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Research Article

Subdivision Schemes Based Collocation Algorithms for Solution of Fourth Order Boundary Value Problems

Syeda Tehmina Ejaz,[1] Ghulam Mustafa,[1] and Faheem Khan[2]

[1] *Department of Mathematics, The Islamia University of Bahawalpur, Punjab 63100, Pakistan*
[2] *Department of Mathematics, University of Sargodha, Punjab 40100, Pakistan*

Correspondence should be addressed to Ghulam Mustafa; ghulam.mustafa@iub.edu.pk

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We present two collocation algorithms based on interpolating and approximating subdivision schemes for the solution of fourth order boundary value problems arising in the mathematical modeling of viscoelastic, and inelastic flows, deformation of beams, arches, and load bearing members like street lights and robotic arms in multipurpose engineering systems. Numerical examples are given to illustrate the algorithms. We conclude that approximating schemes based collocation algorithms give better solution than interpolating schemes based collocation algorithms. Main purpose of this paper is to explore and seek the applications of interpolating and approximating subdivision schemes in the field of boundary value problems along with intrinsic comparison of the results obtained by algorithms based on these schemes. A comparison with other approaches of this type of boundary value problems in order to see the advantages of the proposed methods is also given.

1. Introduction

Subdivision schemes propose consistent and efficient iterative algorithms to produce smooth curves and surfaces from a discrete set of control points by subdividing them according to some refining rules, recursively. In recent years, subdivision techniques have become an integral part of computer graphics due to their wide range of applications in many areas such as engineering, medical science, graphic visualization, and image processing. The idea of subdivision has been initiated by de Rham [1]. Later on, Dyn et al. [2] studied a family of schemes with mask of size four, indexed by a tension parameter. Subdivision schemes can be classified into two important branches, approximating and interpolating ones. Approximating scheme means that the limit curve approximates the initial polygon and that, after subdivision, only the newly generated control points are in the limit curve, while interpolating scheme means that, after subdivision, the control points of the original control polygon and the newly generated control points both lie on the limit curve. Mustafa and Rehman [3] unified existing even-point interpolating and approximating schemes by offering general formula for the mask of $(2m + 4)$-point even-ary subdivision scheme. Aslam et al. [4] presented an explicit formula which unified the mask of $(2m - 1)$-point interpolating as well as approximating schemes. Mustafa et al. [5, 6] presented an explicit formula for the mask of odd-points b-ary interpolating subdivision schemes. Following are the advantages and disadvantages of interpolating and approximating subdivision schemes in the field of geometric modeling:

(i) Interpolating schemes are more useful for engineering applications, especially the schemes with the shape control but approximating schemes do not satisfy the shape control property.

(ii) Interpolating subdivision schemes have the drawback that, in order to create smoother curves, it is necessary to enlarge the support of the mask. The designers in geometric modeling require subdivision schemes to have their masks with a possibly smaller support and to create good smooth curves.

(iii) Approximating schemes yield smoother curves with smaller support as compared to the interpolating schemes.

In this paper, we want to find the answers of the following questions. Are there attractive characteristics of these schemes in the context of solution of boundary value problems? Do approximating (interpolating) schemes play better role than interpolating (approximation) schemes in this case? To seek answers of these questions, we consider the following interpolating [7–9] and approximating [10] subdivision schemes:

$$
\begin{aligned}
p_{2i}^{k+1} &= p_i^k,\\
p_{2i+1}^{k+1} &= \frac{35}{65536}\left(p_{i-4}^k + p_{i+5}^k\right) - \frac{405}{65536}\left(p_{i-3}^k + p_{i+4}^k\right)\\
&\quad + \frac{567}{16384}\left(p_{i-2}^k + p_{i+3}^k\right)\\
&\quad - \frac{2205}{16384}\left(p_{i-3}^k + p_{i+4}^k\right)\\
&\quad + \frac{19845}{32768}\left(p_i^k + p_{i+1}^k\right),
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
p_{2i}^{k+1} &= \frac{23}{16384}\left(p_{i-3}^k + p_{i+3}^k\right) - \frac{285}{8192}\left(p_{i-2}^k + p_{i+2}^k\right)\\
&\quad + \frac{2073}{16384}\left(p_{i-1}^k + p_{i+1}^k\right) + \frac{3333}{4096}p_i^k,\\
p_{2i+1}^{k+1} &= -\frac{3}{32768}\left(p_{i-3}^k + p_{i+4}^k\right) - \frac{33}{32768}\left(p_{i-2}^k + p_{i+3}^k\right)\\
&\quad - \frac{1931}{32768}\left(p_{i-1}^k + p_{i+2}^k\right)\\
&\quad + \frac{18351}{32768}\left(p_i^k + p_{i+1}^k\right)
\end{aligned} \tag{2}
$$

with order of continuity C^4. Schemes (1) and (2) reproduce polynomial curves of degree nine and three by [11] and [10], respectively. Cardinal supports of these schemes are $[-8, 8]$ and $[-6, 6]$, respectively.

We construct collocation algorithms by using the basis functions of the above interpolating and approximating subdivision schemes for the numerical solution of linear fourth order boundary value problems arising in the mathematical modeling of viscoelastic and inelastic flows, deformation of beams, arches, and load bearing members like street lights and robotic arms in multipurpose engineering systems, where elastic members serve as key elements for shedding or transmitting loads and in plate deflection theory and many other areas of engineering and applied mathematics. The mathematical form of these types of problems is given by

$$y^{(\text{iv})}(x) = a(x)\,y(x) + b(x), \quad 0 \leqslant x \leqslant 1, \tag{3}$$

subject to the boundary condition

$$
\begin{aligned}
y(0) &= \alpha,\\
y'(0) &= \beta,\\
y(1) &= \gamma,\\
y'(1) &= \omega,
\end{aligned} \tag{4}
$$

where $a(x)$ and $b(x)$ are continuous and $a(x) \geqslant 0$ on $[0, 1]$. Analytic solution of such type of boundary value problem is possible only in very rare cases. Qu and Agarwal [12, 13] solved this type of problems by interpolatory subdivision scheme based collocation algorithm. But they have computed the solution of second order boundary value problems using 6-point interpolating scheme based collocation algorithm. Mustafa and Ejaz [14] solved third order boundary value problems by using 8-point interpolatory subdivision scheme based collocation algorithm. Until now fourth order boundary value problems have not been solved by subdivision based collocation algorithms. This motivates us to find numerical solution of fourth order boundary value problems by interpolating and approximating subdivision schemes based collocation algorithms.

The outline of the paper is as follows. In Section 2, we construct subdivision matrices of subdivision schemes (1) and (2) for the computation of eigenvalues and their corresponding (right and left) eigenvectors. Basis functions and their derivatives have been also discussed in this section. In Section 3, subdivision based collocation algorithms for solution of (3) are formulated. Approximation properties of these algorithms are also given in Section 3. In Section 4, numerical examples are presented. Comparison of approximate solutions by interpolating and approximating schemes based collocation algorithms is also given. Conclusion is given in Section 5.

2. Basic Properties of the Schemes

In this section, we construct subdivision matrices of the schemes defined in (1) and (2) for the computation of eigenvalues and their corresponding eigenvectors. Basis functions of these schemes and their derivatives have also been discussed in this section.

2.1. Subdivision Matrices. If S_1 and S_2 are subdivision matrices of schemes (1) and (2), then these matrices are defined as

$$S_1 = \begin{pmatrix}
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
L_5&L_4&L_3&L_2&L_1&L_1&L_2&L_3&L_4&L_5&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&L_5&L_4&L_3&L_2&L_1&L_1&L_2&L_3&L_4&L_5&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&L_5&L_4&L_3&L_2&L_1&L_1&L_2&L_3&L_4&L_5&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&L_5&L_4&L_3&L_2&L_1&L_1&L_2&L_3&L_4&L_5&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&L_5&L_4&L_3&L_2&L_1&L_1&L_2&L_3&L_4&L_5&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&L_5&L_4&L_3&L_2&L_1&L_1&L_2&L_3&L_4&L_5&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&L_5&L_4&L_3&L_2&L_1&L_1&L_2&L_3&L_4&L_5&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&L_5&L_4&L_3&L_2&L_1&L_1&L_2&L_3&L_4&L_5\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0
\end{pmatrix}, \tag{5}$$

where $L_1 = 19845/32768$, $L_2 = -2205/32768$, $L_3 = 567/16384$, $L_4 = -405/65536$, $L_5 = 35/65536$, and

$$S_2 = \begin{pmatrix}
s_1&s_2&s_3&s_4&s_3&s_2&s_1&0&0&0&0&0&0\\
s_1'&s_2'&s_3'&s_4'&s_4'&s_3'&s_2'&s_1'&0&0&0&0&0\\
0&s_1&s_2&s_3&s_4&s_3&s_2&s_1&0&0&0&0&0\\
0&s_1'&s_2'&s_3'&s_4'&s_4'&s_3'&s_2'&s_1'&0&0&0&0\\
0&0&s_1&s_2&s_3&s_4&s_3&s_2&s_1&0&0&0&0\\
0&0&s_1'&s_2'&s_3'&s_4'&s_4'&s_3'&s_2'&s_1'&0&0&0\\
0&0&0&s_1&s_2&s_3&s_4&s_3&s_2&s_1&0&0&0\\
0&0&0&s_1'&s_2'&s_3'&s_4'&s_4'&s_3'&s_2'&s_1'&0&0\\
0&0&0&0&s_1&s_2&s_3&s_4&s_3&s_2&s_1&0&0\\
0&0&0&0&s_1'&s_2'&s_3'&s_4'&s_4'&s_3'&s_2'&s_1'&0\\
0&0&0&0&0&s_1&s_2&s_3&s_4&s_3&s_2&s_1&0\\
0&0&0&0&0&s_1'&s_2'&s_3'&s_4'&s_4'&s_3'&s_2'&s_1'\\
0&0&0&0&0&0&s_1&s_2&s_3&s_4&s_3&s_2&s_1
\end{pmatrix}, \tag{6}$$

where $s_1 = 23/16384$, $s_2 = -285/8192$, $s_3 = 2073/16384$, $s_4 = 3333/4096$, $s_1' = -3/32768$, $s_2' = -33/32768$, $s_3' = -1931/32768$, and $s_4' = 18351/32768$.

