) B_2(\theta) D_K(\theta), \\
\widetilde{C}_K(\theta) &= C_K(\theta) F(\theta) + D_K(\theta) C_2(\theta) R(\theta), \\
\widetilde{D}_K(\theta) &= D_K(\theta),
\end{aligned} \tag{64}$$

where $R(\theta) = \sum_{i=1}^N \alpha_i R_i$, $S(\theta) = \sum_{i=1}^N \alpha_i S_i$, $T(\theta) = \sum_{i=1}^N \alpha_i T_i$, $\widetilde{A}_K(\theta) = \sum_{i=1}^N \alpha_i \widetilde{A}_{Ki}$, $\widetilde{B}_K(\theta) = \sum_{i=1}^N \alpha_i \widetilde{B}_{Ki}$, $\widetilde{C}_K(\theta) = \sum_{i=1}^N \alpha_i \widetilde{C}_{Ki}$, $\widetilde{D}_K(\theta) = \sum_{i=1}^N \alpha_i \widetilde{D}_{Ki}$, and $F(\theta)$ and $G(\theta)$ can be obtained by taking any full-rank factorization $F^T(\theta)G(\theta) = T(\theta) - R^T(\theta)S(\theta)$.

Proof. From Theorem 10, an admissible gain-scheduled $\mathscr{H}_2$ output feedback controller in the form of $\mathscr{K}_{\mathscr{DOF}}$ in (8) exists if there exist matrices $\Pi(\theta) > 0$, $\widetilde{P}(\theta) > 0$, $\widetilde{P}_t(\theta) > 0$, $R(\theta), S(\theta), T(\theta), \widetilde{A}_K(\theta), \widetilde{B}_K(\theta), \widetilde{C}_K(\theta)$, and $\widetilde{D}_K(\theta)$ and a scalar $\varepsilon > 0$ satisfying (38)–(40). Now, assume that the above matrix functions are of the following form:

$$\begin{aligned}
&\widetilde{P}(\theta) = \sum_{i=1}^N \alpha_i \widetilde{P}_i = \sum_{i=1}^N \alpha_i \begin{bmatrix} \widetilde{P}_{1i} & \widetilde{P}_{2i} \\ * & \widetilde{P}_{3i} \end{bmatrix}, \\
&\widetilde{P}_t(\theta) = \sum_{k=1}^N \beta_k \widetilde{P}_{tk} = \sum_{k=1}^N \beta_k \begin{bmatrix} \widetilde{P}_{t1k} & \widetilde{P}_{t2k} \\ * & \widetilde{P}_{t3k} \end{bmatrix}, \\
&\Pi(\theta) = \sum_{i=1}^N \alpha_i \Pi_i, \qquad R(\theta) = \sum_{i=1}^N \alpha_i R_i, \\
&S(\theta) = \sum_{i=1}^N \alpha_i S_i, \qquad T(\theta) = \sum_{i=1}^N \alpha_i T_i, \\
&\widetilde{A}_K(\theta) = \sum_{i=1}^N \alpha_i \widetilde{A}_{Ki}, \qquad \widetilde{B}_K(\theta) = \sum_{i=1}^N \alpha_i \widetilde{B}_{Ki}, \\
&\widetilde{C}_K(\theta) = \sum_{i=1}^N \alpha_i \widetilde{C}_{Ki}, \qquad \widetilde{D}_K(\theta) = \sum_{i=1}^N \alpha_i \widetilde{D}_{Ki}.
\end{aligned} \tag{65}$$

Then, with (65), we rewrite $\Phi(\theta)$ and $\Psi(\theta)$ in (39)-(40) as

$$\begin{aligned}
\Phi(\theta) &= \sum_{j=1}^N \sum_{i=1}^N \alpha_i \alpha_j \Phi_{ij} \\
&= \sum_{i=1}^N \alpha_i^2 \Phi_{ii} + \sum_{i=1}^{N-1} \sum_{j=i+1}^N \alpha_i \alpha_j \left(\Phi_{ij} + \Phi_{ji}\right), \\
\Psi(\theta) &= \sum_{k=1}^N \sum_{j=1}^N \sum_{i=1}^N \alpha_i \alpha_j \beta_k \Phi_{ijk}
\end{aligned}$$

$$= \sum_{k=1}^{N} \beta_k \left(\sum_{i=1}^{N} \alpha_i^2 \Psi_{iik} + \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \alpha_i \alpha_j \left(\Psi_{ijk} + \Psi_{jik} \right) \right). \tag{66}$$

