Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 150685, 6 pages
http://dx.doi.org/10.1155/2015/150685

Research Article

Motions of Curves in the Pseudo-Galilean Space $\mathbb{G}_3^1$

Suleyman Cengiz, Esra Betul Koc Ozturk, and Ufuk Ozturk

Department of Mathematics, Faculty of Science, Çankırı Karatekin University, 18100 Çankırı, Turkey

Correspondence should be addressed to Ufuk Ozturk; uuzturk@asu.edu

Received 28 November 2014; Revised 17 February 2015; Accepted 20 February 2015

Academic Editor: Qing-Wen Wang

We study the flows of curves in the pseudo-Galilean 3-space and its equiform geometry without any constraints. We find that the Frenet equations and intrinsic quantities of the inelastic flows of curves are independent of time. We show that the motions of curves in the pseudo-Galilean 3-space and its equiform geometry are described by the inviscid and viscous Burgers' equations.

1. Introduction

In mathematical modeling of many nonlinear events of the natural and the applied sciences such as dynamics of vortex filaments, motions of interfaces, shape control of robot arms, propagation of flame fronts, image processing, supercoiled DNAs, magnetic fluxes, deformation of membranes, and dynamics of proteins, the motions of space curves are being used. The evolutions of these nonlinear phenomena are described by the differential equations which characterize the motions of curves as a family.

The motions of curves have been widely investigated by many authors in different geometries. In 1992 Nakayama and others explained that the close relation between the integrable evolution equations and the motions of curves is based on the equivalence of Frenet equations and the inverse scattering problem at zero eigenvalue [1], so that they identified the evolution equations that govern the 2D and 3D motions of the curves. They also studied the motions of the plane curves in which the curvature obeys the mKDV equation and its hierarchy [2]. Langer and Perline [3] gave the generalization of the motions of curves to n-dimensional Euclidean space. Many well-known integrable equations or their hierarchies related to the motions of space curves can be found in subsequent studies [4–11].

The subject of the curve flows in the pseudo-Galilean space, which is a real Cayley-Klein space with projective signature, is a virgin area to be searched. Inelastic flows of curves in the Galilean and the pseudo-Galilean spaces are studied at [12, 13]. Yoon [14] examined the inextensible flows of curves in the equiform geometry of the Galilean 3-space. Şahin [15] derived the intrinsic equations for a generalized relaxed elastic line on an oriented surface in the Galilean space.

In this study we investigate the motions of curves in the pseudo-Galilean 3-space and in its equiform geometry without any constraints. The first section gives the main definitions and theorems of the pseudo-Galilean 3-space. Next we define the evolution of a one-parameter family of smooth admissible curves in the pseudo-Galilean 3-space and find the flow equations of the curve evolution with use of the Frenet equations. Then we consider some particular cases where the flow of the intrinsic quantities κ and τ induces the inviscid Burgers' equation. Finally we study the curve evolution in the equiform geometry of the pseudo-Galilean 3-space regarding the relations between the Frenet vectors of these spaces.

2. The Pseudo-Galilean Space $\mathbb{G}_3^1$

The pseudo-Galilean space $\mathbb{G}_3^1$ is one of the real Cayley-Klein spaces of projective signature $(0, 0, +, -)$ as explained in [16]. The absolute figure of the pseudo-Galilean space $\mathbb{G}_3^1$ consists of an ordered triple $\{w, f, I\}$ where w is the ideal (absolute) plane, f is the (absolute) line in w, and I is the fixed hyperbolic involution of points of f. The curves in $\mathbb{G}_3^1$ are described in [16, 17].

In the nonhomogeneous affine coordinates for points and vectors (point pairs) the similarity group H_8 of $\mathbb{G}_3^1$ has the following form:

$$\begin{aligned}\bar{x} &= a_{11} + a_{12}x \\ \bar{y} &= a_{21} + a_{22}x + a_{23}y\cosh\varphi + a_{23}z\sinh\varphi \\ \bar{z} &= a_{31} + a_{32}x + a_{23}y\sinh\varphi + a_{23}z\cosh\varphi,\end{aligned} \quad (1)$$

where a_{ij} and φ are real numbers. In particular, for $a_{12} = a_{23} = 1$, the group (1) becomes the group $B_6 \subset H_8$ of isometries of the pseudo-Galilean space $\mathbb{G}_3^1$ as follows:

$$\begin{aligned}\bar{x} &= a_{11} + x \\ \bar{y} &= a_{21} + a_{22}x + y\cosh\varphi + z\sinh\varphi \\ \bar{z} &= a_{31} + a_{32}x + y\sinh\varphi + z\cosh\varphi.\end{aligned} \quad (2)$$

According to the motion group in the pseudo-Galilean space, there are nonisotropic vectors $\mathbf{x} = (x, y, z)$ (for which holds $x \neq 0$) and four types of isotropic vectors: space-like ($x = 0$, $y^2 - z^2 > 0$), time-like ($x = 0$, $y^2 - z^2 < 0$), and two types of light-like vectors ($x = 0$, $y = \pm z$). A non-light-like isotropic vector is a unit vector if $y^2 - z^2 = \pm 1$.

