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Research Article

In-Domain Control of a Heat Equation: An Approach Combining Zero-Dynamics Inverse and Differential Flatness

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This paper addresses the set-point control problem of a one-dimensional heat equation with in-domain actuation. The proposed scheme is based on the framework of zero-dynamics inverse combined with flat system control. Moreover, the set-point control is cast into a motion planning problem of a multiple-input, multiple-output system, which is solved by a Green's function-based reference trajectory decomposition. The validity of the proposed method is assessed through the analysis of the invertibility of the map generated by Green's function and the convergence of the regulation error. The performance of the developed control scheme and the viability of the proposed approach are confirmed by numerical simulation of a representative system.

1. Introduction

Control of parabolic partial differential equations (PDEs) is a long-standing problem in PDE control theory and practice. There exists a very rich literature devoted to this topic, and it is continuing to draw a great attention for both theoretical studies and practical applications. In the existing literature, the majority of work is dedicated to boundary control, which may be represented as a standard Cauchy problem to which functional analytic setting based on semigroup and other related tools can be applied (see, e.g., [1–4]). It is interesting to note that, in recent years, some methods that were originally developed for the control of finite-dimensional nonlinear systems have been successfully extended to the control of parabolic PDEs, such as backstepping (see, e.g., [5–7]), flat systems (see, e.g., [8–14]), and their variations (see, e.g., [15, 16]).

This paper deals with the output regulation problem for set-point control of a one-dimensional heat equation via pointwise in-domain (or interior) actuation. Notice that, due to the fact that the regularity of a pointwise controlled inhomogeneous heat equation is qualitatively different from that of boundary controlled heat equations, the techniques developed for boundary control may not be directly applied to the former case. This constitutes a motivation for the present work. The control scheme developed in this paper is based-on the framework of zero-dynamics inverse (ZDI), which was introduced by Byrnes and Gilliam in [17] and has been exploited and developed in a series of works (see, e.g., [18] and the references therein). It is pointed out in [19] that "*for certain boundary control systems it is very easy to model the system's zero dynamics, which, in turn, provides a simple systematic methodology for solving certain problems of output regulation*." Indeed, the construction of zero-dynamics for output regulation of certain interiorly controlled PDEs is also straightforward (see, e.g., [20]) and hence, the control design can be carried out in a systematic manner. Nevertheless, a main issue related to the application of ZDI method is that it leads to, in general, a dynamic control law. Thus, the implementation of such control schemes requires resolving the corresponding zero-dynamics, which may be very difficult for generic regulation problems, such as set-point control considered in the present work. To overcome this difficulty, we resort to the theory of flat systems [13, 21]. We show that, in the context of ZDI design, the control can be derived from the so-called flat output without explicitly solving the original dynamic equation. Moreover, in the framework of flat systems, set-point control can be cast into a problem of motion planning, which can also be carried out in a systematic manner. Note that it can be expected that the ZDI design is applicable to other systems, such as the interior control of beam and plate equations, as an alternative to the methods proposed in, for example, [11, 22, 23].

The system model used in this work is taken from [20]. In order to perform control design based on the principle of superposition, we present the original system in a form of *parallel connection*. As the control with multiple actuators located in the domain leads to a multiple-input, multiple-output (MIMO) problem, we introduce a Green's function-based reference trajectory decomposition scheme that enables a simple and computational tractable implementation of the proposed control algorithm.

The remainder of the paper is organized as follows. Section 2 describes the model of the considered system and its equivalent settings. Section 3 presents the detailed control design. Section 4 deals with motion planning and addresses the convergence and the solvability of the proposed control scheme. A simulation study is carried out in Section 5, and, finally, some concluding remarks are presented in Section 6.

2. Problem Setting

In the present work, we consider a scalar parabolic equation describing one-dimensional heat transfer with in-domain control, which is studied in [20]. Denote by $u(x,t)$ the temperature distribution over the one-dimensional space, x, and the time, t. The derivatives of $u(x,t)$ with respect to its variables are denoted by u_x and u_t, respectively. Consider m points x_j, $j = 1,\dots,m$, in the interval $(0,1)$ and assume, without loss of generality, that $0 = x_0 < x_1 < x_2 < \cdots < x_m < x_{m+1} = 1$. Let $\Omega \doteq \bigcup_{j=0}^{m}(x_j, x_{j+1})$. The considered heat equation with in-domain control in a normalized coordinate is of the form

$$u_t(x,t) - u_{xx}(x,t) = 0, \quad x \in \Omega,\ t > 0, \tag{1a}$$

$$u(x,0) = \phi(x), \tag{1b}$$

$$\begin{gathered} B_0 u = u_x(0,t) - k_0 u(0,t) = 0, \\ B_1 u = u_x(1,t) + k_1 u(1,t) = 0, \end{gathered} \tag{1c}$$

$$u\left(x_j^+\right) = u\left(x_j^-\right), \quad j = 1,\dots,m, \tag{1d}$$

$$B_{x_j} u = [u_x]_{x_j} = v_j(t), \quad j = 1,2,\dots,m, \tag{1e}$$

where for a function $f(\cdot)$ and a point $x \in [0,1]$ we define

$$[f]_x = f\left(x^+\right) - f\left(x^-\right), \tag{2}$$

with x^- and x^+ denoting, respectively, the usual meaning of left and right hand limits to x. The initial condition is specified in (1b) with $\phi(x) \in L^2(0,1)$. It is assumed that, in system ((1a), (1b), (1c), (1d), and (1e)), we can control the heat flow at the points x_j for $j = 1,\dots,m$; that is,

$$v_j(t) = [u_x]_{x_j} = u_x\left(x_j^+, t\right) - u_x\left(x_j^-, t\right). \tag{3}$$

Note that, in ((1a), (1b), (1c), (1d), and (1e)), $B_{x_j}, x_j \in (0,1)$, represents the pointwise control located in the domain.

The space of weak solutions to system ((1a), (1b), (1c), (1d), and (1e)) is chosen to be $H^1(0,1)$. Note that system ((1a), (1b), (1c), (1d), and (1e)) is exponentially stable in $H^1(0,1)$ if B_0 and B_1 are chosen such that $k_0 > 0$ and $k_1 > 0$ [19].

Denote a set of reference signals corresponding to the control support points of in-domain actuation by $\{u_i^D(x_i,t)\}_{i=1}^m$, where $u_i^D(x_i,t) \in C^\infty$, $i = 1,\dots,m$, for all $t \in (0,T)$ and $T < \infty$. Let $e_i(t) = u(x_i,t) - u_i^D(x_i,t)$ be the regulation errors. Let $e(t) = \{e_i(t)\}_{i=1}^m$ and $v(t) = \{v_i(t)\}_{i=1}^m$.