On the other hand, (59)-(60) are equivalent to

$$\begin{aligned} \Phi_{ij} + \Phi_{ji} &\geqslant \Lambda_{ij} + \Lambda_{ij}^T, \\ \Psi_{ijk} + \Psi_{jik} &\leqslant \Upsilon_{ijk} + \Upsilon_{ijk}^T, \end{aligned} \tag{67}$$

where $1 \leqslant i < j \leqslant N$ and $k = 1, \ldots, N$. Then, from (66)-(67), we have

$$\begin{aligned} \Phi(\theta) &\geqslant \sum_{i=1}^{N} \alpha_i^2 \Phi_{ii} + \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \alpha_i \alpha_j \left(\Lambda_{ij} + \Lambda_{ij}^T \right) = \eta^T \Lambda \eta, \\ \Psi(\theta) &\leqslant \sum_{k=1}^{N} \beta_k \left(\sum_{i=1}^{N} \alpha_i^2 \Psi_{iik} + \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \alpha_i \alpha_j \left(\Upsilon_{ijk} + \Upsilon_{ijk}^T \right) \right) \\ &= \eta^T \Upsilon \eta, \end{aligned} \tag{68}$$

where $\eta \triangleq [\alpha_1 I, \alpha_2 I, \ldots, \alpha_N I]^T$. Inequalities (61)-(62) guarantee $\Phi(\theta) > 0$ and $\Psi(\theta) < 0$, respectively. As to (58), since $\Pi(\theta) = \sum_{i=1}^{N} \alpha_i \Pi_i$ and $\mathrm{Tr}(\Pi(\theta)) = \sum_{i=1}^{N} \alpha_i \mathrm{Tr}(\Pi_i)$, if (58) satisfies, we can get (38). By substituting the matrices defined in (65) into (10), we readily obtain (13), and the proof is completed. □

Remark 14. From the proof of Theorem 13, it can be seen that the Lyapunov function-based matrix for $\mathscr{H}_2$ performance objective is also parameter dependent; that is, $P(\theta)$ takes the form of $P(\theta) = \sum_{i=1}^{N} \alpha_i P_i$. Notably, here in Theorem 13, we do not set any matrix variable to be constant for the whole polytope domain, and therefore, Theorem 13 has the potential to yield less conservative results in the applications of gain-scheduled $\mathscr{H}_2$ LPV control design than the ones presented in Corollary 12. Similar to the case of state feedback control design, the gain-scheduled $\mathscr{H}_2$ dynamic output feedback controller synthesis problem can be also cast into a finite-dimensional convex optimization problem as follows:

$$\begin{aligned} &\text{Minimize} \quad \gamma \\ &\text{subject to} \quad (58)-(62) \text{ for given scalar } \varepsilon. \end{aligned} \tag{69}$$

Remark 15. From the results in the existing literature, it is known that the definition of quadratic Lyapunov function is simple, and the computation cost in the design procedure is low. However, the common quadratic Lyapunov functions tend to be conservative and might not exist for some highly nonlinear systems. To reduce the conservatism and to establish well-performance condition, parameter-dependent Lyapunov functions and new slack variables have been adopted in this paper. It can be seen that the quadratic Lyapunov function is a special case of PDLF. Thus, the proposed method in this paper is more general and less conservative. Meanwhile, the PDLF-based approach also has some disadvantages: the design procedures become more complex, and the computational requirement is usually demanding. The number of PLMIs conditions in Theorems 7 and 13 increases rapidly with the number of system dimensions. Thus, a computational problem might arise for high-order systems. One effective way to solve this problem is to try to reduce the number of variables with the tradeoff between computational burden and conservativeness. For example, Theorem 13 can be replaced by Corollary 12 with increasing the conservativeness and decreasing computational burden.

4. Illustrative Example

In this section, we use Example 1 to show the less conservativeness of the result developed in Theorem 7. Example 2 is provided to show the effectiveness of the gain-scheduled $\mathscr{H}_2$ dynamic output feedback controller proposed in Theorem 13.