The scalar product of two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ can be written as

$$\langle \mathbf{u}, \mathbf{v} \rangle = \begin{cases} u_1 v_1, & \text{if } u_1 \neq 0 \vee v_1 \neq 0 \\ u_2 v_2 - u_3 v_3, & \text{if } u_1 = 0 \wedge v_1 = 0. \end{cases} \quad (3)$$

This scalar product leaves invariant the pseudo-Galilean norm of the vector $\mathbf{u} = (u_1, u_2, u_3)$ defined by

$$\|\mathbf{u}\| = \begin{cases} u_1, & \text{if } u_1 \neq 0 \\ \sqrt{|u_2^2 - u_3^2|}, & \text{if } u_1 = 0. \end{cases} \quad (4)$$

Let α be a spatial curve given first by

$$\begin{aligned}\alpha : I \subseteq \mathbb{R} &\longrightarrow \mathbb{G}_3^1 \\ t &\longrightarrow \alpha(t) = (x(t), y(t), z(t)),\end{aligned} \quad (5)$$

where $x(t), y(t), z(t) \in C^3$. Then the curve $\alpha(t)$ is said to be admissible if $\dot{x}(t) \neq 0$ [16]. For an admissible curve α in $\mathbb{G}_3^1$ parameterized by the arc length $s = x$ with differential form $ds = dx$, given as

$$\alpha(x) = (x, y(x), z(x)), \quad (6)$$

where $y(x), z(x) \in C^3$, the curvature $\kappa(x)$ and the torsion $\tau(x)$ are defined by

$$\kappa(x) = \sqrt{|(y''(x))^2 - (z''(x))^2|} \quad (7)$$

$$\tau(x) = \frac{y''(x) z'''(x) - y'''(x) z''(x)}{\kappa^2(x)}, \quad (8)$$

respectively. The pseudo-Galilean Frenet frame of the admissible curve $\alpha(x)$ parameterized by the arc length has the form

$$\begin{aligned}\mathbf{t}(x) &= \alpha'(x) = (1, y'(x), z'(x)), \\ \mathbf{n}(x) &= \frac{1}{\kappa(x)}\alpha''(x) = \frac{1}{\kappa(x)}(0, y''(x), z''(x)), \\ \mathbf{b}(x) &= \frac{1}{\kappa(x)}(0, \varepsilon z''(x), \varepsilon y''(x)),\end{aligned} \quad (9)$$

where $\mathbf{t}, \mathbf{n},$ and $\mathbf{b}$ are called the tangent vector, principal normal vector, and binormal vector fields of the curve α, respectively. Here $\varepsilon = +1$ or -1 is chosen by the criterion $\det(\mathbf{t}, \mathbf{n}, \mathbf{b}) = 1$. If $\mathbf{n}$ is a space-like or time-like vector, then the curve $\alpha(x)$ given by (6) is time-like or space-like, respectively. Then the Frenet equations of the curve $\alpha(x)$ are given by

$$\begin{bmatrix} \mathbf{t}(x) \\ \mathbf{n}(x) \\ \mathbf{b}(x) \end{bmatrix}_x = \begin{bmatrix} 0 & \kappa(x) & 0 \\ 0 & 0 & \tau(x) \\ 0 & \tau(x) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t}(x) \\ \mathbf{n}(x) \\ \mathbf{b}(x) \end{bmatrix}, \quad (10)$$

where $\mathbf{t}, \mathbf{n},$ and $\mathbf{b}$ are mutually orthogonal vectors [17, 18].

3. Motions of Curves in the Pseudo-Galilean Space $\mathbb{G}_3^1$

In this section we study the curve evolution in the pseudo-Galilean 3-space by using the Frenet frame structure to obtain some related integrable equations.

Let us consider a one-parameter family of smooth admissible curves $\mathbf{r}(u, t)$ in the pseudo-Galilean space $\mathbb{G}_3^1$ where t denotes the time or the scale and u parameterizes each curve of the family. We assume that this family $\mathbf{r}(u, t)$ evolves according to the flow equation

$$\dot{\mathbf{r}} := \frac{d\mathbf{r}}{dt} = a(u,t)\,\mathbf{t} + b(u,t)\,\mathbf{n} + c(u,t)\,\mathbf{b}, \quad (11)$$

$$\mathbf{r}(u, 0) = \mathbf{r}(u), \quad (12)$$

where a, b, c are arbitrary functions.