Problem 1. The considered regulation problem for set-point control is to find a dynamic control $v(t)$ such that the regulation error satisfies $e(t) \to 0$ as $t \to \infty$.

Note that although the model under the form ((1a), (1b), (1c), (1d), and (1e)) allows deducing easily the zero-dynamics, it is not convenient for motion planning and, in particular, for establishing the input-output map, which is essential for feedforward control design. For this reason, we introduce an equivalent formulation of the in-domain control problem described in ((1a), (1b), (1c), (1d), and (1e)) by replacing the jump conditions in (1e) by pointwise controls as source terms. The resulting system will be of the following form:

$$u_t(x,t) - u_{xx}(x,t) = \sum_{j=1}^{m} \delta\left(x - x_j\right) \alpha_j(t), \quad 0 < x < 1,\ t > 0, \tag{4a}$$

$$u(x,0) = \phi(x), \tag{4b}$$

$$\begin{gathered} B_0 u = u_x(0,t) - k_0 u(0,t) = 0, \\ B_1 u = u_x(1,t) + k_1 u(1,t) = 0, \end{gathered} \tag{4c}$$

where $\delta(x - x_j)$ is the Dirac delta function supported at the point x_j, denoting the position of control support, and $\alpha_j : t \mapsto \mathbb{R}$, $j = 1,\dots,m$, are the in-domain control signals.

Lemma 2. *Considering weak solutions in $H^1(H^1(0,1),[0,T])$, $T < \infty$, system ((1a), (1b), (1c), (1d), and (1e)) and system ((4a), (4b), and (4c)) are equivalent if*

$$\alpha_j(t) = -v_j(t) = -[u_x]_{x_j}, \quad j = 1,\dots,m. \tag{5}$$

Proof. The proof follows the idea presented in [24]. Indeed, it suffices to prove "system ((1a), (1b), (1c), (1d), and (1e)) $\Rightarrow$ system ((4a), (4b), and (4c))." Let $X = L^2(0,1)$ be a Hilbert space equipped with the inner product $\langle v, w\rangle = \int_0^1 v(x)w(x)\,dx$, for any $v, w \in X$. Let the operator A be defined by $Av = v_{xx}$, with domain $\mathscr{D}(A) = \{v \in H^2(0,1); B_0 v = B_1 v = 0\}$. It is easy to see that A^*, the adjoint of A, is equal to A. Let $\widetilde{A}$ be an extension of A with the domain $\mathscr{D}(\widetilde{A}) = \{v \in X;\ v \in H^2(\bigcup_{i=0}^m (x_i, x_{i+1})), B_0 v = B_1 v = 0, v(x_j^+) = v(x_j^-), j = 1,\dots,m\}$. Let $v \in \mathscr{D}(\widetilde{A})$,

$w \in \mathscr{D}(A^*) = \mathscr{D}(A)$. Using integration by parts we obtain that

$$\left\langle \widetilde{A}v, w \right\rangle = \langle v, Av \rangle + \sum_{j=1}^{m} \left(v_x\left(x_j^-\right) - v_x\left(x_j^+\right) \right) w\left(x_j\right). \quad (6)$$

Let $X_{-1} = (\mathscr{D}(A^*))'$, the dual space of $\mathscr{D}(A)$. We need to define another extension for A. Let $\widehat{A} : H^1(0,1) \to X_{-1}$ be defined by

$$\left\langle \widehat{A}v, w \right\rangle = \langle v, A^* w \rangle \quad \forall w \in \mathscr{D}\left(A^*\right), \quad (7)$$

with $\mathscr{D}(\widehat{A}) = H^1(0,1)$. Note that $\delta(\cdot - x_j)$ is not in X, but in the larger space X_{-1}. It follows from (6), (7), and $A = A^*$ that

$$\widetilde{A}v = \widehat{A}v + \sum_{j=1}^{m} \left(v_x\left(x_j^-\right) - v_x\left(x_j^+\right) \right) \delta\left(x - x_j\right) \quad (8)$$

in X_{-1}. If v satisfies system ((1a), (1b), (1c), (1d), and (1e)), then $\dot{v}(t) = \widetilde{A}v(t)$, which yields, considering (8), $\dot{v}(t) = \widehat{A}v + \sum_{j=1}^{m}(v_x(x_j^-) - v_x(x_j^+))\delta(x - x_j)$. Finally, we can see that system ((1a), (1b), (1c), (1d), and (1e)) becomes system ((4a), (4b), and (4c)) with $\alpha_j(t) = -v_j(t) = -[z_x]_{x_j}$, $j = 1, \ldots, m$, where we look for generalized solutions $v(\cdot, t) \in \mathscr{D}(\widehat{A}) = H^1(0,1)$ such that (8) is true in X_{-1}. □

To establish in-domain control at every actuation point, we will proceed in the way of *parallel connection*; that is, for every $x_j \in (0,1)$, consider the following two systems:

$$u_t(x,t) - u_{xx}(x,t) = 0, \quad x \in \left(0, x_j\right) \cup \left(x_j, 1\right), \; t > 0, \quad (9a)$$

$$u(x,0) = \phi_j(x), \quad (9b)$$

$$B_0 u = u_x(0,t) - k_0 u(0,t) = 0, \quad B_1 u = u_x(1,t) + k_1 u(1,t) = 0, \quad (9c)$$

$$u\left(x_j^+\right) = u\left(x_j^-\right), \quad (9d)$$

$$B_{x_j} u = [u_x]_{x_j} = w_j(t), \quad (9e)$$

$$u_t(x,t) - u_{xx}(x,t) = \delta\left(x - x_j\right) \beta_j(t), \quad 0 < x < 1, \; t > 0, \quad (10a)$$

$$u(x,0) = \phi_j(x), \quad (10b)$$

$$B_0 u = u_x(0,t) - k_0 u(0,t) = 0, \quad B_1 u = u_x(1,t) + k_1 u(1,t) = 0, \quad (10c)$$

with $\sum_{j=1}^{m} \phi_j(x) = \phi(x)$. Similarly, systems ((9a), (9b), (9c), (9d), and (9e)) and ((10a), (10b), and (10c)) are equivalent provided $u \in H^1(0,1)$ and $\beta_j = -w_j = -[u_x]_{x_j}$. Let $\alpha_j = -v_j = \beta_j = -w_j = -[u_x^j]_{x_j}$ for all $j = 1, 2, \ldots, m$, where u^j denotes the solution to system ((10a), (10b), and (10c)). It follows that $u(x,t) = \sum_{j=1}^{m} u^j(x,t)$ is a solution to system ((4a), (4b), and (4c)). Moreover,