The first ten real eigenvalues of matrices S_1 and S_2 are the same which are given as follows:

$$\lambda_i = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}, \frac{1}{512}, \qquad i = 0, 1, 2, \ldots, 9. \tag{7}$$

The remaining eigenvalues are complex which are not required. For above eigenvalues λ_i, the eigenvectors υ_{R_i} and υ_{L_i} that satisfies $S_1\upsilon_{R_i} = \lambda_i\upsilon_{R_i}$ and $\upsilon_{L_i}S_1^T = \upsilon_{L_i}\lambda_i$ are called right and left eigenvectors of the matrix S_1, respectively. We can also define the right ν_{R_i} and left ν_{L_i} eigenvectors of S_2 in a similar way. The normalized left and right eigenvectors corresponding to first five eigenvalues of S_1 and S_2 are given in Tables 1 and 2, respectively.

2.2. Basis Functions. The basis functions for the convergent subdivision schemes (1) and (2) are the limit curves $\phi(x)$ and $\Phi(x)$ generated from the cardinal data $\{p_i = (i, \delta_0)^T\}$. $\phi(x)$ and $\Phi(x)$ are also known as fundamental solutions of the subdivision schemes, so

$$\phi(i) = \Phi(i) = \begin{cases} 1, & i = 0, \\ 0, & i \neq 0. \end{cases} \tag{8}$$

Table 1: Eigenvalues and eigenvectors of the matrix S_1.

Eigenvalues λ_i	Corresponding right and left eigenvectors
1	$\upsilon_{R_0} = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)^T$ $\upsilon_{L_0} = (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0)^T$
$\frac{1}{2}$	$\upsilon_{R_1} = (-8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8)^T$ $\upsilon_{L_1} = (-1575, -1474560, -315738080, -1397587968, 43588613880, -4311679549440, 1336741045920, -4824847319040,$ $0, 4824847319040, -1336741045920, 4311679549440, -43588613880, 1397587968, 315738080,$ $1474560, 1575)^T/5841884245680$
$\frac{1}{4}$	$\upsilon_{R_2} = (64, 49, 36, 25, 16, 9, 4, 1, 0, 1, 4, 9, 16, 25, 36, 49, 64)^T$ $\upsilon_{L_2} = (459375, 215040000, 47660030080, 103151616000, -3882427261296, 23490017902592, -84313449846912,$ $313469774708736, -497829885297150, 313469774708736, -84313449846912, 23490017902592, -3882427261296,$ $103151616000, 47660030080, 215040000, 459375)^T/259624836213120$
$\frac{1}{8}$	$\upsilon_{R_3} = (-512, -343, -216, -125, -64, -27, -8, -1, 0, 1, 8, 27, 64, 125, 216, 343, 512)^T$ $\upsilon_{L_3} = (-104125, -24371200, 1177382520, -5986263040, -10571778214, 207884427264, -972244098856, -1386160480256,$ $0, 1386160480256, 972244098856, -207884427264, 10571778214, 5986263040, -1177382520, 24371200,$ $104125)^T/9460416859904$
$\frac{1}{16}$	$\upsilon_{R_4} = (4096, 2401, 1296, 625, 256, 81, 16, 1, 0, 1, 16, 81, 256, 625, 1296, 2401, 4096)^T$ $\upsilon_{L_4} = (392875, 45977600, -1296269280, 5912719360, 1180083476, -86261280768, 332951715808, -67767008256,$ $850467338370, -67767008256, 332951715808, -86261280768, 1180083476, 5912719360, -1296269280,$ $45977600, 392875)^T/183768238080$

Table 2: Eigenvalues and eigenvectors of the matrix S_2.

Eigenvalues λ_i	Corresponding right and left eigenvectors
1	$\nu_{R_0} = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)^T$ $\nu_{L_0} = (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0)^T$
$\frac{1}{2}$	$\nu_{R_1} = (-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6)^T$ $\nu_{L_1} = (-271, 1475866, 286711948, -3680077858, 73546185237, -507541369116, 0, 507541369116,$ $-73546185237, 3680077858, -286711948, -1475866, 271)^T/740670013440$
$\frac{1}{4}$	$\nu_{R_2} = (36, 25, 16, 9, 4, 1, 0, 1, 4, 9, 16, 25, 36)^T$ $\nu_{L_2} = (-3927, 10663114, -582891346, -38170879262, 32176934487, 2719571096148, -5426009838428,$ $2719571096148, 32176934487, -38170879262, -582891346, 10663114, -3927)^T/4991362191360$
$\frac{1}{8}$	$\nu_{R_3} = (-216, -125, -64, -27, -8, -1, 0, 1, 8, 27, 64, 125, 216)^T$ $\nu_{L_3} = (23, -31050, -1351836, 21998258, -507201588, 193168476, 0,$ $-193168476, 507201588, -21998258, 1351836, 31050, -23)^T/635351040$
$\frac{1}{16}$	$\nu_{R_4} = (1/27)(51867, 25027, 10267, 3267, 667, 67, 27, 67, 667, 3267, 10267, 25027, 51867)^T$ $\nu_{L_4} = (-6693, 4466462, 117537274, -1293454266, 6372836901, -15842276196, 21281793036,$ $-15842276196, 6372836901, -1293454266, 117537274, 4466462, -6693)^T/42150297600$

Furthermore, $\phi(x) = \Phi(x)$ satisfies the following two-scale equation:

$$\phi(x) = \sum_{j=-p}^{p} a_i \phi(2x-i), \qquad \text{Sup}(\phi) = (-p-1, p+1). \tag{9}$$

The lth derivative of basis function at integers satisfies the relation

$$\phi^{(l)}(x) = \sum_{j=-p}^{p} a_i \phi^{(l)}(2x-i), \qquad \text{Sup}(\phi) = [-p, p]. \tag{10}$$

Since $\upsilon_{R_i}^T \upsilon_{L_j} = 1$ and $\nu_{R_i}^T \nu_{L_j} = 1$ for $i = j$ and 0 otherwise, then, by [16], we get the following result.

Lemma 1. *The fundamental solution (cardinal basis) $\phi(x)$ of the subdivision scheme (1) is four times continuously differentiable, supported on $[-8, 8]$, and its derivatives at integers are given by*

$$\begin{aligned}
&\phi'(t) = 2\,\text{sgn}(t)\, e_{|t|}^T \upsilon_{L_1},\\
&\phi''(t) = 2^2 e_{|t|}^T \upsilon_{L_2},\\
&\phi'''(t) = 2^3\,\text{sgn}(t)\, e_{|t|}^T \upsilon_{L_3},\\
&\phi^{iv}(t) = 2^4 e_{|t|}^T \upsilon_{L_4},\\
&\qquad -8 \leqslant t \leqslant 8,
\end{aligned} \tag{11}$$

where υ_{L_i}, $0 \leqslant i \leqslant 4$, are defined in Table 1; the sgn *function of a real number t is defined as*

$$\text{sgn}(t) = \begin{cases} -1, & t < 0,\\ 0, & t = 0,\\ 1, & t > 0. \end{cases} \tag{12}$$

e_t's are the column matrices defined as

$$\begin{aligned}
e_t = (&a_{8t}, a_{7t}, a_{6t}, a_{5t}, a_{4t}, a_{3t}, a_{2t}, a_{1t}, a_{0t}, a_{-1t}, a_{-2t},\\
&a_{-3t}, a_{-4t}, a_{-5t}, a_{-6t}, a_{-7t}, a_{-8t})^T,
\end{aligned} \tag{13}$$

where $0 \leq t \leq 8$ and

$$a_{it} = \begin{cases} 1, & i = t,\\ 0, & i \neq t. \end{cases} \tag{14}$$

Lemma 2. *The fundamental solution $\Phi(x)$ of subdivision scheme (2) defined in (9) is four times continuously differentiable, supported on $[-6, 6]$, and its derivatives at integers are defined as*

$$\begin{aligned}
&\Phi'(t) = 2\,\text{sgn}(t)\, e_{|t|}^T \nu_{L_1},\\
&\Phi''(t) = 2^2 e_{|t|}^T \nu_{L_2},\\
&\Phi'''(t) = 2^3\,\text{sgn}(t)\, e_{|t|}^T \nu_{L_3},\\
&\Phi^{iv}(t) = 2^4 e_{|t|}^T \nu_{L_4}\\
&\qquad -6 \leqslant t \leqslant 6,
\end{aligned} \tag{15}$$

where ν_{L_i} for $0 \leqslant i \leqslant 4$ are defined in Table 2; the sgn *function of a real number t is defined by (12); e_t's are the column matrices defined as*

$$\begin{aligned}
e_t = (&a_{6t}, a_{5t}, a_{4t}, a_{3t}, a_{2t}, a_{1t}, a_{0t}, a_{-1t}, a_{-2t}, a_{-3t}, a_{-4t},\\
&a_{-5t}, a_{-6t})^T, \quad 0 \leqslant t \leqslant 6,
\end{aligned} \tag{16}$$

where a_{it} are defined by (14).