Let

$$g(u,t) := \left\|\frac{\partial \mathbf{r}}{\partial u}\right\| = \sqrt{\left|\left\langle \frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial u} \right\rangle\right|} \quad (13)$$

denote the length along the curve. The arc length parameter s is given by

$$s(u,t) := \int_0^u g(u', t)\, du'. \quad (14)$$

From (10) we can express the Frenet vectors and the intrinsic quantities as

$$\mathbf{t} := \frac{\partial \mathbf{r}}{\partial s} = \frac{1}{g}\frac{\partial \mathbf{r}}{\partial u},$$
$$\mathbf{n} := \frac{1}{\kappa}\frac{\partial \mathbf{t}}{\partial s} = \frac{1}{\kappa g}\frac{\partial \mathbf{t}}{\partial u}, \quad (15)$$
$$\mathbf{b} := \frac{1}{\tau}\frac{\partial \mathbf{n}}{\partial s} = \frac{1}{\tau g}\frac{\partial \mathbf{n}}{\partial u},$$

$$\kappa := \left\| \frac{\partial \mathbf{t}}{\partial s} \right\| = \frac{1}{g}\left\| \frac{\partial \mathbf{t}}{\partial u} \right\|,$$
$$\tau := \left\langle \mathbf{n}, \frac{\partial \mathbf{b}}{\partial s} \right\rangle, \quad (16)$$

respectively.

Now we will derive the flow equations for the Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$, the metric g, the curvature κ, and the torsion τ for the curve evolution $\mathbf{r}(u, t)$ satisfying (12). Since $g^2 = \langle \partial \mathbf{r}/\partial u, \partial \mathbf{r}/\partial u \rangle$ taking the derivatives of both sides and using (11) and (15) we can compute the flow of the metric g as

$$\begin{aligned} 2g\frac{dg}{dt} &= 2\left\langle \frac{\partial \mathbf{r}}{\partial u}, \frac{\partial}{\partial u}\left(\frac{d\mathbf{r}}{dt}\right)\right\rangle \\ &= 2g\left\langle \mathbf{t}, \frac{\partial a}{\partial u}\mathbf{t} + a\kappa g\mathbf{n} + \frac{\partial b}{\partial u}\mathbf{n} + b\tau g\mathbf{b} + \frac{\partial c}{\partial u}\mathbf{b} + c\tau g\mathbf{n}\right\rangle \\ &= 2g\left\langle \mathbf{t}, \frac{\partial a}{\partial u}\mathbf{t} + \left(\frac{\partial b}{\partial u} + a\kappa g + c\tau g\right)\mathbf{n} + \left(\frac{\partial c}{\partial u} + b\tau g\right)\mathbf{b}\right\rangle \\ &= 2g\frac{\partial a}{\partial u}. \end{aligned} \quad (17)$$

So the flow of the metric equals

$$\frac{dg}{dt} = \frac{\partial a}{\partial u}. \quad (18)$$

It is important to notice that the variables u and t are independent but s and t are not. As a consequence, we have

$$\frac{d}{dt}\frac{\partial}{\partial s} = \frac{d}{dt}\left(\frac{1}{g}\frac{\partial}{\partial u}\right) = -\frac{\partial a}{\partial s}\frac{\partial}{\partial s} + \frac{\partial}{\partial s}\frac{d}{dt}. \quad (19)$$

We can evaluate the flow equation of the unit tangent vector $\mathbf{t}$ as

$$\begin{aligned} \frac{d\mathbf{t}}{dt} &= \frac{d}{dt}\frac{\partial \mathbf{r}}{\partial s} \\ &= -\frac{\partial a}{\partial s}\frac{\partial \mathbf{r}}{\partial s} + \frac{\partial}{\partial s}\frac{d\mathbf{r}}{dt} \\ &= -\frac{\partial a}{\partial s}\mathbf{t} + \frac{\partial a}{\partial s}\mathbf{t} + a\kappa\mathbf{n} + \frac{\partial b}{\partial s}\mathbf{n} + b\tau\mathbf{b} + \frac{\partial c}{\partial s}\mathbf{b} + c\tau\mathbf{n} \\ &= \left(\frac{\partial b}{\partial s} + a\kappa + c\tau\right)\mathbf{n} + \left(\frac{\partial c}{\partial s} + b\tau\right)\mathbf{b}. \end{aligned} \quad (20)$$