$$[u_x]_{x_i} = \sum_{j=1}^{m} \left[u_x^j\right]_{x_i} = \left[u_x^i\right]_{x_i} = v_i, \quad (11)$$

for all $i = 1, 2, \ldots, m$. Hence $u(x,t) = \sum_{j=1}^{m} u^j(x,t)$ is a solution to system ((1a), (1b), (1c), (1d), and (1e)). Therefore, throughout this paper, we assume $\alpha_j = -v_j = \beta_j = -w_j = -[u_x^j]_{x_j}$ for all $j = 1, 2, \ldots, m$. Due to the equivalences of systems ((1a), (1b), (1c), (1d), and (1e)) and ((4a), (4b), and (4c)), and systems ((9a), (9b), (9c), (9d), and (9e)) and ((10a), (10b), and (10c)), we may consider ((4a), (4b), and (4c)) and system ((9a), (9b), (9c), (9d), and (9e)) in the following parts.

It is worth noting that, in general, the internal pointwise control of the heat equation cannot be allocated at arbitrary points. In particular, the approximate controllability may be lost if the support of the control is located on a nodal set of eigenfunctions (see, e.g., [25, 26]). Therefore, it is important to ensure that the controllability property of the system is insensitive to the location of control support, so that the expected performance can be achieved. Indeed, the loss of controllability will not happen to the considered system if the parameters k_0 and k_1 are chosen appropriately. Precisely, we call a point $x \in (0,1)$ strategic if it is not located on a nodal set of eigenfunctions of the corresponding PDE [25, 26]. We have then the following.

Proposition 3. *For system (4a) with the boundary conditions (4c), all the points $x \in (0,1)$ are strategic for any $k_0 > 0$ and $k_1 > 0$. Furthermore, the approximate controllability of such a system is insensitive to the position of control support for all $x \in (k_1/(k_0 + k_1), 1)$.*

Proof. The eigenfunctions of system (4a) with the boundary conditions (4c) are given by [27]

$$\psi_n(x) = \cos\left(\mu_n x\right) + \frac{k_0}{\mu_n} \sin\left(\mu_n x\right), \quad n = 1, 2, \ldots, \quad (12)$$

where μ_n are positive roots of the transcendental equation

$$\tan(\mu) = \frac{\mu\left(k_0 + k_1\right)}{\left(\mu^2 - k_0 k_1\right)}, \quad k_0 > 0, \; k_1 > 0. \quad (13)$$

The solution of $\psi_n(x) = 0$ is given by

$$\tan\left(\mu_n\right) = -\frac{\mu_n}{x k_0}, \quad \forall x \in (0,1), \; k_0 > 0. \quad (14)$$

Obviously, for any fixed $x \in (0,1)$, the solution of (14) cannot be that of (13) for all $n = 1, 2, \ldots$. Therefore, all the points $x \in (0,1)$ are strategic for any $k_0 > 0$ and $k_1 > 0$. Furthermore, for μ to verify simultaneously (13) and (14), it must hold

$$\mu^2 = k_0 k_1 - x k_0\left(k_0 + k_1\right). \quad (15)$$

Therefore, μ is real-valued if and only if $x \le k_1/(k_0 + k_1) < 1$. □

Proposition 3 implies that the approximate controllability of the considered system could hold in almost the whole domain by choosing $k_0 \gg k_1$.

3. Control Design and Implementation

In the framework of zero-dynamics inverse, the in-domain control is derived from the so-called forced zero-dynamics, or zero-dynamics for short. To work with the *parallel connected* system ((9a), (9b), (9c), (9d), and (9e)), we first split the reference signal as

$$u^D(x,t)=\sum_{j=1}^{m}\gamma_j\left(x,x_j\right)u_j^d\left(x_j,t\right),\qquad u_j^d\left(x_j,t\right)\in H^1(0,T),\ \text{for any } T<\infty, \tag{16}$$

where $\gamma_j(x,x_j)$ will be determined in Theorem 7 (see Section 4). Denoting by $\varepsilon^j(t) = u^j(x_j,t)-u_j^d(x_j,t)$ the regulation error corresponding to system ((9a), (9b), (9c), (9d), and (9e)), the zero-dynamics can be obtained by replacing the input constraints in (9e) by the requirement that the regulation errors vanish identically; that is, $\varepsilon^j(t)=0$. Thus, we obtain for a fixed index j

$$\xi_t(x,t)=\xi_{xx}(x,t),\qquad x\in\left(0,x_j\right)\cup\left(x_j,1\right),\ t>0, \tag{17a}$$

$$\xi(x,0)=0, \tag{17b}$$

$$\begin{aligned}&\xi_x(0,t)-k_0\xi(0,t)=0,\\&\xi_x(1,t)+k_1\xi(1,t)=0,\end{aligned} \tag{17c}$$

$$\xi\left(x_j,t\right)=u_j^d\left(x_j,t\right), \tag{17d}$$

where $u_j^d(x_j,t)\in H^1(0,T)$ for any $T<\infty$. It should be noticed that, by construction, we can always choose suitable initial data of $u^d(x,0)$ so that the zero-dynamics will have a homogenous initial condition. This will significantly facilitate control design. Then, we can get from (9e)

$$w_j=\left[u_x^j\right]_{x_j}=\left[\xi_x^j\right]_{x_j}, \tag{18}$$

which shows that the in-domain control for system ((9a), (9b), (9c), (9d), and (9e)) can be derived from the solution of the zero-dynamics. Hence, ((17a), (17b), (17c), and (17d)) and (18) form a dynamic control scheme. This is indeed the basic idea of zero-dynamic inverse design. The convergence of regulation errors with ZDI-based control is given in following theorem.

Theorem 4. *The regulation error corresponding to system ((9a), (9b), (9c), (9d), and (9e)), $\varepsilon^j(t)=u^j(x_j,t)-u_j^d(x_j,t)$, tends to 0 as t tends to ∞ for any $u_j^d(x_j,t)\in H^1(0,T)$ and $T<\infty$.*

The proof of Theorem 4 can follow the development presented in Section III of [20] for the case of a heat equation with one in-domain actuator and hence, it is omitted. Note that a key fact used in the proof of this theorem is that the system given in ((9a), (9b), (9c), (9d), and (9e)) without interior control is exponentially stable for any initial data $\phi_j(x)\in L^2(0,1)$ if $k_0>0$ and $k_1>0$.