From (11) and (15), we get values of derivatives at the integers given in Tables 3 and 4, respectively.

3. Description of Numerical Algorithms

In this section, first we formulate two collocation algorithms which are based on interpolating (1) and approximating (2) subdivision schemes for the solution of (3). Then we settle down the boundary conditions to get unique solution.

3.1. Collocation Algorithms. Here we formulate two collocation algorithms based on two subdivision schemes. These collocation algorithms are defined in coming subsections.

3.1.1. Interpolating Collocation Algorithm. The collocation algorithm based on interpolating scheme (1), say, interpolating collocating algorithm, is given below. In this algorithm, we assume approximate solution $Z_1(x)$ of (3) as

$$Z_1(x) = \sum_{i=-8}^{N+8} z_i \phi\left(\frac{x - x_i}{h}\right), \quad 0 \leqslant x \leqslant 1, \tag{17}$$

where N is the positive integer $N \geqslant 8$, $h = 1/N$ and $x_i = i/N = ih$, and $\{z_i\}$ are the unknown to be determined for the solution of (3). The collocation algorithm, together with the boundary conditions to be discussed, is given by

$$\begin{aligned}
&Z_1^{\text{iv}}(x_j) = a(x_j)\, Z_1(x_j) + b(x_j),\\
&\qquad j = 0, 1, 2, \ldots, N,
\end{aligned} \tag{18}$$

with the following type of boundary conditions:

$$\begin{aligned}
Z_1(0) &= \alpha,\\
Z_1'(0) &= \beta,\\
Z_1(N) &= \gamma,\\
Z_1'(N) &= \omega,
\end{aligned} \tag{19}$$

where α, β, γ, and ω are constants. Let $a_j = a(x_j)$, $b_j = b(x_j)$; then (18) can be written as

$$Z_1^{\text{iv}}(x_j) = a_j Z_1(x_j) + b_j, \quad j = 0, 1, 2, \ldots, N, \tag{20}$$

where

$$Z_1^{\text{iv}}(x_j) = \frac{1}{h^4} \sum_{i=-8}^{N+8} z_i \phi^{\text{iv}}\left(\frac{x_j - x_i}{h}\right). \tag{21}$$

Using (17) and (21) in (20), we get the following $N + 1$ system of equations:

$$\begin{aligned}
&\sum_{i=-8}^{N+8} z_i \phi^{\text{iv}}\left(\frac{x_j - x_i}{h}\right) - h^4 a_j \sum_{i=-8}^{N+8} z_i \phi\left(\frac{x_j - x_i}{h}\right) = h^4 b_j,\\
&\qquad j = 0, 1, 2, \ldots, N.
\end{aligned} \tag{22}$$

3.1.2. Approximating Collocation Algorithm. In approximating collocating algorithm (i.e., algorithm based on approximating scheme (2)), we assume approximate solution $Z_2(x)$ of (3) as

$$Z_2(x) = \sum_{i=-6}^{N+6} z_i \Phi\left(\frac{x - x_i}{h}\right), \quad 0 \leqslant x \leqslant 1, \tag{23}$$

Table 3: Derivatives of ϕ at cardinal data by (11).

At\Derivatives	ϕ'	ϕ''	ϕ'''	ϕ^{iv}
0	0	$-\frac{2370618501415}{309077185968}$	0	$\frac{33869667}{457408}$
±1	$\mp\frac{1914621952}{1159104017}$	$\frac{3265310153216}{676106344305}$	$\pm\frac{43317515008}{15295995855}$	$-\frac{5295054752}{89730585}$
±2	$\pm\frac{530452796}{1159104017}$	$-\frac{878265102572}{676106344305}$	$\mp\frac{121530512357}{61183983420}$	$\frac{10404741119}{358922340}$
±3	$\mp\frac{1470464}{13780629}$	$\frac{734063059456}{2028319032915}$	$\pm\frac{240606976}{566518365}$	$-\frac{74879584}{9970065}$
±4	$\pm\frac{17297069}{1159104017}$	$-\frac{80883901277}{1352212688610}$	$\mp\frac{5285889107}{244735933680}$	$\frac{295020869}{2871378720}$
±5	$\mp\frac{2772992}{5795520085}$	$\frac{214899200}{135221268861}$	$\mp\frac{37414144}{3059199171}$	$\frac{9238624}{17946117}$
±6	$\mp\frac{1127636}{10431936153}$	$\frac{297875188}{405663806583}$	$\pm\frac{1090169}{453214692}$	$-\frac{900187}{7976052}$
±7	$\mp\frac{4096}{8113728119}$	$\frac{64000}{19317324123}$	$\mp\frac{21760}{437028453}$	$\frac{71840}{17946117}$
±8	$\mp\frac{5}{9272832136}$	$\frac{4375}{618154371936}$	$\mp\frac{2975}{13984910496}$	$\frac{11225}{328157568}$

Table 4: Derivatives of Φ at cardinal data by (15).

At\Derivatives	Φ'	Φ''	Φ'''	Φ^{iv}
0	0	$-\frac{1356502459607}{311960136960}$	0	$\frac{1773482753}{219532800}$
±1	$\mp\frac{42295114093}{30861250560}$	$\frac{226630924679}{103986712320}$	$\pm\frac{596199}{245120}$	$-\frac{1320189683}{219532800}$
±2	$\pm\frac{3502199297}{17635000320}$	$\frac{3575214943}{138648949760}$	$\mp\frac{1565437}{980480}$	$\frac{708092989}{292710400}$
±3	$\mp\frac{1840038929}{185167503360}$	$-\frac{19085439631}{623920273920}$	$\pm\frac{10999129}{39709440}$	$-\frac{215575711}{439065600}$
±4	$\pm\frac{71677987}{92583751680}$	$-\frac{291445673}{623920273920}$	$\mp\frac{12517}{735360}$	$\frac{58768637}{1317196800}$
±5	$\pm\frac{105419}{26452500480}$	$\frac{5331557}{623920273920}$	$\mp\frac{115}{294144}$	$\frac{2233231}{1317196800}$
±6	$\mp\frac{271}{370335006720}$	$-\frac{1309}{415946849280}$	$\pm\frac{23}{79418880}$	$-\frac{2231}{878131200}$

where N is the positive integer $N \geqslant 6$, $h = 1/N$ and $x_i = i/N = ih$, and $\{z_i\}$ are the unknown to be determined for the solution of (3). The collocation algorithm, together with the boundary conditions to be discussed, is given by

$$Z_2^{\text{iv}}(x_j) = a(x_j) Z_2(x_j) + b(x_j), \quad j = 0, 1, 2, \ldots, N, \tag{24}$$

with the following type of boundary conditions:

$$Z_2(0) = \alpha, \quad Z_2'(0) = \beta, \quad Z_2(N) = \gamma, \quad Z_2'(N) = \omega. \tag{25}$$

Equation (24) can be written as

$$Z_2^{\text{iv}}(x_j) = a_j Z_2(x_j) + b_j, \quad j = 0, 1, 2, \ldots, N, \tag{26}$$

where

$$Z_2^{\text{iv}}(x_j) = \frac{1}{h^4} \sum_{i=-6}^{N+6} z_i \Phi^{\text{iv}}\left(\frac{x_j - x_i}{h}\right). \tag{27}$$

Using (23) and (27) in (26), we get the following $N+1$ system of equations:

$$\sum_{i=-6}^{N+6} z_i \Phi^{\text{iv}}\left(\frac{x_j - x_i}{h}\right) - h^4 a_j \sum_{i=-6}^{N+6} z_i \Phi\left(\frac{x_j - x_i}{h}\right) = h^4 b_j, \quad j = 0, 1, 2, \ldots, N. \tag{28}$$

Now we simplify systems (22) and (28) in the following theorems.

Theorem 3 (interpolating collocation algorithm). *For $j = 0$ by (22), one gets*

$$\begin{aligned} & z_{-8}\phi_{-8}^{iv} + z_{-7}\phi_{-7}^{iv} + z_{-6}\phi_{-6}^{iv} + z_{-5}\phi_{-5}^{iv} + z_{-4}\phi_{-4}^{iv} \\ &\quad + z_{-3}\phi_{-3}^{iv} + z_{-2}\phi_{-2}^{iv} + z_{-1}\phi_{-1}^{iv} + z_0 q_0 + z_1\phi_1^{iv} \\ &\quad + z_2\phi_2^{iv} + z_3\phi_3^{iv} + z_4\phi_4^{iv} + z_5\phi_5^{iv} + z_6\phi_6^{iv} + z_7\phi_7^{iv} \\ &\quad + z_8\phi_8^{iv} = h^4 b_0, \end{aligned} \tag{29}$$

where $\phi_j^{iv} = \phi^{iv}(j)$ and $q_0 = \phi_0^{iv} - a_0 h^4$.