Similarly for the flow of the unit normal vector $\mathbf{n}$ we have

$$\begin{aligned} \frac{d\mathbf{n}}{dt} = &\left(\frac{1}{\kappa}\frac{\partial}{\partial s}\left(\frac{\partial b}{\partial s} + a\kappa + c\tau\right)\right. \\ &\left. -\frac{1}{\kappa}\frac{d\kappa}{dt} - \frac{\partial a}{\partial s} + \frac{\tau}{\kappa}\left(\frac{\partial c}{\partial s} + b\tau\right)\right)\mathbf{n} \\ &+ \left(\frac{\tau}{\kappa}\left(\frac{\partial b}{\partial s} + a\kappa + c\tau\right) + \frac{1}{\kappa}\frac{\partial}{\partial s}\left(\frac{\partial c}{\partial s} + b\tau\right)\right)\mathbf{b}. \end{aligned} \quad (21)$$

Since $\langle d\mathbf{n}/dt, \mathbf{n}\rangle = 0$ we obtain

$$\frac{d\mathbf{n}}{dt} = \left(\frac{\tau}{\kappa}\left(\frac{\partial b}{\partial s} + a\kappa + c\tau\right) + \frac{1}{\kappa}\frac{\partial}{\partial s}\left(\frac{\partial c}{\partial s} + b\tau\right)\right)\mathbf{b}, \quad (22)$$

$$\frac{d\kappa}{dt} = \frac{\partial}{\partial s}\left(\frac{\partial b}{\partial s} + a\kappa + c\tau\right) - \kappa\frac{\partial a}{\partial s} + \tau\left(\frac{\partial c}{\partial s} + b\tau\right). \quad (23)$$

Also the flow of the binormal vector $\mathbf{b}$ becomes

$$\begin{aligned} \frac{d\mathbf{b}}{dt} = &\left(\frac{\tau}{\kappa}\left(\frac{\partial b}{\partial s} + a\kappa + c\tau\right) + \frac{1}{\kappa}\frac{\partial}{\partial s}\left(\frac{\partial c}{\partial s} + b\tau\right)\right)\mathbf{n} \\ &+ \left(\frac{1}{\tau}\frac{\partial}{\partial s}\left(\frac{\tau}{\kappa}\left(\frac{\partial b}{\partial s} + a\kappa + c\tau\right) + \frac{1}{\kappa}\frac{\partial}{\partial s}\left(\frac{\partial c}{\partial s} + b\tau\right)\right)\right. \\ &\left. -\frac{1}{\tau}\frac{d\tau}{dt} - \frac{\partial a}{\partial s}\right)\mathbf{b}. \end{aligned} \quad (24)$$

From the equation $\langle d\mathbf{b}/dt, \mathbf{b}\rangle = 0$ we obtain

$$\frac{d\mathbf{b}}{dt} = \left(\frac{\tau}{\kappa}\left(\frac{\partial b}{\partial s} + a\kappa + c\tau\right) + \frac{1}{\kappa}\frac{\partial}{\partial s}\left(\frac{\partial c}{\partial s} + b\tau\right)\right)\mathbf{n}, \quad (25)$$

$$\frac{d\tau}{dt} = \frac{\partial}{\partial s}\left(\frac{\tau}{\kappa}\left(\frac{\partial b}{\partial s} + a\kappa + c\tau\right) + \frac{1}{\kappa}\frac{\partial}{\partial s}\left(\frac{\partial c}{\partial s} + b\tau\right)\right) - \tau\frac{\partial a}{\partial s}. \quad (26)$$

Since $\langle \mathbf{t}, \mathbf{n}\rangle = 0$ and $\langle \mathbf{t}, \mathbf{b}\rangle = 0$ we have

$$\begin{aligned} \left\langle \frac{d\mathbf{t}}{dt}, \mathbf{n}\right\rangle + \left\langle \mathbf{t}, \frac{d\mathbf{n}}{dt}\right\rangle &= 0, \\ \left\langle \frac{d\mathbf{t}}{dt}, \mathbf{b}\right\rangle + \left\langle \mathbf{t}, \frac{d\mathbf{b}}{dt}\right\rangle &= 0. \end{aligned} \quad (27)$$

Then by (20), (22), and (25) we can write

$$\begin{aligned} \frac{\partial b}{\partial s} + a\kappa + c\tau &= 0, \\ \frac{\partial c}{\partial s} + b\tau &= 0. \end{aligned} \quad (28)$$

Hence the flow equations of the Frenet frame take the form

$$\begin{aligned} \frac{d\mathbf{t}}{dt} &= 0, \\ \frac{d\mathbf{n}}{dt} &= 0, \\ \frac{d\mathbf{b}}{dt} &= 0, \end{aligned} \quad (29)$$

and for the intrinsic quantities the flow equations become

$$\frac{d\kappa}{dt} = -\kappa \frac{\partial a}{\partial s}, \qquad \frac{d\tau}{dt} = -\tau \frac{\partial a}{\partial s}. \tag{30}$$

Therefore, we have the following theorem.