To implement the dynamic control scheme composed of ((17a), (17b), (17c), and (17d)) and (18), we resort to the technique of flat systems [11, 13, 28, 29]. In particular, we apply a standard procedure of Laplace transform-based method to find the solution to ((17a), (17b), (17c), and (17d)). Henceforth, we denote by $\widehat{f}(x,s)$ the Laplace transform of a function $f(x,t)$ with respect to the time variable. Since ((17a), (17b), (17c), and (17d)) has a homogeneous initial condition, then for fixed $x_j\in(0,1)$, the transformed equations of ((17a), (17b), (17c), and (17d)) in the Laplace domain read as

$$s\widehat{\xi}(x,s)=\widehat{\xi}_{xx}(x,s),\qquad x\in\left(0,x_j\right)\cup\left(x_j,1\right),\ s\in\mathbb{C}, \tag{19a}$$

$$\begin{aligned}&\widehat{\xi}_x(0,s)-k_0\widehat{\xi}(0,s)=0,\\&\widehat{\xi}_x(1,s)+k_1\widehat{\xi}(1,s)=0,\end{aligned} \tag{19b}$$

$$\widehat{\xi}\left(x_j,s\right)=\widehat{u}_j^d\left(x_j,s\right). \tag{19c}$$

We divide ((19a), (19b), and (19c)) into two subsystems, that is, for fixed $x_j\in(0,1)$, considering

$$s\widehat{\xi}(x,s)=\widehat{\xi}_{xx}(x,s),\qquad 0<x<x_j,\ s\in\mathbb{C}, \tag{20a}$$

$$\widehat{\xi}_x(0,s)-k_0\widehat{\xi}(0,s)=0, \tag{20b}$$

$$\widehat{\xi}\left(x_j,s\right)=\widehat{u}_j^d\left(x_j,s\right), \tag{20c}$$

$$s\widehat{\xi}(x,s)=\widehat{\xi}_{xx}(x,s),\qquad x_j<x<1,\ s\in\mathbb{C}, \tag{21a}$$

$$\widehat{\xi}_x(1,s)+k_1\widehat{\xi}(x,s)=0, \tag{21b}$$

$$\widehat{\xi}\left(x_j,s\right)=\widehat{u}_j^d\left(x_j,s\right). \tag{21c}$$

Let $\widehat{\xi}_-^j(x,s)$ and $\widehat{\xi}_+^j(x,s)$ be the general solutions to ((20a), (20b), and (20c)) and ((21a), (21b), and (21c)), respectively, and denote their inverse Laplace transforms by $\xi_-^j(x,t)$ and $\xi_+^j(x,t)$. The solution to ((17a), (17b), (17c), and (17d)) can be written as

$$\xi^j(x,t)=\xi_-^j(x,t)\,\chi_{\{(0,x_j)\}}+\xi_+^j(x,t)\,\chi_{\{[x_j,1)\}}, \tag{22}$$

where

$$\chi(x)_{\{\Omega_j\}}=\begin{cases}1, & x\in\Omega_j\subseteq(0,1);\\ 0, & \text{otherwise.}\end{cases} \tag{23}$$

Then at each point $x_i\in(0,1)$, by (18) and the argument of "*parallel connection*" (see Section 2), we have $[u_x]_{x_i}=\sum_{j=1}^{m}[u_x^j]_{x_i}=[u_x^i]_{x_i}=[\xi_x^i]_{x_i}$, $i=1,\ldots,m$. Hence the in-domain control signals of system ((1a), (1b), (1c), (1d), and (1e)) can be computed by

$$v_i = [u_x]_{x_i} = [\xi^i_x]_{x_i}, \quad i = 1, \ldots, m. \tag{24}$$

In the following steps, we present the computation of the solution to system ((17a), (17b), (17c), and (17d)), ξ^j. Issues related to the generation reference trajectory $u^D(x,t)$ for system ((1a), (1b), (1c), (1d), and (1e)) will be addressed in Section 4.

Note that $\hat{\xi}^j_-(x,s)$ and $\hat{\xi}^j_+(x,s)$, the general solutions to ((20a), (20b), and (20c)) and ((21a), (21b), and (21c)), are given by

$$\begin{aligned}\hat{\xi}^j_-(x,s) &= C_1\phi_1(x,s) + C_2\phi_2(x,s),\\ \hat{\xi}^j_+(x,s) &= C_3\phi_1(x,s) + C_4\phi_2(x,s),\end{aligned} \tag{25}$$

with

$$\begin{aligned}\phi_1(x,s) &= \frac{\sinh(\sqrt{s}x)}{\sqrt{s}},\\ \phi_2(x,s) &= \cosh(\sqrt{s}x).\end{aligned} \tag{26}$$

We obtain by applying (20b) and (20c)

$$\begin{aligned}C_1\phi_1(x_j,s) + C_2\phi_2(x_j,s) &= \hat{u}^d_j(x_j,s),\\ C_1 - k_0C_2 &= 0,\end{aligned} \tag{27}$$

which can be written as

$$\begin{pmatrix}\phi_1(x_j,s) & \phi_2(x_j,s)\\ 1 & -k_0\end{pmatrix}\begin{pmatrix}C_1\\ C_2\end{pmatrix} = \begin{pmatrix}\hat{u}^d_j(x_j,s)\\ 0\end{pmatrix}. \tag{28}$$

Let

$$R^j_- = \begin{pmatrix}\phi_1(x_j,s) & \phi_2(x_j,s)\\ 1 & -k_0\end{pmatrix}, \tag{29}$$

$$\hat{u}^d_j(x_j,s) = -\det(R^j_-)\,\hat{y}^j_-(x_j,s). \tag{30}$$

We obtain

$$\begin{pmatrix}C_1\\ C_2\end{pmatrix} = \frac{\operatorname{adj}(R^j_-)}{\det(R^j_-)}\begin{pmatrix}\hat{u}^d_j(x_j,s)\\ 0\end{pmatrix} = \begin{pmatrix}k_0\hat{y}^j_-(x_j,s)\\ \hat{y}^j_-(x_j,s)\end{pmatrix}. \tag{31}$$

Therefore, the solution to ((20a), (20b), and (20c)) can be expressed as

$$\hat{\xi}^j_-(x,s) = (k_0\phi_1(x) + \phi_2(x))\,\hat{y}^j_-(x_j,s). \tag{32}$$