Example 1. Consider the following numerical example borrowed from [26, 29]. The problem is to control the yaw angles of a satellite system. The satellite system consisting of two rigid bodies joined by a flexible link has the state-space representation as follows:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & -f & f \\ k & -k & f & -f \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u, \\ z &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} u, \end{aligned} \tag{70}$$

where $x = \begin{bmatrix} \theta_1 & \theta_2 & \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix}^T$ and k and f are torque constant and viscous damping, and they vary in the following uncertainty ranges: $k \in [0.09 \;\; 0.4]$; $f \in [0.0038 \;\; 0.04]$.

For this system, our purpose is to design a gain-scheduled $\mathscr{H}_2$ state feedback control $u(t)$ in the form of (7), such that the closed-loop system is exponentially stable with a minimized $\mathscr{H}_2$ disturbance attenuation level γ.

Define

$$\begin{aligned} &x \triangleq \frac{0.4 - k(t)}{0.4 - 0.09}, \qquad y \triangleq \frac{f(t) - 0.0038}{0.04 - 0.0038}, \\ &\alpha_1(t) = xy, \qquad \alpha_2(t) = (1 - x)\, y, \\ &\alpha_3(t) = x(1 - y), \qquad \alpha_4(t) = (1 - x)(1 - y). \end{aligned} \tag{71}$$

It is easy to check that $\alpha_i(t)$, $i = 1, \ldots, 4$, are convex coordinates, since they satisfy $0 \leqslant \alpha_i(t) \leqslant 1$ and $\sum_{i=1}^{4} \alpha_i(t) = 1$. It should be noted that the choice of scalar ε is important to converge to minimum $\mathscr{H}_2$ performance [26]. In this example, the value of minimum guaranteed $\mathscr{H}_2$ performance γ^* is 1.0883 with fixed $\varepsilon = 0.11$ and 1.4156 with fixed $\varepsilon = 0.5$ by the method in [26], and 0.8892 with fixed $\varepsilon = 0.11$ and 1.0531 with fixed $\varepsilon = 0.5$ by using Theorem 7. Table 1 shows the minimum $\mathscr{H}_2$ performance and the numbers of decision variables when different methods are used. It is clearly shown in Table 1 that the guaranteed performance obtained by our

Table 1: Minimum $\mathscr{H}_2$ performance for different cases.

ε	0.11		0.5	
Method	[26]	Theorem 7	[26]	Theorem 7
γ^*	1.0883	0.8892	1.4156	1.0531
Complexity	14	257	14	257

approach is much better than that obtained by the method in [26], which indicates the less conservativeness of the controller design result developed in this paper, even though this procedure increases some numerical complexity. From Table 1, we can also see that the smaller the value of ε, the better the value of γ^*. Although the computational complexity of Theorem 7 for this example is relatively more than the approach in [26], the reduced conservatism is significant. With the rapid development of computer technology and computational method, the computational burden problem may be solved.

Example 2. Consider a continuous-time LPV system $\mathcal{S}$ in (1) with the following matrix functions:

$$A(\theta)=\begin{bmatrix}1.5+0.5\theta & 3 & \theta\\ -2.2+\theta & -1.8+0.5\theta & 0.2\theta\\ 0.1 & 0.5 & -\theta\end{bmatrix},$$
$$B_1(\theta)=\begin{bmatrix}0.2\\0.02\\0.1\end{bmatrix},\qquad B_2(\theta)=\begin{bmatrix}2\theta\\0.1+\theta\\0.2\end{bmatrix},\tag{72}$$
$$C_1(\theta)=\begin{bmatrix}1&1&0\\0&0&1\end{bmatrix},\qquad D_1(\theta)=\begin{bmatrix}-1\\1\end{bmatrix},$$
$$C_2(\theta)=\begin{bmatrix}0&1&0\\1&0&1\end{bmatrix},$$

where $\theta(t)=\sin(0.2t)$ is a time-varying parameter, $|\theta(t)|\leqslant 1$, and $|\dot{\theta}(t)|\leqslant 0.2$. Let the exogenous disturbance input $w(t)$ be

$$w(t)=\exp(-t)\sin(0.5t),\qquad t\geqslant 0.\tag{73}$$

It can be checked that the above system with $u(t)=0$ is unstable, and the states of open-loop system are shown in Figure 1 with the initial condition given by $x(0)=\begin{bmatrix}-0.1 & -0.1 & 0.1\end{bmatrix}^T$. Therefore, our purpose is to design a gain-scheduled $\mathscr{H}_2$ dynamic output feedback control $u(t)$ in the form of (8), such that the closed-loop system is exponentially stable with a minimized $\mathscr{H}_2$ disturbance attenuation level γ.