Proof. Substituting $j = 0$ in (22), we get

$$\begin{aligned} & \left\{ z_{-8}\phi^{\text{iv}}\left(\frac{x_0 - x_{-8}}{h}\right) + z_{-7}\phi^{\text{iv}}\left(\frac{x_0 - x_{-7}}{h}\right) + \cdots \right. \\ &\quad \left. + z_{N+7}\phi^{\text{iv}}\left(\frac{x_0 - x_{N+7}}{h}\right) + z_{N+8}\phi^{\text{iv}}\left(\frac{x_0 - x_{N+8}}{h}\right) \right\} \\ &\quad - a_0 h^4 \left\{ z_{-8}\phi\left(\frac{x_0 - x_{-8}}{h}\right) + z_{-7}\phi\left(\frac{x_0 - x_{-7}}{h}\right) \right. \\ &\quad + \cdots + z_{N+7}\phi\left(\frac{x_0 - x_{N+7}}{h}\right) \\ &\quad \left. + z_{N+8}\phi\left(\frac{x_0 - x_{N+8}}{h}\right) \right\} = h^4 b_0. \end{aligned} \tag{30}$$

For $x_i = ih$, $i = 0, 1, 2, \ldots, N$, this implies

$$\begin{aligned} & z_{-8}\phi^{\text{iv}}(8) + z_{-7}\phi^{\text{iv}}(7) + \cdots + z_{N+7}\phi^{\text{iv}}(-N-7) \\ &\quad + z_{N+8}\phi^{\text{iv}}(-N-8) - a_0 h^4 \{ z_{-8}\phi(8) + z_{-7}\phi(7) \\ &\quad + \cdots + z_{N+7}\phi(-N-7) + z_{N+8}\phi(-N-8) \} = h^4 b_0. \end{aligned} \tag{31}$$

Since the cardinal support of basis function $\phi(x)$ is $[-8, 8]$, so $\phi'(x)$, $\phi''(x)$, $\phi'''(x)$, and $\phi^{\text{iv}}(x)$ are zero outside the interval $[-8, 8]$; also, by Table 3, we get

$$\begin{aligned} & z_{-8}\phi^{\text{iv}}(8) + z_{-7}\phi^{\text{iv}}(7) + z_{-6}\phi^{\text{iv}}(6) + z_{-5}\phi^{\text{iv}}(5) \\ &\quad + z_{-4}\phi^{\text{iv}}(4) + z_{-3}\phi^{\text{iv}}(3) + z_{-2}\phi^{\text{iv}}(2) \\ &\quad + z_{-1}\phi^{\text{iv}}(1) + z_0\phi^{\text{iv}}(0) + z_1\phi^{\text{iv}}(-1) \\ &\quad + z_2\phi^{\text{iv}}(-2) + z_3\phi^{\text{iv}}(-3) + z_4\phi^{\text{iv}}(-4) \\ &\quad + z_5\phi^{\text{iv}}(-5) + z_6\phi^{\text{iv}}(-6) + z_7\phi^{\text{iv}}(-7) \\ &\quad + z_8\phi^{\text{iv}}(-8) - a_0 h^4 z_0 \phi(0) = h^4 b_0. \end{aligned} \tag{32}$$

If $\phi_i^{\text{iv}} = \phi^{\text{iv}}(i)$, then

$$\begin{aligned} & z_{-8}\phi_8^{\text{iv}} + z_{-7}\phi_7^{\text{iv}} + z_{-6}\phi_6^{\text{iv}} + z_{-5}\phi_5^{\text{iv}} + z_{-4}\phi_4^{\text{iv}} + z_{-3}\phi_3^{\text{iv}} \\ &\quad + z_{-2}\phi_2^{\text{iv}} + z_{-1}\phi_1^{\text{iv}} + z_0\left(\phi_0^{\text{iv}} - a_0 h^4\right) + z_1\phi_{-1}^{\text{iv}} \\ &\quad + z_2\phi_{-2}^{\text{iv}} + z_3\phi_{-3}^{\text{iv}} + z_4\phi_{-4}^{\text{iv}} + z_5\phi_{-5}^{\text{iv}} + z_6\phi_{-6}^{\text{iv}} \\ &\quad + z_7\phi_{-7}^{\text{iv}} + z_8\phi_{-8}^{\text{iv}} = h^4 b_0. \end{aligned} \tag{33}$$

As we observe from Table 3, $\phi_{-i}^{\text{iv}} = \phi_i^{\text{iv}}$; we have

$$\begin{aligned} & z_{-8}\phi_{-8}^{\text{iv}} + z_{-7}\phi_{-7}^{\text{iv}} + z_{-6}\phi_{-6}^{\text{iv}} + z_{-5}\phi_{-5}^{\text{iv}} + z_{-4}\phi_{-4}^{\text{iv}} \\ &\quad + z_{-3}\phi_{-3}^{\text{iv}} + z_{-2}\phi_{-2}^{\text{iv}} + z_{-1}\phi_{-1}^{\text{iv}} + z_0\left(\phi_0^{\text{iv}} - a_0 h^4\right) \\ &\quad + z_1\phi_1^{\text{iv}} + z_2\phi_2^{\text{iv}} + z_3\phi_3^{\text{iv}} + z_4\phi_4^{\text{iv}} + z_5\phi_5^{\text{iv}} + z_6\phi_6^{\text{iv}} \\ &\quad + z_7\phi_7^{\text{iv}} + z_8\phi_8^{\text{iv}} = h^4 b_0. \end{aligned} \tag{34}$$

For $q_0 = \phi_0^{\text{iv}} - a_0 h^4$, we get (29). This completes the proof. □

Theorem 4 (interpolating collocation algorithm). *For $j = 1, 2, 3, \ldots, N$, system (22) is equivalent to*

$$\begin{aligned} & z_{-8}\phi_{-j-8}^{\text{iv}} + z_{-7}\phi_{-j-7}^{\text{iv}} + \cdots + z_0\phi_{-j}^{\text{iv}} \\ &\quad + z_1\left(\phi_{1-j}^{\text{iv}} - a_j h^4 \phi_{1-j}\right) + z_2\left(\phi_{2-j}^{\text{iv}} - a_j h^4 \phi_{2-j}\right) \\ &\quad + \cdots + z_N\left(\phi_{N-j}^{\text{iv}} - a_j h^4 \phi_{N-j}\right) + z_{N+1}\phi_{N+1-j}^{\text{iv}} \\ &\quad + \cdots + z_{N+8}\phi_{N+8-j}^{\text{iv}} = h^4 b_j. \end{aligned} \tag{35}$$

Proof. By expanding (22), we get

$$\begin{aligned} & z_{-8}\phi^{\text{iv}}\left(\frac{x_j - x_{-8}}{h}\right) + z_{-7}\phi^{\text{iv}}\left(\frac{x_j - x_{-7}}{h}\right) + \cdots \\ &\quad + z_{N+7}\phi^{\text{iv}}\left(\frac{x_j - x_{N+7}}{h}\right) + z_{N+8}\phi^{\text{iv}}\left(\frac{x_j - x_{N+8}}{h}\right) \\ &\quad - a_j h^4 \left\{ z_{-8}\phi\left(\frac{x_j - x_{-8}}{h}\right) + z_{-7}\phi\left(\frac{x_j - x_{-7}}{h}\right) \right. \\ &\quad + \cdots + z_{N+7}\phi\left(\frac{x_j - x_{N+7}}{h}\right) \\ &\quad \left. + z_{N+8}\phi\left(\frac{x_j - x_{N+8}}{h}\right) \right\} = h^4 b_j. \end{aligned} \tag{36}$$

For $x_j = jh$, $j = 1, 2, \ldots, N$, we get

$$\begin{aligned} & z_{-8}\phi^{\text{iv}}(j+8) + z_{-7}\phi^{\text{iv}}(j+7) + \cdots + z_{N+7}\phi^{\text{iv}}(j-N \\ &\quad -7) + z_{N+8}\phi^{\text{iv}}(j-N-8) - a_j h^4 \{ z_{-8}\phi(j+8) \\ &\quad + z_{-7}\phi(j+7) + \cdots + z_{N+7}\phi(j-N-7) \\ &\quad + z_{N+8}\phi(j-N-8) \} = h^4 b_j. \end{aligned} \tag{37}$$

This implies

$$\begin{aligned} & z_{-8}\left(\phi^{\text{iv}}(j+8)-a_j h^4 \phi(j+8)\right) \\ & \quad + z_{-7}\left(\phi^{\text{iv}}(j+7)-a_j h^4 \phi(j+7)\right)+\cdots \\ & \quad + z_{N+7}\left(\phi^{\text{iv}}(j-N-7)-a_j h^4 \phi(j-N-7)\right) \\ & \quad + z_{N+8}\left(\phi^{\text{iv}}(j-N-8)-a_j h^4 \phi(j-N-8)\right) \\ & = h^4 b_j. \end{aligned} \tag{38}$$

If $\phi_j^{\text{iv}} = \phi^{\text{iv}}(j)$, for $j = 1, 2, \ldots, N$, then

$$\begin{aligned} & z_{-8}\left(\phi_{j+8}^{\text{iv}}-a_j h^4 \phi_{j+8}\right)+z_{-7}\left(\phi_{j+7}^{\text{iv}}-a_j h^4 \phi_{j+7}\right)+\cdots \\ & \quad + z_{N+7}\left(\phi_{j-N-7}^{\text{iv}}-a_j h^4 \phi_{j-N-7}\right) \\ & \quad + z_{N+8}\left(\phi_{j-N-8}^{\text{iv}}-a_j h^4 \phi_{j-N-8}\right)=h^4 b_j. \end{aligned} \tag{39}$$