We may proceed in the same way to deal with ((21a), (21b), and (21c)). Indeed, letting

$$\begin{aligned}R^j_+ &= \begin{pmatrix}\phi_1(x_j,s) & \phi_2(x_j,s)\\ \phi_2(1,s) + k_1\phi_1(1,s) & s\phi_1(1,s) + k_1\phi_2(1,s)\end{pmatrix},\end{aligned} \tag{33}$$

$$\hat{u}^d_j(x_j,s) = \det(R^j_+)\,\hat{y}^j_+(x_j,s), \tag{34}$$

we get from ((21a), (21b), and (21c))

$$\begin{pmatrix}C_3\\ C_4\end{pmatrix} = \begin{pmatrix}(s\phi_1(1,s) + k_1\phi_2(1,s))\,\hat{y}^j_+(x_j,s)\\ -(\phi_2(1,s) + k_1\phi_1(1,s))\,\hat{y}^j_+(x_j,s)\end{pmatrix}, \tag{35}$$

$$\begin{aligned}\hat{\xi}^j_+(x,s) &= ((s\phi_1(1,s) + k_1\phi_2(1,s))\,\phi_1(x)\\ &\quad + (\phi_2(1,s) + k_1\phi_1(1,s))\,\phi_2(x))\,\hat{y}^j_+(x_j,s).\end{aligned} \tag{36}$$

Applying the results from [30, 31], which are based on module theory, to (30) and (34), we may choose $\hat{y}_j(x_j,s)$ as a basic output such that

$$\begin{aligned}\hat{y}^j_+(x_j,s) &= -\det(R^j_-)\,\hat{y}_j(x_j,s),\\ \hat{y}^j_-(x_j,s) &= \det(R^j_+)\,\hat{y}_j(x_j,s).\end{aligned} \tag{37}$$

It should be noticed that the concept of basic outputs (or flat outputs) plays a central role in flat system theory, because the system trajectory and the input can be directly computed from a basic output and its time derivatives [13, 21].

Using the property of hyperbolic functions, we obtain from (32) and (36) that

$$\begin{aligned}\hat{\xi}^j_-(x,s) &= \left(k_1\frac{\sinh(\sqrt{s}x_j - \sqrt{s})}{\sqrt{s}} - \cosh(\sqrt{s}x_j - \sqrt{s})\right)\\ &\quad\cdot\left(k_0\frac{\sinh(\sqrt{s}x)}{\sqrt{s}} + \cosh(\sqrt{s}x)\right)\hat{y}_j(x_j,s),\\ \hat{\xi}^j_+(x,s) &= \left(k_1\frac{\sinh(\sqrt{s}x - \sqrt{s})}{\sqrt{s}} - \cosh(\sqrt{s}x - \sqrt{s})\right)\\ &\quad\cdot\left(k_0\frac{\sinh(\sqrt{s}x_j)}{\sqrt{s}} + \cosh(\sqrt{s}x_j)\right)\hat{y}_j(x_j,s).\end{aligned} \tag{38}$$

Note that

$$\hat{\xi}^j(x,s) = \hat{\xi}^j_-(x,s)\,\chi_{\{(0,x_j)\}} + \hat{\xi}^j_+(x,s)\,\chi_{\{[x_j,1)\}} \tag{39}$$

is a solution to ((19a), (19b), and (19c)). Using the fact

$$\begin{aligned}\sinh x &= \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!},\\ \cosh x &= \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!},\end{aligned} \tag{40}$$

we obtain the time-domain solution to ((17a), (17b), (17c), and (17d)), which is given by

$$\begin{aligned}\xi^j(x,t) = \Bigg[\Bigg(& k_0k_1\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{x^{2k+1}\left(x_j-1\right)^{2(n-k)+1}}{(2k+1)!\,[2(n-k)+1]!}y_j^{(n)} \\ &- k_0\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{x^{2k+1}\left(x_j-1\right)^{2(n-k)}}{(2k+1)!\,[2(n-k)]!}y_j^{(n)} \\ &+ k_1\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{x^{2k}\left(x_j-1\right)^{2(n-k)+1}}{(2k)!\,[2(n-k)+1]!}y_j^{(n)} \\ &- \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{x^{2k}\left(x_j-1\right)^{2(n-k)}}{(2k)!\,[2(n-k)]!}y_j^{(n)}\Bigg)\chi_{\{(0,x_j)\}} \\ &+ \Bigg(k_0k_1\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{x_j^{2k+1}(x-1)^{2(n-k)+1}}{(2k+1)!\,[2(n-k)+1]!}y_j^{(n)} \\ &- k_0\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{x_j^{2k+1}(x-1)^{2(n-k)}}{(2k+1)!\,[2(n-k)]!}y_j^{(n)} \\ &+ k_1\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{x_j^{2k}(x-1)^{2(n-k)+1}}{(2k)!\,[2(n-k)+1]!}y_j^{(n)} \\ &- \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{x_j^{2k}(x-1)^{2(n-k)}}{(2k)!\,[2(n-k)]!}y_j^{(n)}\Bigg)\chi_{\{[x_j,1)\}}\Bigg].\end{aligned} \tag{41}$$

Furthermore, by a direct computation we get

$$\begin{aligned}\left[\hat{\xi}_x^j\right]_{x_j} &= \left[\left(\frac{k_0k_1}{\sqrt{s}}+\sqrt{s}\right)\sinh\left(\sqrt{s}\right) + \left(k_0+k_1\right)\cosh\left(\sqrt{s}\right)\right]\hat{y}_j\left(x_j,s\right) \\ &= \left(k_0k_1\sum_{n=0}^{\infty}\frac{s^n}{(2n+1)!} + \left(k_0+k_1\right)\sum_{n=0}^{\infty}\frac{s^n}{(2n)!} + \sum_{n=0}^{\infty}\frac{s^{n+1}}{(2n+1)!}\right)\hat{y}_j\left(x_j,s\right).\end{aligned} \tag{42}$$

It follows from (24) that, in time domain, the control is given by

$$\begin{aligned}v_j(t) &= \left[\xi_x^j\right]_{x_j} \\ &= k_0k_1\sum_{n=0}^{\infty}\frac{y_j^{(n)}\left(x_j,t\right)}{(2n+1)!} + \left(k_0+k_1\right)\sum_{n=0}^{\infty}\frac{y_j^{(n)}\left(x_j,t\right)}{(2n)!} + \sum_{n=0}^{\infty}\frac{y_j^{(n+1)}\left(x_j,t\right)}{(2n+1)!}.\end{aligned} \tag{43}$$

Finally, provided $u_j^d(x_j,t) = \xi^j(x_j,t)$, for $j = 1,\dots,m$, the reference trajectory $u^D(x,t)$ can be determined from (16) and (41).

4. Motion Planning

For control purpose, we have to choose appropriate reference trajectories, or equivalently the basic outputs. Denote now by $\overline{u}^D(x)$ the desired steady-state profile. Without loss of generality, we consider a set of basic outputs of the form

$$y_j(t) = \overline{y}\left(x_j\right)\varphi_j(t), \quad j = 1,\dots,m, \tag{44}$$

where $\varphi_j(t)$ is a smooth function evolving from 0 to 1. Motion planning amounts then to deriving $\overline{y}(x_j)$ from $\overline{u}^D(x)$ and to determining appropriate functions $\varphi_j(t)$, for $j = 1,\dots,m$.