To solve the synthesis problem, we transform system (72) into the polytopic form. The system matrices $A(\theta)$ and $B_2(\theta)$ of LPV system (72) can be expressed as

$$A(\theta)=\sum_{i=1}^{2}\alpha_i A_i,\qquad B(\theta)=\sum_{i=1}^{2}\alpha_i B_i,\tag{74}$$

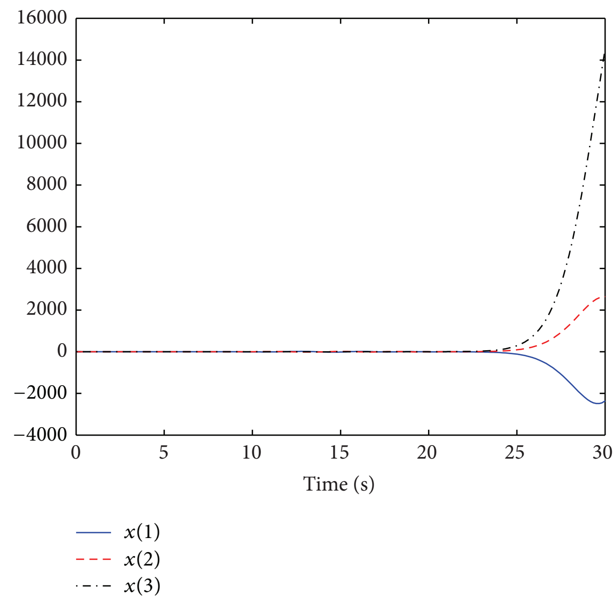

Figure 1: States of the open-loop system.

where

$$A_1=\begin{bmatrix}1&3&-1\\-3.2&-2.3&-0.2\\0.1&0.5&1\end{bmatrix},\qquad B_{21}=\begin{bmatrix}-2\\-0.9\\0.2\end{bmatrix},$$
$$A_2=\begin{bmatrix}2&3&1\\-1.2&-1.3&0.2\\0.1&0.5&-1\end{bmatrix},\qquad B_{22}=\begin{bmatrix}2\\1.1\\0.2\end{bmatrix}\tag{75}$$

with

$$\alpha_1(t)=\frac{\theta_{\max}-\theta}{\theta_{\max}-\theta_{\min}},\qquad \alpha_2(t)=\frac{\theta-\theta_{\min}}{\theta_{\max}-\theta_{\min}}.\tag{76}$$

It is easy to check that $\alpha_i(t)$, $i=1,2$, are convex coordinates, since they satisfy $0\leq\alpha_1(t)\leq 1$, $0\leq\alpha_2(t)\leq 1$, and $\alpha_1(t)+\alpha_2(t)=1$. In this example, we set $\varepsilon=0.5$. Using Corollary 12, it is found that the LMIs (55) are infeasible. However, using Theorem 13 and solving the LMIs (58)–(62) by using the same standard LMI-Toolbox in the Matlab environment [29], we obtain that the LMIs constraints (58)–(62) are feasible. Furthermore, by solving the convex optimization problem of (69) in Remark 14, we obtain that the minimum achievable noise attenuation level for the gain-scheduled $\mathscr{H}_2$ dynamic output feedback control problem is $\gamma^*=0.5806$ and the corresponding matrices are as follows:

$$R_1=\begin{bmatrix}0.1096&0.0090&0.0292\\-0.1255&0.1156&-0.0502\\0.0289&0.0046&0.0101\end{bmatrix},$$

$$R_2=\begin{bmatrix}0.1469&-0.0229&0.0384\\-0.1468&0.1029&0.0219\\0.0045&0.0141&0.0324\end{bmatrix},$$

$$S_1 = 10^5 \times \begin{bmatrix} 1.9878 & -0.7346 & -3.8287 \\ 3.8991 & 8.3985 & -9.4741 \\ -4.7517 & -0.2094 & 9.5476 \end{bmatrix},$$