Theorem 1. *Let $\mathbf{r} = \mathbf{r}(u,t)$ be a one-parameter family of smooth admissible curves in the pseudo-Galilean space $\mathbb{G}_3^1$. If $\mathbf{r}$ evolves according to (11), then, the Frenet frame $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ of $\mathbf{r}$ is not time dependent and the intrinsic quantities κ and τ of $\mathbf{r}$ satisfy the equations*

$$\frac{d\kappa}{dt} = -\kappa \frac{\partial a}{\partial s}, \qquad \frac{d\tau}{dt} = -\tau \frac{\partial a}{\partial s}, \tag{31}$$

where s is the arc length parameter of $\mathbf{r}$.

Remark 2. Burgers' equations describe various kinds of phenomena such as a mathematical model of turbulence and the approximate theory of flow through a shock wave traveling in viscous fluid. The inviscid Burgers' equation is a model for the nonlinear wave propagation, especially in fluid mechanics. It takes the form

$$\frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi}{\partial s} = 0, \tag{32}$$

where $\psi(s,t)$ is a solution of the equation.

From Remark 2, if we choose the curvature $\kappa = a$ or the torsion $\tau = a$ in (30), then we have that the intrinsic quantities κ and τ evolve according to the inviscid Burgers' equation. So, we obtain the following corollary.

Corollary 3. *Let $\mathbf{r} = \mathbf{r}(u,t)$ be a curve evolution in the pseudo-Galilean space $\mathbb{G}_3^1$ with the intrinsic quantities κ and τ given by (11). If one sets $\kappa = a$ or $\tau = a$, then the intrinsic quantities κ and τ satisfy the inviscid Burgers' equation.*

3.1. Inextensible Curve Flows in the Pseudo-Galilean Space. In this section, we investigate some properties of the inextensible flows in the pseudo-Galilean space $\mathbb{G}_3^1$.

Definition 4. A curve evolution $\mathbf{r}(u,t)$ and its flow $d\mathbf{r}/dt$ in the pseudo-Galilean space $\mathbb{G}_3^1$ are said to be inextensible if

$$\frac{d}{dt}\left\| \frac{\partial \mathbf{r}}{\partial u} \right\| = 0. \tag{33}$$

According to Definition 4 and (11), in case the family of curves $\mathbf{r}(u,t)$ is inextensible, from (18) we get

$$\frac{\partial a}{\partial u} = 0, \qquad g(u,t) = \xi(u) \tag{34}$$

for some single variable function ξ. Therefore, we have the following corollary.

Corollary 5. *The curve evolution $\mathbf{r}(u,t)$ which is given by (11) is inextensible if and only if $\partial a/\partial u = 0$.*

If we now restrict ourselves to the arc length parameterized admissible curves that undergo purely inextensible deformations, that is, $g(u,t) = \xi(u) = 1$ and $\partial a/\partial s = 0$, then the local coordinate u corresponds to the arc length parameter s. Thus the flow of the curve is expressed as

$$\dot{\mathbf{r}} := \frac{d\mathbf{r}}{dt} = a(s,t)\,\mathbf{t} + b(s,t)\,\mathbf{n} + c(s,t)\,\mathbf{b} \tag{35}$$

and the flow of the Frenet frame $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ with the intrinsic quantities κ and τ is given by

$$\frac{d\mathbf{t}}{dt} = 0, \quad \frac{d\mathbf{n}}{dt} = 0, \quad \frac{d\mathbf{b}}{dt} = 0, \quad \frac{d\kappa}{dt} = 0, \quad \frac{d\tau}{dt} = 0. \tag{36}$$

So, we get the following corollary.

Corollary 6. *Let $\mathbf{r} = \mathbf{r}(u,t)$ be a curve evolution in the pseudo-Galilean space $\mathbb{G}_3^1$ with its flow $d\mathbf{r}/dt$ given by (11). If the curve flow $\mathbf{r}(u,t)$ is inextensible, then the Frenet vectors $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$, the curvature κ, and the torsion τ are not time dependent.*

4. Motions of Curves in the Equiform Geometry of $\mathbb{G}_3^1$

Similarity group (1) matches an ordinary (formal) line element $(dx = 0, dy, dz)$ in a pseudo-Euclidean plane (i.e., $x =$ const.) into a segment of length proportional to the original one with the coefficient of proportionality a_{23}. Other line elements (dx, dy, dz), which lie on an isotropic plane $(dx = 0)$, are matched into proportional ones with the coefficient a_{12}. So, all line segments are matched into proportional ones with the same coefficient of proportionality if and only if $a_{12} = a_{23}$. Then we obtain a subgroup $H_7 \subset H_8$ which preserves length ratio of segments and angles between planes and lines, respectively. This group is called the group of equiform transformations of the pseudo-Galilean space.