As we observe from Table 3, $\phi_{-j}^{\text{iv}} = \phi_j^{\text{iv}}$, $j = 1, 2, \ldots, N$; then the above equation can be written as

$$\begin{aligned} & z_{-8}\left(\phi_{-j-8}^{\text{iv}}-a_j h^4 \phi_{-j-8}\right)+z_{-7}\left(\phi_{-j-7}^{\text{iv}}-a_j h^4 \phi_{-j-7}\right) \\ & \quad +\cdots+z_{N+7}\left(\phi_{N+7-j}^{\text{iv}}-a_j h^4 \phi_{N+7-j}\right) \\ & \quad + z_{N+8}\left(\phi_{N+8-j}^{\text{iv}}-a_j h^4 \phi_{N+8-j}\right)=h^4 b_j. \end{aligned} \tag{40}$$

Since $\phi'(x)$, $\phi''(x)$, $\phi'''(x)$, and $\phi^{\text{iv}}(x)$ are zero outside the interval $[-8, 8]$, then by Table 3, we get (35). □

Theorem 5 (approximating collocation algorithm). *For $j = 0$, by (28), one gets*

$$\begin{aligned} & z_{-6}\Phi_{-6}^{iv}+z_{-5}\Phi_{-5}^{iv}+z_{-4}\Phi_{-4}^{iv}+z_{-5}\Phi_{-5}^{iv}+z_{-4}\Phi_{-4}^{iv} \\ & \quad + z_{-3}\Phi_{-3}^{iv}+z_{-2}\Phi_{-2}^{iv}+z_{-1}\Phi_{-1}^{iv}+z_0 q_0+z_1\Phi_1^{iv} \\ & \quad + z_2\Phi_2^{iv}+z_3\Phi_3^{iv}+z_4\Phi_4^{iv}+z_5\Phi_5^{iv}+z_6\Phi_6^{iv} \\ & = h^4 b_0, \end{aligned} \tag{41}$$

where $\Phi_j^{iv} = \Phi^{iv}(j)$ and $\Upsilon_0 = \Phi_0^{iv} - a_0 h^4$.

Proof. Substituting $j = 0$ in (28), we get

$$\begin{aligned} & \left\{z_{-6}\Phi^{\text{iv}}\left(\frac{x_0-x_{-6}}{h}\right)+z_{-5}\Phi^{\text{iv}}\left(\frac{x_0-x_{-5}}{h}\right)+\cdots\right. \\ & \quad + z_{N+5}\Phi^{\text{iv}}\left(\frac{x_0-x_{N+5}}{h}\right) \\ & \quad \left. + z_{N+6}\Phi^{\text{iv}}\left(\frac{x_0-x_{N+6}}{h}\right)\right\} \\ & \quad - a_0 h^4\left\{z_{-6}\Phi\left(\frac{x_0-x_{-6}}{h}\right)+z_{-5}\Phi\left(\frac{x_0-x_{-5}}{h}\right)\right. \\ & \quad +\cdots+z_{N+5}\Phi\left(\frac{x_0-x_{N+5}}{h}\right) \\ & \quad \left. + z_{N+6}\Phi\left(\frac{x_0-x_{N+6}}{h}\right)\right\}=h^4 b_0. \end{aligned} \tag{42}$$

For $x_i = ih$, $i = 0, 1, 2, \ldots, N$, this implies

$$\begin{aligned} & z_{-6}\Phi^{\text{iv}}(6)+z_{-5}\Phi^{\text{iv}}(5)+\cdots+z_{N+5}\Phi^{\text{iv}}(-N-5) \\ & \quad + z_{N+6}\Phi^{\text{iv}}(-N-6)-a_0 h^4\left\{z_{-6}\Phi(6)+z_{-5}\Phi(5)\right. \\ & \quad \left.+\cdots+z_{N+5}\Phi(-N-5)+z_{N+6}\Phi(-N-6)\right\} \\ & \quad = h^4 b_0. \end{aligned} \tag{43}$$

Since the cardinal support of basis function $\Phi(x)$ is $[-6, 6]$, so $\Phi'(x)$, $\Phi''(x)$, $\Phi'''(x)$, and $\Phi^{\text{iv}}(x)$ are zero outside the interval $[-6, 6]$; also, by Table 4, we get

$$\begin{aligned} & z_{-6}\Phi^{\text{iv}}(6)+z_{-5}\Phi^{\text{iv}}(5)+z_{-4}\Phi^{\text{iv}}(4)+z_{-3}\Phi^{\text{iv}}(3) \\ & \quad + z_{-2}\Phi^{\text{iv}}(2)+z_{-1}\Phi^{\text{iv}}(1)+z_0\Phi^{\text{iv}}(0) \\ & \quad + z_1\Phi^{\text{iv}}(-1)+z_2\Phi^{\text{iv}}(-2)+z_3\Phi^{\text{iv}}(-3) \\ & \quad + z_4\Phi^{\text{iv}}(-4)+z_5\Phi^{\text{iv}}(-5)+z_6\Phi^{\text{iv}}(-6) \\ & \quad - a_0 h^4 z_0\Phi(0)=h^4 b_0. \end{aligned} \tag{44}$$

If $\Phi_i^{\text{iv}} = \Phi^{\text{iv}}(i)$, then

$$\begin{aligned} & z_{-6}\Phi_6^{\text{iv}}+z_{-5}\Phi_5^{\text{iv}}+z_{-4}\Phi_4^{\text{iv}}+z_{-3}\Phi_3^{\text{iv}}+z_{-2}\Phi_2^{\text{iv}}+z_{-1}\Phi_1^{\text{iv}} \\ & \quad + z_0\left(\Phi_0^{\text{iv}}-a_0 h^4\right)+z_1\Phi_{-1}^{\text{iv}}+z_2\Phi_{-2}^{\text{iv}}+z_3\Phi_{-3}^{\text{iv}} \\ & \quad + z_4\Phi_{-4}^{\text{iv}}+z_5\Phi_{-5}^{\text{iv}}+z_6\Phi_{-6}^{\text{iv}}=h^4 b_0. \end{aligned} \tag{45}$$

As we observe from Table 4, $\Phi_{-i}^{\text{iv}} = \Phi_i^{\text{iv}}$; we have

$$\begin{aligned} & z_{-6}\Phi_{-6}^{\text{iv}}+z_{-5}\Phi_{-5}^{\text{iv}}+z_{-4}\Phi_{-4}^{\text{iv}}+z_{-3}\Phi_{-3}^{\text{iv}}+z_{-2}\Phi_{-2}^{\text{iv}} \\ & \quad + z_{-1}\Phi_{-1}^{\text{iv}}+z_0\left(\Phi_0^{\text{iv}}-a_0 h^4\right)+z_1\Phi_1^{\text{iv}}+z_2\Phi_2^{\text{iv}} \\ & \quad + z_3\Phi_3^{\text{iv}}+z_4\Phi_4^{\text{iv}}+z_5\Phi_5^{\text{iv}}+z_6\Phi_6^{\text{iv}}+z_7\Phi_7^{\text{iv}} \\ & \quad + z_8\Phi_8^{\text{iv}}=h^4 b_0. \end{aligned} \tag{46}$$

For $\Upsilon_0 = \Phi_0^{\text{iv}} - a_0 h^4$, we get (41). This completes the proof. □

Theorem 6 (approximating collocation algorithm). *For $j = 1, 2, 3, \ldots, N$, system (28) is equivalent to*

$$\begin{aligned} & z_{-6}\Phi_{-j-6}^{iv}+z_{-5}\Phi_{-j-5}^{iv}+\cdots+z_0\Phi_{-j}^{iv} \\ & \quad + z_1\left(\Phi_{1-j}^{iv}-a_j h^4\Phi_{1-j}\right)+z_2\left(\Phi_{2-j}^{iv}-a_j h^4\Phi_{2-j}\right) \\ & \quad +\cdots+z_N\left(\Phi_{N-j}^{iv}-a_j h^4\Phi_{N-j}\right)+z_{N+1}\Phi_{N+1-j}^{iv} \\ & \quad +\cdots+z_{N+6}\Phi_{N+6-j}^{iv}=h^4 b_j. \end{aligned} \tag{47}$$

Proof. By expanding (28), we get

$$\begin{aligned} & z_{-6}\Phi^{\text{iv}}\left(\frac{x_j - x_{-6}}{h}\right) + z_{-5}\Phi^{\text{iv}}\left(\frac{x_j - x_{-5}}{h}\right) + \cdots \\ & + z_{N+5}\Phi^{\text{iv}}\left(\frac{x_j - x_{N+5}}{h}\right) + z_{N+6}\Phi^{\text{iv}}\left(\frac{x_j - x_{N+6}}{h}\right) \\ & - a_j h^4 \left\{ z_{-8}\Phi\left(\frac{x_j - x_{-8}}{h}\right) + z_{-7}\Phi\left(\frac{x_j - x_{-7}}{h}\right) \right. \\ & + \cdots + z_{N+5}\Phi\left(\frac{x_j - x_{N+5}}{h}\right) \\ & \left. + z_{N+6}\Phi\left(\frac{x_j - x_{N+6}}{h}\right) \right\} = h^4 b_j. \end{aligned} \tag{48}$$

For $x_j = jh$, $j = 1, 2, \ldots, N$, we get

$$\begin{aligned} & z_{-6}\Phi^{\text{iv}}(j+6) + z_{-5}\Phi^{\text{iv}}(j+5) + \cdots + z_{N+5}\Phi^{\text{iv}}(j \\ & - N - 5) + z_{N+6}\Phi^{\text{iv}}(j - N - 6) \\ & - a_j h^4 \{ z_{-6}\Phi(j+6) + z_{-5}\Phi(j+5) + \cdots \\ & + z_{N+5}\Phi(j - N - 5) + z_{N+6}\Phi(j - N - 6) \} \\ & = h^4 b_j. \end{aligned} \tag{49}$$