To this aim and due to the equivalence of systems ((1a), (1b), (1c), (1d), and (1e)) and ((4a), (4b), and (4c)), we consider the steady-state heat equation corresponding to system ((4a), (4b), and (4c)):

$$\overline{u}_{xx}(x) = \sum_{j=1}^{m}\delta\left(x-x_j\right)\overline{\alpha}_j, \quad 0 < x < 1,\ t > 0, \tag{45a}$$

$$\begin{aligned}\overline{u}_x(0) - k_0\overline{u}(0) &= 0, \\ \overline{u}_x(1) + k_1\overline{u}(1) &= 0.\end{aligned} \tag{45b}$$

Based on the principle of superposition for linear systems, the solution to the steady-state heat equation ((45a) and (45b)) can be expressed as

$$\begin{aligned}\overline{u}(x) &= \int_0^1\sum_{j=1}^{m}G(x,\zeta)\,\delta\left(\zeta-x_j\right)\overline{\alpha}_j\,\mathrm{d}\zeta \\ &= \sum_{j=1}^{m}G\left(x,x_j\right)\overline{\alpha}_j,\end{aligned} \tag{46}$$

where $G(x,\zeta)$ is the Green's function corresponding to ((45a) and (45b)), which is of the form

$$G(x,\zeta) = \begin{cases}\dfrac{\left(k_1\zeta-k_1-1\right)\left(k_0x+1\right)}{k_0+k_1+k_0k_1}, & 0 \le x < \zeta; \\[2ex] \dfrac{\left(k_1x-k_1-1\right)\left(k_0\zeta+1\right)}{k_0+k_1+k_0k_1}, & \zeta \le x \le 1.\end{cases} \tag{47}$$

Indeed, it is easy to check that $G_{xx}(x,\zeta) = \delta(x-\zeta)$ and $G(x,\zeta)$ satisfies the boundary conditions, $G_x(0,\zeta) - k_0G(0,\zeta) = 0$ and $G_x(1,\zeta) + k_1G(1,\zeta) = 0$, the joint condition, $G(\zeta^+,\zeta) = G(\zeta^-,\zeta)$, and the jump condition, $[G_x(x,\zeta)]_\zeta = 1$.

Taking m distinguished points along the solution to ((45a) and (45b)), $\overline{u}(x_1), \ldots, \overline{u}(x_m)$, we get

$$\begin{pmatrix} \overline{u}(x_1) \\ \vdots \\ \overline{u}(x_m) \end{pmatrix} = \begin{pmatrix} G(x_1, x_1) & \cdots & G(x_1, x_m) \\ \vdots & \ddots & \vdots \\ G(x_m, x_1) & \cdots & G(x_m, x_m) \end{pmatrix} \begin{pmatrix} \overline{\alpha}_1 \\ \vdots \\ \overline{\alpha}_m \end{pmatrix}. \quad (48)$$

Note that, in (48), the matrix formed by the Green's function defined an input-output map in steady-state, which is also called the *influence matrix*.

Lemma 5. *The influence matrix chosen as in (48) is invertible. Thus,*

$$\begin{pmatrix} \overline{\alpha}_1 \\ \vdots \\ \overline{\alpha}_m \end{pmatrix} = \begin{pmatrix} G(x_1, x_1) & \cdots & G(x_1, x_m) \\ \vdots & \ddots & \vdots \\ G(x_m, x_1) & \cdots & G(x_m, x_m) \end{pmatrix}^{-1} \begin{pmatrix} \overline{u}(x_1) \\ \vdots \\ \overline{u}(x_m) \end{pmatrix}. \quad (49)$$

Proof. For $m = 1$, since $k_0 > 0$, $k_1 > 0$, and $x_1 \in (0, 1)$, it follows that $G(x_1, x_1) = (k_1 x_1 - k_1 - 1)(k_0 x_1 + 1)/(k_0 + k_1 + k_0 k_1) < 0$. Hence it is invertible. We prove the claim for $m > 1$ by contradiction. Suppose that the influence matrix is not invertible; then it is of rank $m - 1$ or less. Without loss of generality, we may assume that, for some $x_n > x_i$ with $i = 1, \ldots, n-1$, there exist $n - 1$ constants $l_1, l_2, \ldots, l_{n-1}$ such that

$$G(x_j, x_n) = \sum_{i=1}^{n-1} l_i G(x_1, x_i), \quad j = 1, \ldots, n, \quad (50)$$

where $1 < n \leq m$ and $\sum_{i=1}^{n-1} l_i^2 > 0$. Let

$$G(x) = G(x, x_n), \qquad F(x) = \sum_{i=1}^{n-1} l_i G(x, x_i). \quad (51)$$

Equation (50) shows that $F(x) = G(x)$ at every boundary point of $[x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n]$. Note that $F(x)$ is a linear function in $[x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n]$ and that $G(x) = (k_1 x_n - k_1 - 1)(k_0 x + 1)/(k_0 + k_1 + k_0 k_1)$ in $[x_1, x_n]$; that is, $G(x)$ is a linear function in $[x_1, x_n]$. Hence $F(x) \equiv G(x)$ in $[x_1, x_n]$.

By $F(x_1) = G(x_1)$, we get

$$k_1 x_n - k_1 - 1 = \sum_{i=1}^{n-1} l_i (k_1 x_i - k_1 - 1). \quad (52)$$

By $F(x_n) = G(x_n)$, we get

$$k_0 x_n + 1 = \sum_{i=1}^{n-1} l_i (k_0 x_i + 1). \quad (53)$$

Therefore

$$\sum_{i=1}^{n-1} l_i = 1. \quad (54)$$

By $F_x(x_1^+) = G_x(x_1^+)$ and $F_x(x_n^-) = G_x(x_n^-)$, we get

$$\begin{aligned} k_0 (k_1 x_n - k_1 - 1) &= k_0 k_1 \sum_{i=1}^{n-1} l_i x_i - k_0 (k_1 + 1) \sum_{i=2}^{n-1} l_i \\ &\quad + l_1 k_1 = k_0 k_1 \sum_{i=1}^{n-1} l_i x_i + k_1 \sum_{i=1}^{n-1} l_i. \end{aligned} \quad (55)$$