Definition 7. Geometry of the pseudo-Galilean space $\mathbb{G}_3^1$ induced by the 7-parameter equiform group H_7 is called the equiform geometry of the space $\mathbb{G}_3^1$.

Let $\alpha : I \to \mathbb{G}_3$ be an admissible curve with the arc length parameter s. We define the equiform invariant parameter of α by

$$\sigma = \int \frac{1}{p} ds, \tag{37}$$

where $p = 1/\kappa$ is the radius of the curvature of the curve α. It follows that

$$\frac{d\sigma}{ds} = \frac{1}{p}. \tag{38}$$

We then have the new equiform invariant Frenet equations as

$$\begin{bmatrix} \mathbf{T}(\sigma) \\ \mathbf{N}(\sigma) \\ \mathbf{B}(\sigma) \end{bmatrix}_{\sigma} = \begin{bmatrix} \widetilde{\kappa} & 1 & 0 \\ 0 & \widetilde{\kappa} & \widetilde{\tau} \\ 0 & \widetilde{\tau} & \widetilde{\kappa} \end{bmatrix} \begin{bmatrix} \mathbf{T}(\sigma) \\ \mathbf{N}(\sigma) \\ \mathbf{B}(\sigma) \end{bmatrix}, \tag{39}$$

where $\widetilde{\kappa}$ is called the *equiform curvature* and $\widetilde{\tau}$ is called the *equiform torsion* of the curve α [12]. These are related to the curvature κ and torsion τ by the equations

$$\widetilde{\kappa} = -\frac{\kappa_s}{\kappa^2}, \qquad \widetilde{\tau} = \frac{\tau}{\kappa}. \tag{40}$$

Also the equiformly invariant Frenet vectors $\mathbf{T}, \mathbf{N},$ and $\mathbf{B}$ are related to the pseudo-Galilean Frenet vectors $\mathbf{t}, \mathbf{n},$ and $\mathbf{b}$ as

$$\begin{aligned} \mathbf{T} &= \frac{\mathbf{t}}{\kappa} = p\mathbf{t}, \\ \mathbf{N} &= \frac{\mathbf{n}}{\kappa} = p\mathbf{n}, \\ \mathbf{B} &= \frac{\mathbf{b}}{\kappa} = p\mathbf{b}. \end{aligned} \tag{41}$$

The equiformly invariant arc length parameter of the curve evolution $\mathbf{r}(u,t)$ can be defined as a function of u by

$$\sigma(u) = \int_0^u \frac{1}{p} g(u', t)\, du'. \tag{42}$$

So the operator $\partial/\partial\sigma$ is equal to $p(\partial/\partial u)$. The flow of the curve evolution $\mathbf{r}(u,t)$ can be expressed in the form

$$\frac{d\mathbf{r}}{dt} = W\mathbf{T} + U\mathbf{N} + V\mathbf{B}, \tag{43}$$

where $W, U,$ and V are arbitrary functions. The preceding flow of $\mathbf{r}(u,t)$ is related to flow (11) in the pseudo-Galilean space $\mathbb{G}_3^1$ as

$$\frac{d\mathbf{r}}{dt} = a\mathbf{t} + b\mathbf{n} + c\mathbf{b}, \tag{44}$$

with $a = pW$, $b = pU$, and $c = pV$. Then using the formulas in Section 3 we obtain the flow of the metric

$$\frac{dg}{dt} = \frac{\partial a}{\partial u} = \frac{\partial}{\partial u}(pW) = p\frac{\partial W}{\partial u} + g\widetilde{\kappa}W \tag{45}$$

or

$$\frac{dg}{dt} = \frac{\partial a}{\partial u} = g\left(\frac{\partial W}{\partial \sigma} + \widetilde{\kappa}W\right). \tag{46}$$

The partial derivatives $\partial/\partial\sigma$ and d/dt do not commute in general while the partials $\partial/\partial u$ and d/dt commute:

$$\frac{d}{dt}\frac{\partial}{\partial\sigma} = -\left(\frac{\partial W}{\partial\sigma} + \widetilde{\kappa}W - \frac{1}{p}\frac{dp}{dt}\right)\frac{\partial}{\partial\sigma} + \frac{\partial}{\partial\sigma}\frac{d}{dt}. \tag{47}$$