$$S_2 = 10^6 \times \begin{bmatrix} 0.2857 & -0.1633 & -0.5386 \\ 0.2507 & 0.9838 & -0.6979 \\ -0.6214 & 0.1299 & 1.2169 \end{bmatrix},$$

$$T_1 = \begin{bmatrix} -1.8487 & -1.8817 & 3.3794 \\ 4.9104 & 3.5711 & -7.6522 \\ -0.4836 & -0.8968 & 1.0365 \end{bmatrix},$$

$$T_2 = \begin{bmatrix} 1.8140 & 0.0634 & -1.0335 \\ -2.0647 & -0.4457 & 3.1531 \\ 0.3720 & 0.8677 & 0.3924 \end{bmatrix},$$

$$\widetilde{A}_{K1} = \begin{bmatrix} -6.4403 & -7.0131 & -2.0779 \\ -3.5879 & -4.9302 & -0.1886 \\ 11.6185 & 11.0381 & 2.1956 \end{bmatrix},$$

$$\widetilde{A}_{K2} = \begin{bmatrix} -0.7134 & 2.3105 & -3.0523 \\ -0.8594 & 0.6710 & -3.0487 \\ -3.4059 & -6.9564 & 1.4208 \end{bmatrix},$$

$$\widetilde{B}_{K1} = 10^6 \times \begin{bmatrix} 0.8267 & 0.9623 \\ 0.3636 & 2.0002 \\ -1.7249 & -2.3245 \end{bmatrix},$$

$$\widetilde{B}_{K2} = 10^6 \times \begin{bmatrix} -0.0142 & -0.4723 \\ 0.1378 & 1.1325 \\ -0.0000 & 0.7169 \end{bmatrix},$$

$$\widetilde{C}_{K1} = \begin{bmatrix} -0.0118 & -0.0185 & -0.0224 \end{bmatrix},$$

$$\widetilde{C}_{K2} = \begin{bmatrix} -0.0082 & -0.0355 & -0.0344 \end{bmatrix},$$

$$\widetilde{D}_{K1} = \begin{bmatrix} -1.3604 & -1.3838 \end{bmatrix},$$

$$\widetilde{D}_{K2} = \begin{bmatrix} -0.3560 & -0.5457 \end{bmatrix}. \tag{77}$$

Setting $F(\theta) = I_3$, we obtain $G(\theta) = T(\theta) - R^T(\theta)S(\theta)$ by the process of proof in Theorem 10. Therefore, from (13) and Theorem 13, the matrices $(A_K(\theta), B_K(\theta), C_K(\theta), D_K(\theta))$ for the desired $\mathscr{H}_2$ dynamic output feedback controller $\mathscr{K}_{\mathscr{DOF}}$ in (8) can be obtained by the Matlab symbolic computation.

Figures 2 and 3 give the state responses of the closed-loop system $\mathscr{C}_{\mathscr{DOF}}$ and the dynamic output feedback control system $\mathscr{K}_{\mathscr{DOF}}$, respectively. The control input u in (8) is shown in Figure 4. From the above results, we can conclude that the desired stability of the closed-loop system is verified.

Based on the results of disturbance attenuation in Example 1 and the characteristic curves of Figures 2–4 in Example 2, it is shown that $\mathscr{H}_2$ performance can be used to capture both the response to stationary noise and the transient response of the closed-loop system.

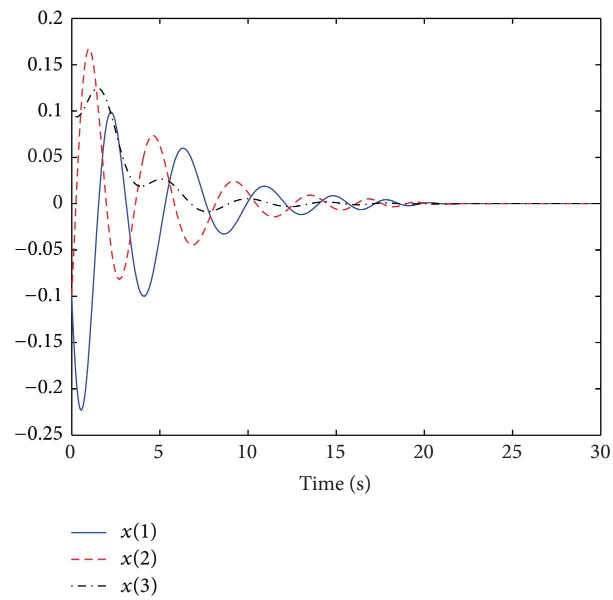

Figure 2: States of the closed-loop system.