It follows that $\sum_{i=2}^{n-1} l_i = 0$, which yields, considering (54), $l_1 = 1$. By $F_x(x_2^+) = G_x(x_2) = F_x(x_2^-)$, we deduce

$$\begin{aligned} & l_1 k_1 (k_0 x_1 + 1) + l_2 k_1 (k_0 x_2 + 1) \\ &\quad + k_0 \sum_{i=3}^{n-1} l_i (k_1 x_i - k_1 - 1) \\ &= l_1 k_1 (k_0 x_1 + 1) + k_0 \sum_{i=2}^{n-1} l_i (k_1 x_i - k_1 - 1), \end{aligned} \quad (56)$$

which gives $l_2 = 0$. Similarly, by $F_x(x_j^+) = G_x(x_j) = F_x(x_j^-)$ $(j = 3, 4, \ldots, n-2)$, we obtain $l_3 = l_4 = \cdots = l_{n-2} = 0$. Hence $l_{n-1} = 0$. Then we deduce from (55) that

$$k_0 (k_1 x_n - k_1 - 1) = k_1 (k_0 x_1 + 1). \quad (57)$$

It follows from (53) that $k_0 x_1 + 1 = k_0 x_n + 1$. We conclude then by (57) that $k_0 + k_1 + k_0 k_1 = 0$, which is a contradiction to $k_0 > 0$ and $k_1 > 0$. □

In steady-state, we can obtain from (43) that

$$\overline{v}_j = (k_0 k_1 + k_0 + k_1) \overline{y}(x_j) = -\overline{\alpha}_j. \quad (58)$$

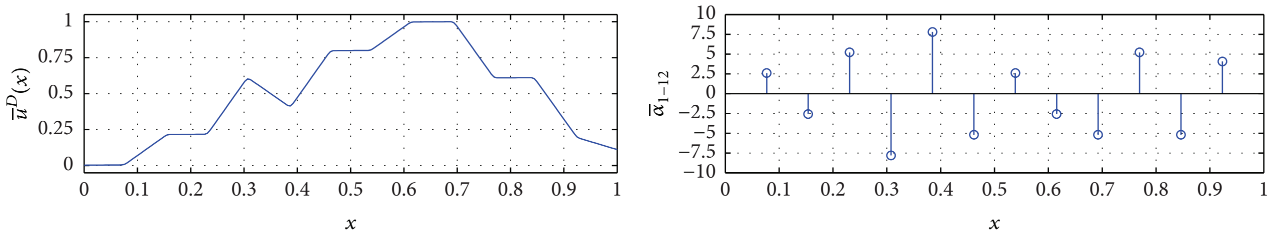
Finally, $\overline{y}(x_j)$ can be computed by (49) and (58) for a given $\overline{u}^D(x)$.

It is worth noting that (49) provides a simple and straightforward way to compute the static control from the prescribed steady-state profile. Indeed, a direct computation can show that applying (49) will result in the same static control obtained in [20] where a *serially connected* model is used.

To ensure the convergence of (41) and (43), we choose the following smooth function as $\varphi_j(t)$:

$$\varphi_j(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ \dfrac{\int_0^t \exp(-1/(\tau(1-\tau)))^{\varepsilon} \, d\tau}{\int_0^T \exp(-1/(\tau(1-\tau)))^{\varepsilon} \, d\tau}, & \text{if } t \in (0, T) \\ 1, & \text{if } t \geq T \end{cases} \quad (59)$$

which is known as Gevrey function of order $\sigma = 1 + 1/\varepsilon$, $\varepsilon > 0$ (see, e.g., [13]).



(a) (b)

Figure 1: Controller characteristics: (a) desired profile; (b) static control signals.

Lemma 6. *If the basic outputs $\varphi_j(t)$, $j = 1, \ldots, m$, are chosen as Gevrey functions of order $1 < \sigma < 2$, then the infinite series (41) and (43) are convergent.*

The claim of Lemma 6 can be proved by following a standard procedure using the bounds of Gevrey functions (see, e.g., [13]):

$$\left|\varphi^{(k+1)}(t)\right| \leq M\frac{(k!)^{\sigma}}{K^{k}}, \qquad \exists K, M > 0,\ \forall k \in \mathbb{Z}_{\geq 0},\ \forall t \in \left[t_0, T\right]. \tag{60}$$

Therefore, the details of proof are omitted.

Theorem 7. *Assume $k_0 > 0$ and $k_1 > 0$. Let the basic outputs $\varphi_j(t)$, $j = 1, \ldots, m$, be chosen as (59) with an order $1 < \sigma < 2$. Let the reference trajectory of system ((1a), (1b), (1c), (1d), and (1e)) be given by (16) with*

$$\gamma_j\left(x, x_j\right) = -\frac{\left(k_0k_1 + k_0 + k_1\right)G\left(x, x_j\right)}{\left(k_0x_j + 1\right)\left(k_0\left(x_j - 1\right) - 1\right)}, \qquad j = 1, \ldots, m, \tag{61}$$

where $G(x, \zeta)$ is the Green's function defined in (47). Then the regulation error of system ((1a), (1b), (1c), (1d), and (1e)) with the control given in (43) tends to zero; that is, $e_i(t) = u(x_i, t) - u_i^D(x_i, t) \to 0$ as $t \to \infty$, for $i = 1, 2, \ldots, m$.

Proof. By a direct computation we have

$$\begin{aligned}
\left|e_i(t)\right| &= \left|(x_i, t) - u_i^D(x_i, t)\right| = \Bigg|u(x_i, t) + \sum_{j=1}^{m}\frac{\left(k_0k_1 + k_0 + k_1\right)G\left(x_i, x_j\right)u_j^d\left(x_j, t\right)}{\left(k_0x_j + 1\right)\left(k_0\left(x_j - 1\right) - 1\right)}\Bigg| \\
&= \Bigg|u(x_i, t) + \sum_{j=1}^{m}\frac{\left(k_0k_1 + k_0 + k_1\right)G\left(x_i, x_j\right)\xi^j\left(x_j, t\right)}{\left(k_0x_j + 1\right)\left(k_0\left(x_j - 1\right) - 1\right)}\Bigg| \\
&\leq \left|u(x_i, t) - \overline{u}(x_i)\right| + \Bigg|\overline{u}(x_i) + \left(k_0k_1 + k_0 + k_1\right)\cdot\sum_{j=1}^{m}G\left(x_i, x_j\right)\overline{y}\left(x_j\right)\Bigg| + \Bigg|\left(k_0k_1 + k_0 + k_1\right) \\
&\quad\cdot\sum_{j=1}^{m}\frac{G\left(x_i, x_j\right)\xi^j\left(x_j, t\right)}{\left(k_0x_j + 1\right)\left(k_0\left(x_j - 1\right) - 1\right)} - \left(k_0k_1 + k_0 + k_1\right)\sum_{j=1}^{m}G\left(x_i, x_j\right)\overline{y}\left(x_j\right)\Bigg|.
\end{aligned} \tag{62}$$