Using (41) and (29) the flow equation of the equiformly invariant tangent vector field $\mathbf{T}$ is calculated as

$$\begin{aligned} \frac{d\mathbf{T}}{dt} &= \frac{d}{dt}(p\mathbf{t}) \\ &= -\frac{1}{g}\frac{\partial a}{\partial u}\mathbf{T} \\ &= -\left(\frac{\partial W}{\partial\sigma} + \widetilde{\kappa}W\right)\mathbf{T}. \end{aligned} \tag{48}$$

Similarly, we can write the flows of the equiformly invariant principal normal and binormal vector fields, the equiform curvature, and the equiform torsion, respectively, as follows:

$$\frac{d\mathbf{N}}{dt} = \frac{d}{dt}(p\mathbf{n}) = -\left(\frac{\partial W}{\partial\sigma} + \widetilde{\kappa}W\right)\mathbf{N}, \tag{49}$$

$$\frac{d\mathbf{B}}{dt} = \frac{d}{dt}(p\mathbf{n}) = -\left(\frac{\partial W}{\partial\sigma} + \widetilde{\kappa}W\right)\mathbf{B}, \tag{50}$$

$$\frac{d\widetilde{\kappa}}{dt} = -2\left(\frac{\partial W}{\partial\sigma} + \widetilde{\kappa}W\right)\widetilde{\kappa} - \frac{\partial}{\partial\sigma}\left(\frac{\partial W}{\partial\sigma} + \widetilde{\kappa}W\right), \tag{51}$$

$$\frac{d\widetilde{\tau}}{dt} = 0. \tag{52}$$

Therefore, we obtain the following theorem.

Theorem 8. *Let $\mathbf{r} = \mathbf{r}(u,t)$ be an admissible curve in the equiform geometry of $\mathbb{G}_3^1$ with the equiform invariant Frenet frame (39). If $\mathbf{r}$ evolves according to (43), then the flows of*

(i) the equiform invariant Frenet vectors $\mathbf{T}, \mathbf{N},$ and $\mathbf{B}$ of $\mathbf{r}$ are, respectively, given as

$$\begin{aligned} \frac{d\mathbf{T}}{dt} &= -\left(\frac{\partial W}{\partial\sigma} + \widetilde{\kappa}W\right)\mathbf{T}, \\ \frac{d\mathbf{N}}{dt} &= -\left(\frac{\partial W}{\partial\sigma} + \widetilde{\kappa}W\right)\mathbf{N}, \\ \frac{d\mathbf{B}}{dt} &= -\left(\frac{\partial W}{\partial\sigma} + \widetilde{\kappa}W\right)\mathbf{B}, \end{aligned} \tag{53}$$

(ii) the equiform curvature $\widetilde{\kappa}$ and the equiform torsion $\widetilde{\tau}$ of $\mathbf{r}$ are, respectively, given as

$$\begin{aligned} \frac{d\widetilde{\kappa}}{dt} &= -2\left(\frac{\partial W}{\partial\sigma} + \widetilde{\kappa}W\right)\widetilde{\kappa} - \frac{\partial}{\partial\sigma}\left(\frac{\partial W}{\partial\sigma} + \widetilde{\kappa}W\right), \\ \frac{d\widetilde{\tau}}{dt} &= 0, \end{aligned} \tag{54}$$

where σ is the equiform invariant parameter and W is an arbitrary function.

Remark 9. Viscous Burgers' equation can be regarded as a one-dimensional analog of the Navier-Stokes equations which model the behavior of viscous fluids. It is given by the equation

$$\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi}{\partial s} = \nu\frac{\partial^2\psi}{\partial s^2}, \tag{55}$$

where $\psi(s,t)$ is a solution of the equation.

From Remark 9, if we choose $(\partial W/\partial\sigma) + \tilde{\kappa}W = (1/2)(\partial\tilde{\kappa}/\partial\sigma)$ in (51), then we see that the intrinsic quantity $\tilde{\kappa}$ evolves according to the viscous Burgers' equation. So, we have the following corollary.

Corollary 10. *Let $\mathbf{r} = \mathbf{r}(u,t)$ be an equiform invariant curve evolution in the equiform geometry of $\mathbb{G}_3^1$ with the intrinsic quantity $\tilde{\kappa}$ given by (39). If the equality $\partial W/\partial\sigma + \tilde{\kappa}W = (1/2)(\partial\tilde{\kappa}/\partial\sigma)$ holds, then the intrinsic quantity $\tilde{\kappa}$ satisfies the viscous Burgers' equation.*

4.1. Inextensible Curve Flows in the Equiform Geometry of $\mathbb{G}_3^1$. In this section, we investigate some properties of the inextensible flows in the equiform geometry of $\mathbb{G}_3^1$.