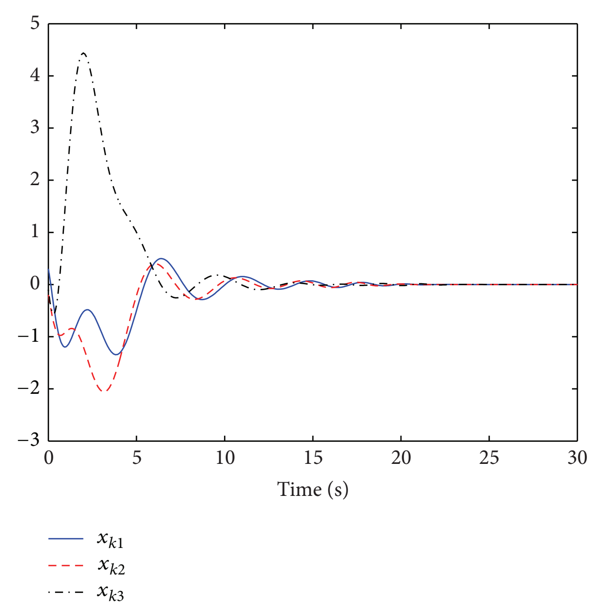

Figure 3: Controller states of $\mathscr{DOF}$ system.

5. Conclusions

In this paper, the problems of gain-scheduled $\mathscr{H}_2$ controller designs for continuous-time polytopic LPV systems have been addressed. Based on a basis-dependent Lyapunov function and the introduction of some auxiliary slack variables, sufficient conditions for both state feedback and dynamic output feedback controller synthesis problems have been established in terms of PLMIs, which guarantee the exponential stability and a prescribed $\mathscr{H}_2$ performance level of the closed-loop system over the given polytope. Moreover, the controller design problems have been cast into a convex optimization problem on the basis of the polytopic characteristic of the dependent parameters and the convexification method, which can be readily solved via standard LMI Toolbox.

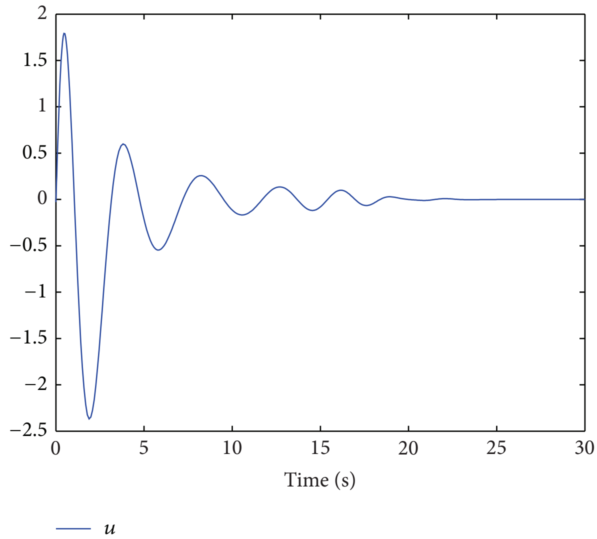

FIGURE 4: Control input.

Numerical examples have been provided to illustrate the effectiveness and advantage of the proposed design methods.

Several works may be needed in the future to improve the current results. First of all, in this paper, sufficient conditions for both state feedback and dynamic output feedback controller synthesis problems have been established in terms of PLMIs. With the increasing number of system dimension, how to solve these complicated PLMIs conditions quickly and efficiently is a good problem which should be further studied. Second, the developed results are expected to extend to the domain of practical application, such as designing the stabilized controller for the flight control system with good performances.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported in part by the National Natural Science Foundation of China under Grant nos. 61203007 and 61025014 and in part by the Natural Science Foundation of Shaanxi Province under Grant no. 2013JM8045.

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