This implies

$$\begin{aligned} & z_{-6}\left(\Phi^{\text{iv}}(j+6) - a_j h^4 \Phi(j+6)\right) \\ & \quad + z_{-5}\left(\Phi^{\text{iv}}(j+5) - a_j h^4 \Phi(j+5)\right) + \cdots \\ & \quad + z_{N+5}\left(\Phi^{\text{iv}}(j - N - 5) - a_j h^4 \Phi(j - N - 5)\right) \\ & \quad + z_{N+6}\left(\Phi^{\text{iv}}(j - N - 6) - a_j h^4 \Phi(j - N - 6)\right) \\ & = h^4 b_j. \end{aligned} \tag{50}$$

If $\Phi_j^{\text{iv}} = \Phi^{\text{iv}}(j)$, for $j = 1, 2, \ldots, N$, then

$$\begin{aligned} & z_{-6}\left(\Phi_{j+6}^{\text{iv}} - a_j h^4 \Phi_{j+6}\right) + z_{-5}\left(\Phi_{j+5}^{\text{iv}} - a_j h^4 \Phi_{j+5}\right) + \cdots \\ & \quad + z_{N+5}\left(\Phi_{j-N-5}^{\text{iv}} - a_j h^4 \Phi_{j-N-5}\right) \\ & \quad + z_{N+6}\left(\Phi_{j-N-6}^{\text{iv}} - a_j h^4 \Phi_{j-N-6}\right) = h^4 b_j. \end{aligned} \tag{51}$$

As we observe from Table 4, $\Phi_{-j}^{\text{iv}} = \Phi_j^{\text{iv}}$, $j = 1, 2, \ldots, N$; then the above equation can be written as

$$\begin{aligned} & z_{-6}\left(\Phi_{-j-6}^{\text{iv}} - a_j h^4 \Phi_{-j-6}\right) + z_{-5}\left(\Phi_{-j-5}^{\text{iv}} - a_j h^4 \Phi_{-j-5}\right) \\ & \quad + \cdots + z_{N+5}\left(\Phi_{N+5-j}^{\text{iv}} - a_j h^4 \Phi_{N+5-j}\right) \\ & \quad + z_{N+6}\left(\Phi_{N+6-j}^{\text{iv}} - a_j h^4 \Phi_{N+6-j}\right) = h^4 b_j. \end{aligned} \tag{52}$$

Since $\Phi'(x)$, $\Phi''(x)$, $\Phi'''(x)$, and $\Phi^{\text{iv}}(x)$ are zero outside the interval $[-6, 6]$, then, by Table 4, we get (47). □

3.2. Boundary Conditions at End Points. We have two different systems of $(N + 1)$ equations defined by (22) and (28). In order to get unique solution of these systems, we need sixteen more conditions for system (22) and twelve more conditions for system (28). Four conditions can be achieved from boundary conditions given in (4) for both systems of linear equations in which first order derivatives are involved and remaining conditions are achieved by some extrapolation method at the end points. First we find the approximation of the first derivative by difference operators and after that we define the extrapolation method at end points for both systems of linear equations.

3.2.1. Approximation of Derivative Boundary Conditions. In this section, we approximate the derivative boundary conditions by difference operators. Since approximation order of interpolating scheme (1) and approximating scheme (2) is ten and four, respectively, so we approximate derivative boundary conditions at end points with approximation orders ten and four for interpolating and approximating collocation algorithms.

If we use interpolating collocation algorithm for the solution of (3), then approximation of derivative conditions at ends point is defined as

$$\begin{aligned} Z_1'(0) = {} & \left(\frac{N}{2520}\right)\{-7381z_0 + 25200z_1 - 56700z_2 \\ & + 100800z_3 - 132300z_4 + 127008z_5 - 88200z_6 \\ & + 43200z_7 - 14175z_8 + 28800z_9 - 252z_{10}\} \\ & + O\left(h^{10}\right), \end{aligned} \tag{53}$$

$$\begin{aligned} Z_1'(N) = {} & \left(\frac{N}{2520}\right)\{7381z_N - 25200z_{N-1} \\ & + 56700z_{N-2} - 100800z_{N-3} + 132300z_{N-4} \\ & - 127008z_{N-5} + 88200z_{N-6} - 43200z_{N-7} \\ & + 14175z_{N-8} - 28800z_{N-9} + 252z_{N-10}\} + O\left(h^{10}\right) \end{aligned} \tag{54}$$

and if we use approximating collocation algorithm for the solution of (3), then approximation of derivative conditions at end points is defined as

$$\begin{aligned} Z_2'(0) = {} & \left(\frac{N}{12}\right)\{-25z_0 + 48z_1 - 36z_2 + 16z_3 - 3z_4\} \\ & + O\left(h^4\right), \end{aligned} \tag{55}$$

$$\begin{aligned} Z_2'(N) = {} & \left(\frac{N}{12}\right) \\ & \cdot \{25z_N - 48z_{N-1} + 36z_{N-2} - 16z_{N-3} + 3z_{N-4}\} \\ & + O\left(h^4\right). \end{aligned} \tag{56}$$

3.2.2. Adjustment of Boundary Conditions. Still we need twelve and eight more conditions for systems (22) and (28), respectively, to get stable systems for the solution of (3). For this we made some adjustment of boundary conditions for systems (22) and (28), which are defined below.

Case 1. If we use interpolating collocation algorithm for the approximate solution of (3), then we define six conditions at left end points and six conditions at the right end points. Since subdivision scheme (1) reproduces nine-degree (i.e., tenth order) polynomials, so we define boundary conditions of order ten for solution of (22). For simplicity only left end points $z_{-7}, z_{-6}, z_{-5}, z_{-4}, z_{-3}, z_{-2}$ are discussed and the values of right end points $z_{N+2}, z_{N+3}, z_{N+4}, z_{N+5}, z_{N+6}, z_{N+7}$ can be treated similarly.

The values $z_{-7}, z_{-6}, z_{-5}, z_{-4}, z_{-3}, z_{-2}$ can be determined by the nonic polynomial $q(x)$ interpolating (x_i, z_i), $2 \le i \le 7$. Precisely, we have

$$z_{-i} = q(-x_i), \quad i = 2,3,4,5,6,7, \tag{57}$$

where

$$q(x_i) = \sum_{j=1}^{10} \binom{10}{j} (-1)^{j+1} Z_1(x_{i-j}). \tag{58}$$

Since by (17) $Z_1(x_i) = z_i$ for $i = 2,3,4,5,6,7$, then, by replacing x_i by $-x_i$, we have

$$q(-x_i) = \sum_{j=1}^{10} \binom{10}{j} (-1)^{j+1} z_{-i+j}. \tag{59}$$

Hence the following boundary conditions can be employed at the left end:

$$\sum_{j=0}^{10} \binom{10}{j} (-1)^{j} z_{-i+j} = 0, \quad i = 7,6,5,4,3,2. \tag{60}$$

Similarly, for the right end, we can define $z_i = q(-x_i)$, $i = N+2, N+3, N+4, N+5, N+6, N+7$, and

$$q(x_i) = \sum_{j=1}^{10} \binom{10}{j} (-1)^{j+1} z_{i-j}. \tag{61}$$

So we have the following boundary conditions at the right end:

$$\sum_{j=0}^{10} \binom{10}{j} (-1)^{j} z_{i-j} = 0, \quad i = N+2, N+3, N+4, N+5, N+6, N+7. \tag{62}$$

Finally, we get the following new system of $(N + 17)$ linear equations with $(N + 17)$ unknowns $\{z_i\}$, in which $N + 1$ equations are obtained from (29) and (35), four equations from boundary conditions (19), and twelve from boundary conditions (60) and (62).

Case 2. If we use approximating collocation algorithm (28) for the solution of (3) then we need eight more conditions. So in this case, we define four extra conditions at the left end points and four conditions at the right end points by some extrapolation method. Since the subdivision scheme reproduces cubic (i.e., fourth order) polynomial, so we define quartic polynomial for the adjustment of boundary treatment. The values $z_{-4}, z_{-3}, z_{-2}, z_{-1}$ are determined by the quartic polynomial $p(x)$ interpolating (x_i, z_i). This polynomial is defined as

$$z_{-i+1} = p(-x_{i+1}), \quad i = 1,2,3,4, \tag{63}$$

where

$$p(x_{i+1}) = \sum_{j=1}^{4} \binom{4}{j} (-1)^{j+1} Z_2(x_{i-j+1}). \tag{64}$$

Since by (23) $Z_2(x_i) = z_i$ for $i = 1,2,3,4$, then, by replacing x_i by $-x_i$, we have

$$p(-x_{i+1}) = \sum_{j=1}^{4} \binom{4}{j} (-1)^{j+1} z_{-i+j+1}. \tag{65}$$

Hence the following boundary conditions can be employed at the left end:

$$\sum_{j=0}^{4} \binom{4}{j} (-1)^{j} z_{-i+j+1} = 0, \quad i = 1,2,3,4. \tag{66}$$

Similarly, for the right end, we can define $z_{i+1} = p(x_{i+1})$, $i = N+1, N+2, N+3, N+4$, and

$$p(x_{i+1}) = \sum_{j=1}^{4} \binom{4}{j} (-1)^{j+1} z_{i-j+1}. \tag{67}$$

So we have the following boundary conditions at the right end:

$$\sum_{j=0}^{4} \binom{4}{j} (-1)^{j} z_{i-j+1} = 0, \quad i = N+1, N+2, N+3, N+4. \tag{68}$$

Finally, we get a following new system of $(N + 13)$ linear equations with $(N + 13)$ unknowns $\{z_i\}$, in which $N + 1$ equations are obtained from (41) and (47), four equations from boundary conditions (25), and eight from boundary conditions (66) and (68).