Let $\mathbf{r}(u,t)$ be an inextensible curve evolution in the equiform geometry of $\mathbb{G}_3^1$ given by (43). Then, from Definition 4, we have

$$\frac{\partial W}{\partial\sigma} + \tilde{\kappa}W = 0 \tag{56}$$

and from this equation we get

$$W = \frac{C}{\kappa}, \tag{57}$$

where C is an integration constant. So, we get the following corollary.

Corollary 11. *The curve evolution $\mathbf{r}(u,t)$, which is given by (43), is inextensible if and only if $W = C/\kappa$ for some integration constant C.*

From Theorem 8 and Corollary 11, we have the following corollary.

Corollary 12. *If the curve evolution $\mathbf{r}(u,t)$, which is given by (43), is inextensible, then the Frenet vectors $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$, the curvature $\tilde{\kappa}$, and the torsion $\tilde{\tau}$ of $\mathbf{r}$ are not time dependent.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The authors would like to thank the referees for the helpful suggestions.

References

[1] K. Nakayama, H. Segur, and M. Wadati, "Integrability and the motion of curves," *Physical Review Letters*, vol. 69, no. 18, pp. 2603–2606, 1992.

[2] K. Nakayama and M. Wadati, "Motion of curves in the plane," *Journal of the Physical Society of Japan*, vol. 62, no. 2, pp. 473–479, 1993.

[3] J. Langer and R. Perline, "Curve motion inducing modified Korteweg-de Vries systems," *Physics Letters A*, vol. 239, no. 1-2, pp. 36–40, 1998.

[4] R. E. Goldstein and D. M. Petrich, "The Korteweg-de Vries hierarchy as dynamics of closed curves in the plane," *Physical Review Letters*, vol. 67, no. 23, pp. 3203–3206, 1991.

[5] M. Gürses, "Motion of curves on two-dimensional surfaces and soliton equations," *Physics Letters A*, vol. 241, no. 6, pp. 329–334, 1998.

[6] K. Nakayama, "Motion of curves in hyperboloids in the Minkowski space II," *Journal of the Physical Society of Japan*, vol. 68, no. 10, pp. 3214–3218, 1999.

[7] W. K. Schief and C. Rogers, "Binormal motion of curves of constant curvature and torsion. Generation of soliton surfaces," *The Royal Society of London. Proceedings. Series A. Mathematical, Physical and Engineering Sciences*, vol. 455, no. 1988, pp. 3163–3188, 1999.

[8] K.-S. Chou and C. Qu, "The KdV equation and motion of plane curves," *Journal of the Physical Society of Japan*, vol. 70, no. 7, pp. 1912–1916, 2001.

[9] K.-S. Chou and C. Qu, "Integrable equations arising from motions of plane curves," *Physica D: Nonlinear Phenomena*, vol. 162, no. 1-2, pp. 9–33, 2002.

[10] K.-S. Chou and C. Qu, "Integrable motions of space curves in affine geometry," *Chaos, Solitons & Fractals*, vol. 14, no. 1, pp. 29–44, 2002.

[11] K.-S. Chou and C. Qu, "Motions of curves in similarity geometries and Burgers-mKdV hierarchies," *Chaos, Solitons and Fractals*, vol. 19, no. 1, pp. 47–53, 2004.

[12] Z. Erjavec and B. Divjak, "The equiform differential geometry of curves in the pseudo-Galilean space," *Mathematical Communications*, vol. 13, no. 2, pp. 321–332, 2008.

[13] A. O. Öğrenmiş, M. Yeneroğlu, and M. Külahcı, "Inelastic admissible curves in the pseudo-Galilean space G_3^1," *International Journal of Open Problems in Computer Science and Mathematics*, vol. 4, no. 3, pp. 199–207, 2011.

[14] D. W. Yoon, "Inelastic ows of curves according to equiform in Galilean space," *Journal of the Chungcheong Mathematical Society*, vol. 24, no. 4, pp. 665–673, 2011.

[15] T. Şahin, "Intrinsic equations for a generalized relaxed elastic line on an oriented surface in the Galilean space," *Acta Mathematica Scientia. Series B. English Edition*, vol. 33, no. 3, pp. 701–711, 2013.

[16] B. Divjak and Ž. Milin-Šipuš, "Special curves on ruled surfaces in Galilean and pseudo-Galilean spaces," *Acta Mathematica Hungarica*, vol. 98, no. 3, pp. 203–215, 2003.

[17] B. Divjak, "Curves in pseudo-Galilean geometry," *Annales Universitatis Scientiarum Budapestinensis de Rolando Eötvös Nominatae. Sectio Mathematica*, vol. 41, pp. 117–128, 1998.

[18] B. Divjak and Ž. Milin-šipuš, "Minding isometries of ruled surfaces in pseudo-Galilean space," *Journal of Geometry*, vol. 77, no. 1-2, pp. 35–47, 2003.