By (58) and (46), it follows $\overline{u}(x_i) = -(k_0k_1 + k_0 + k_1)\sum_{j=1}^{m} G(x_i, x_j)\overline{y}(x_j)$. Based on (41), (44), and the property of $\varphi_j(t)$ we have

$$\frac{\xi^j\left(x_j, t\right)}{\left(k_0x_j + 1\right)\left(k_0\left(x_j - 1\right) - 1\right)} \longrightarrow \overline{y}\left(x_j\right) \qquad \text{as } t \longrightarrow \infty. \tag{63}$$

Note that $u(x_i, t) \to \overline{u}(x_i)$ as $t \to \infty$. Therefore $|e_i(t)| \to 0$ as $t \to \infty$. □

Remark 8. For any $x \in (0, 1)$, replacing x_i by x in the proof of Theorem 7, we can get $|u(x, t) - u^D(x, t)| \to 0$ as $t \to \infty$, which shows that the solution $u(x, t)$ of system ((1a), (1b), (1c), (1d), and (1e)) converges to the reference trajectory $u^D(x, t)$ at every point $x \in (0, 1)$.

5. Simulation Study

In the simulation, we implement system ((4a), (4b), and (4c)) with 12 actuators evenly distributed in the domain at the spot points $\{1/13, 2/13, \ldots, 12/13\}$. The numerical implementation is based on a PDE solver, pdepe, in Matlab PDE Toolbox. In numerical simulation, 200 points in space and 100 points in time are used for the region $[0, 1] \times [0, 0.5]$. The basic outputs $\varphi_j(t)$ used in the simulation are Gevrey functions of the same order. In order to meet the convergence condition given in Lemma 6, the parameter of the Gevrey function is set to $\varepsilon = 1.1$. The feedback boundary control gains are chosen as $k_0 = 10$ and $k_1 = 1$. A perturbation of the form $u(x, 0) = \cos(\pi x)$ is applied at $t = 0$ in the simulation.

The desired steady-state temperature distribution is a piecewise linear curve, depicted in Figure 1(a), which is

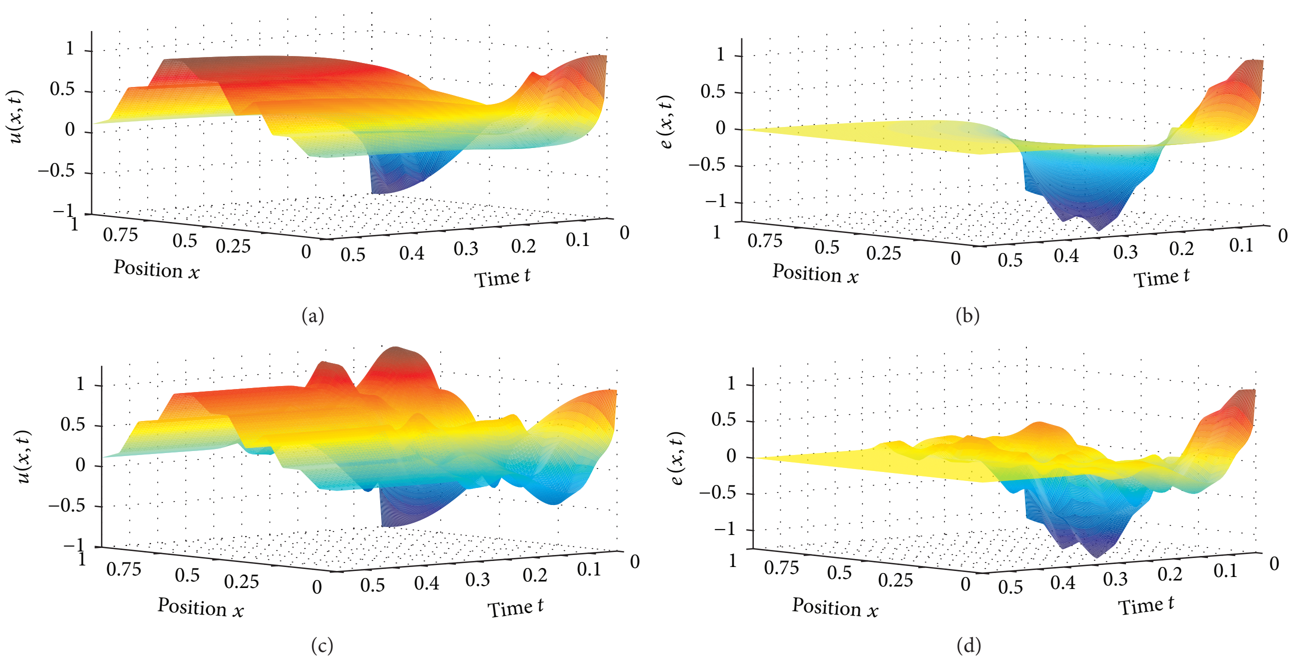


FIGURE 2: Evolution of temperature distribution: (a) response with static control; (b) regulation error with static control; (c) response with dynamic control; (d) regulation error with dynamic control.

a solution to ((45a) and (45b)). The corresponding static controls, $\overline{\alpha}_1, \dots, \overline{\alpha}_{12}$, are shown in Figure 1(b). Note that the dynamic control signals, $\alpha_i(t)$, are smooth functions connecting 0 to $\overline{\alpha}_i$ for $i = 1, \dots, 12$. The evolution of temperature distribution with static and dynamic control, as well as the corresponding regulation errors with respect to the static profile defined as $e(x,t) = u(x,t) - \overline{u}^D(x)$, is depicted in Figure 2. The simulation results show that the system performs well with the developed control scheme. It can also be seen that the dynamic control provides a faster response time compared to the static one.

6. Conclusion

This paper presented a solution to the problem of set-point control of temperature distribution with in-domain actuation described by an inhomogeneous parabolic PDE. To apply the principle of superposition, the system is presented in a *parallel connection* form. The dynamic control problem introduced by the ZDI design is solved by using the technique of flat systems motion planning. As the control with multiple in-domain actuators results in a MIMO problem, a Green's function-based reference trajectory decomposition is introduced, which considerably simplifies the control design and implementation. Convergence and solvability analysis confirms the validity of the control algorithm and the simulation results demonstrate the viability of the proposed approach. Finally, as both ZDI design and flatness-based control can be carried out in a systematic manner, we can expect that the approach developed in this work may be applicable to a broader class of distributed parameter systems.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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