3.3. Stable Systems of Linear Equations. In this section, we present stable systems of linear equations for both interpolating and approximating collocation algorithms.

3.3.1. Stable System for Interpolating Collocation Algorithm. From (29) and (35), we get the following undetermined system of $(N + 1)$ equations with $(N + 17)$ unknowns $\{z_i\}$:

$$A_1 \mathbb{Z}_1 = G_1, \tag{69}$$

where the matrices $A_1, \mathbb{Z}_1$, and G_1 of orders $(N+1)\times(N+17)$, $N + 17$, and $N + 1$, respectively, are given by

$$A_1=\begin{pmatrix}\phi^{iv}_{-8}&\phi^{iv}_{-7}&\phi^{iv}_{-6}&\phi^{iv}_{-5}&\phi^{iv}_{-4}&\phi^{iv}_{-3}&\phi^{iv}_{-2}&\phi^{iv}_{-1}&q_0&\phi^{iv}_{1}&\phi^{iv}_{2}&\phi^{iv}_{3}&\phi^{iv}_{4}&\phi^{iv}_{5}&\phi^{iv}_{6}&\phi^{iv}_{7}&\phi^{iv}_{8}&\cdots&0&0&0\\ 0&\phi^{iv}_{-8}&\phi^{iv}_{-7}&\phi^{iv}_{-6}&\phi^{iv}_{-5}&\phi^{iv}_{-4}&\phi^{iv}_{-3}&\phi^{iv}_{-2}&\phi^{iv}_{-1}&q_1&\phi^{iv}_{1}&\phi^{iv}_{2}&\phi^{iv}_{3}&\phi^{iv}_{4}&\phi^{iv}_{5}&\phi^{iv}_{6}&\phi^{iv}_{7}&\cdots&0&0&0\\ 0&0&\phi^{iv}_{-8}&\phi^{iv}_{-7}&\phi^{iv}_{-6}&\phi^{iv}_{-5}&\phi^{iv}_{-4}&\phi^{iv}_{-3}&\phi^{iv}_{-2}&\phi^{iv}_{-1}&q_2&\phi^{iv}_{1}&\phi^{iv}_{2}&\phi^{iv}_{3}&\phi^{iv}_{4}&\phi^{iv}_{5}&\phi^{iv}_{6}&\cdots&0&0&0\\ 0&0&0&\phi^{iv}_{-8}&\phi^{iv}_{-7}&\phi^{iv}_{-6}&\phi^{iv}_{-5}&\phi^{iv}_{-4}&\phi^{iv}_{-3}&\phi^{iv}_{-2}&\phi^{iv}_{-1}&q_3&\phi^{iv}_{1}&\phi^{iv}_{2}&\phi^{iv}_{3}&\phi^{iv}_{4}&\phi^{iv}_{5}&\cdots&0&0&0\\ \cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&\cdots&\cdots&\cdots&\cdots&\cdots&\phi'''_7&\phi'''_8&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&\cdots&\cdots&\cdots&\cdots&\cdots&\phi'''_6&\phi'''_7&\phi'''_8\end{pmatrix}, \quad (70)$$

$$\mathbb{Z}_1=\left(z_{-8},z_{-7},z_{-6},z_{-5},z_{-4},\dots,z_{N+8}\right)^T, \quad (71)$$

$$G_1=\left(b_0h^4,b_1h^4,b_2h^4,b_4h^3,\dots,b_Nh^4\right)^T, \quad (72)$$

where $\phi^{iv}_j=\phi^{iv}(j)$ and $q_j=\phi^{iv}_0-a_jh^4$.

For obtaining the unique solution of (69), we made some adjustment of boundary conditions in previous section which is defined in (53), (54), (60), and (62). By using this adjustment, we get the following system of $(N+17)$ linear equations with $(N+17)$ unknowns $\{z_i\}$, defined as

$$D_1\mathbb{Z}_1=R_1, \quad (73)$$

where the coefficient matrix

$$D_1=\left(B_0^T,A_1^T,B_1^T\right)^T. \quad (74)$$

A_1 is defined by (70); B_0 and B_1 are defined as

$$B_0=\begin{pmatrix}0&1&-10&45&-120&210&-252&210&-120&45&-10&1&0&0&0&0&0&0&0&\cdots&0&0\\ 0&0&1&-10&45&-120&210&-252&210&-120&45&-10&1&0&0&0&0&0&0&\cdots&0&0\\ 0&0&0&1&-10&45&-120&210&-252&210&-120&45&-10&1&0&0&0&0&0&\cdots&0&0\\ 0&0&0&0&1&-10&45&-120&210&-252&210&-120&45&-10&1&0&0&0&0&\cdots&0&0\\ 0&0&0&0&0&1&-10&45&-120&210&-252&210&-120&45&-10&1&0&0&0&\cdots&0&0\\ 0&0&0&0&0&0&1&-10&45&-120&210&-252&210&-120&45&-10&1&0&0&\cdots&0&0\\ 0&0&0&0&0&0&0&0&\frac{7381N}{2520}&\frac{25200N}{2520}&-\frac{56700N}{2520}&\frac{100800N}{2520}&-\frac{132300N}{2520}&\frac{127008N}{2520}&-\frac{88200N}{2520}&\frac{43200N}{2520}&-\frac{14175N}{2520}&\frac{2800N}{2520}&-\frac{252N}{2520}&\cdots&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&\cdots&0&0\end{pmatrix}. \quad (75)$$

First six rows of B_0 are obtained from (60), second last row is obtained from (53), and last row is taken from given boundary conditions $Z_1(0)$ which is defined in (19) and

$$B_1=\begin{pmatrix}0&0&\cdots&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&\cdots&\frac{N}{10}&-\frac{10N}{9}&\frac{45N}{8}&-\frac{120N}{7}&35N&-\frac{252N}{5}&\frac{105N}{2}&-40N&\frac{45N}{2}&-10N&\frac{7381N}{2520}&0&0&0&0&0&0&0&0\\ 0&0&\cdots&0&0&1&-10&45&-120&210&-252&210&-120&45&-10&1&0&0&0&0&0&0\\ 0&0&\cdots&0&0&0&1&-10&45&-120&210&-252&210&-120&45&-10&1&0&0&0&0&0\\ 0&0&\cdots&0&0&0&0&1&-10&45&-120&210&-252&210&-120&45&-10&1&0&0&0&0\\ 0&0&\cdots&0&0&0&0&0&1&-10&45&-120&210&-252&210&-120&45&-10&1&0&0&0\\ 0&0&\cdots&0&0&0&0&0&0&1&-10&45&-120&-120&210&-252&210&-120&45&-10&1&0\\ 0&0&\cdots&0&0&0&0&0&0&0&1&-10&45&-120&210&-252&210&-120&45&-10&1&0\end{pmatrix}. \quad (76)$$

First row of B_1 is obtained from $Z_1(N)$ which is defined in (19), second row is obtained from (54), and the last six rows are obtained from (62) and $\mathbb{Z}_1$ which is defined in (71) and R_1 is defined as

$$R_1 = \left(0, 0, 0, 0, 0, 0, \beta, \alpha, G_1^T, \gamma, \omega, 0, 0, 0, 0, 0, 0\right)^T, \quad (77)$$

where G_1 is defined by (72).

3.3.2. Stable System for Approximating Collocation Algorithm. From (41) and (47), we get the following undetermined system of $(N + 1)$ equations with $(N + 13)$ unknowns $\{z_i\}$:

$$A_2\mathbb{Z}_2 = G_2, \quad (78)$$

where the matrices A_2, $\mathbb{Z}_2$, and G_2 of orders $(N+1)\times(N+13)$, $N + 13$, and $N + 1$, respectively, are given by

$$A_2 = \begin{pmatrix} \Phi^{iv}_{-6} & \Phi^{iv}_{-5} & \Phi^{iv}_{-4} & \Phi^{iv}_{-3} & \Phi^{iv}_{-2} & \Phi^{iv}_{-1} & \Upsilon_0 & \Phi^{iv}_{1} & \Phi^{iv}_{2} & \Phi^{iv}_{3} & \Phi^{iv}_{4} & \Phi^{iv}_{5} & \Phi^{iv}_{6} & \cdots & 0 & 0 & 0 \\ 0 & \Phi^{iv}_{-6} & \Phi^{iv}_{-5} & \Phi^{iv}_{-4} & \Phi^{iv}_{-3} & \Phi^{iv}_{-2} & \Phi^{iv}_{-1} & \Upsilon_1 & \Phi^{iv}_{1} & \Phi^{iv}_{2} & \Phi^{iv}_{3} & \Phi^{iv}_{4} & \Phi^{iv}_{5} & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \Phi^{iv}_{-6} & \Phi^{iv}_{-5} & \Phi^{iv}_{-4} & \Phi^{iv}_{-3} & \Phi^{iv}_{-2} & \Phi^{iv}_{-1} & \Upsilon_2 & \Phi^{iv}_{1} & \Phi^{iv}_{3} & \Phi^{iv}_{4} & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Phi^{iv}_{-6} & \Phi^{iv}_{-5} & \Phi^{iv}_{-4} & \Phi^{iv}_{-3} & \Phi^{iv}_{-2} & \Phi^{iv}_{-1} & \Upsilon_3 & \Phi^{iv}_{2} & \Phi^{iv}_{3} & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \Phi'''_5 & \Phi'''_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \Phi'''_4 & \Phi'''_5 & \Phi'''_6 \end{pmatrix}, \quad (79)$$

$$\mathbb{Z}_2 = \left(z_{-6}, z_{-5}, z_{-4}, z_{-3}, z_{-2}, \dots, z_{N+6}\right)^T, \quad (80)$$

$$G_2 = \left(b_0h^4, b_1h^4, b_2h^4, b_4h^3, \dots, b_Nh^4\right)^T, \quad (81)$$

where $\Phi^{iv}_j = \Phi^{iv}(j)$ and $\Upsilon_j = \Phi^{iv}_0 - a_jh^4$.

In order to get the unique solution of system (78), we have defined some extra conditions in (55), (56), (66), and (68). By using these extra conditions we get the following system of $(N + 13)$ linear equations with $(N + 13)$ unknowns $\{z_i\}$:

$$D_2\mathbb{Z}_2 = R_2, \quad (82)$$

where the coefficient matrix

$$D_2 = \left(\mathbb{B}_0^T, A_2^T, \mathbb{B}_1^T\right)^T. \quad (83)$$

A_2 is defined by (79); $\mathbb{B}_0$ and $\mathbb{B}_1$ are defined as

$$\mathbb{B}_0 = \begin{pmatrix} 0 & 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 6 & 4 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{-251N}{12} & \dfrac{48N}{12} & \dfrac{-36N}{12} & \dfrac{16N}{12} & \dfrac{-3N}{12} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \quad (84)$$

First four rows of $\mathbb{B}_0$ are obtained from (66), second last row is obtained from (55), and the last row is taken from the given boundary conditions $Z_2(0)$ which is defined in (25) and

$$\mathbb{B}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \dfrac{3N}{12} & -\dfrac{16N}{12} & \dfrac{36N}{12} & -\dfrac{48N}{12} & \dfrac{25N}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 \end{pmatrix}. \quad (85)$$

First row of $\mathbb{B}_1$ is obtained from $Z_2(N)$ which is defined in (25), second row is obtained from (56), and the last four rows are obtained from (68); $\mathbb{Z}_2$ is defined in (80) and R_2 is defined as

$$R_2 = \left(0,0,0,0,0,0,\beta,\alpha,G_2^T,\gamma,\omega,0,0,0,0,0,0\right)^T, \quad (86)$$

where G_2 is defined by (81). Hence to obtain the approximate solution of the fourth order boundary value problem (3) by interpolating and approximating collocation algorithms we need to solve systems (73) and (82), respectively.

3.4. Existence of the Solution. In this section, we discuss the nonsingularity of the coefficient matrices D_1 and D_2 defined in (74) and (83), respectively. We observe that the coefficient matrices D_1 and D_2 are neither symmetric nor diagonally dominant. However it can be shown that D_1 and D_2 are nonsingular. Since D_1 and D_2 are band matrices with half bandwidth 9 and 7, numerical complexities for solving the linear systems using Gaussian elimination are about $81(N+17)$ and $49(N+13)$ multiplications, respectively. For large N, the matrices are almost symmetric except the first and last eight rows and columns of D_1 and first and last four rows and columns of D_2 due to the boundary conditions. Therefore we first consider their symmetric part, that is, square symmetric matrices E_1 and E_2 of orders $N+1$ defined as

$$E_1 = \begin{pmatrix} \phi_0^{iv} & \phi_1^{iv} & \phi_2^{iv} & \phi_3^{iv} & \phi_4^{iv} & \phi_5^{iv} & \phi_6^{iv} & \phi_7^{iv} & \phi_8^{iv} & \cdots & 0 & 0 & 0 & 0 \\ \phi_{-1}^{iv} & \phi_0^{iv} & \phi_1^{iv} & \phi_2^{iv} & \phi_3^{iv} & \phi_4^{iv} & \phi_5^{iv} & \phi_6^{iv} & \phi_7^{iv} & \cdots & 0 & 0 & 0 & 0 \\ \phi_{-2}^{iv} & \phi_{-1}^{iv} & \phi_0^{iv} & \phi_1^{iv} & \phi_2^{iv} & \phi_3^{iv} & \phi_4^{iv} & \phi_5^{iv} & \phi_6^{iv} & \cdots & 0 & 0 & 0 & 0 \\ \phi_{-3}^{iv} & \phi_{-2}^{iv} & \phi_{-1}^{iv} & \phi_0^{iv} & \phi_1^{iv} & \phi_2^{iv} & \phi_3^{iv} & \phi_4^{iv} & \phi_5^{iv} & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \phi_4^{iv} & \phi_3^{iv} & \phi_2^{iv} & \phi_1^{iv} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \phi_3^{iv} & \phi_2^{iv} & \phi_1^{iv} & \phi_0^{iv} \end{pmatrix},$$

$$E_2 = \begin{pmatrix} \Phi_0^{iv} & \Phi_1^{iv} & \Phi_2^{iv} & \Phi_3^{iv} & \Phi_4^{iv} & \Phi_5^{iv} & \Phi_6^{iv} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \Phi_{-1}^{iv} & \Phi_0^{iv} & \Phi_1^{iv} & \Phi_2^{iv} & \Phi_3^{iv} & \Phi_4^{iv} & \Phi_5^{iv} & \Phi_6^{iv} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \Phi_{-2}^{iv} & \Phi_{-1}^{iv} & \Phi_0^{iv} & \Phi_1^{iv} & \Phi_2^{iv} & \Phi_3^{iv} & \Phi_4^{iv} & \Phi_5^{iv} & \Phi_6^{iv} & \cdots & 0 & 0 & 0 & 0 \\ \Phi_{-3}^{iv} & \Phi_{-2}^{iv} & \Phi_{-1}^{iv} & \Phi_0^{iv} & \Phi_1^{iv} & \Phi_2^{iv} & \Phi_3^{iv} & \Phi_4^{iv} & \Phi_5^{iv} & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \Phi_4^{iv} & \Phi_3^{iv} & \Phi_2^{iv} & \Phi_1^{iv} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \Phi_3^{iv} & \Phi_2^{iv} & \Phi_1^{iv} & \Phi_0^{iv} \end{pmatrix}. \quad (87)$$

So E_1 and E_2 are symmetric matrices obtained from D_1 and D_2, respectively. It can be shown that E_1 and E_2 are nonsingular when N increase. The nonsingularity of the matrices E_1 and E_2 is shown in Table 5 by finding their determinants. The matrices E_1 and E_2 remain nonsingular for $N \leqslant 500$ and $N \leqslant 100$, respectively. For large N the determinants of the matrices may or may not be equal to zero. The nonsingularity of the matrices D_1 and D_2 has been checked by finding their eigenvalues for $N \leqslant 500$ and for $N \leqslant 100$, respectively. Since the eigenvalues for both the matrices are nonzero, then by [17] matrices D_1 and D_2 are nonsingular. However the matrices D_1 for $N \geqslant 500$ and D_2 for $N \geqslant 100$ may or may not be nonsingular. Therefore we claim that systems of (73) and (82) are stable.

3.5. Error Estimation. In this section, we discuss the approximation properties of the interpolating and approximating collocation algorithms. Since schemes (1) and (2) reproduce polynomial curves of degree nine and three, so by [10] schemes have approximation orders ten and four, respectively. Here we present our main results for error estimation. Proof of these results is similar to the proof of Proposition [12, 14].

Proposition 7. *Suppose the exact solution $y(x) \in C^4[0,1]$ and $\{z_i\}$ are obtained by (73); then absolute error by interpolating collocation algorithm is*

$$\left\|\mathrm{err}_1(x)\right\|_\infty = \left\|Z_1^{(l)}(x) - y^{(l)}(x)\right\|_\infty = O\left(h^{3-l}\right), \quad l = 0,1,2,3, \quad (88)$$

where l denotes the order of derivative.

Proposition 8. *Suppose the exact solution $y(x) \in C^4[0,1]$ and $\{z_i\}$ are obtained by (82); then absolute error by approximating collocation algorithm is*

$$\left\|\mathrm{err}_2(x)\right\|_\infty = \left\|Z_2^{(l)}(x) - y^{(l)}(x)\right\|_\infty = O\left(h^{3-l}\right), \quad l = 0,1,2,3. \quad (89)$$

Table 5: Determinants of the matrices.

N	E_1	E_2
10	$-8667/56$	1.11401292×10^3
50	-177183	3.2517495×10^2
100	-552709050	0.753776508473953
200	$-5.033491472 \times 10^{36}$	...
300	$-4.477989536 \times 10^{71}$	...
400	3987757210720454	...
500	3987757210720454	...

4. Numerical Examples and Comparison

In this section, the interpolating and approximating collocation algorithms described in Section 3 are tested on the problems given below. Absolute errors between exact and approximate solutions are also calculated. For the sake of comparisons, we also tabulated the results in this section. Graphical illustrations of solutions are presented.

4.1. Numerical Examples. Here we find the numerical solutions of some of the boundary value problems arising in the mathematical modeling of viscoelastic and inelastic flows and so forth.

Example 1. Consider the fourth order linear boundary value problem

$$y^{\text{iv}}(x) + xy = -\left(8 + 7x + x^3\right)e^x, \quad 0 < x < 1, \tag{90}$$

subject to the boundary conditions

$$\begin{aligned} y(0) &= y(1) = 0, \\ y'(0) &= 1, \\ y'(1) &= -e. \end{aligned} \tag{91}$$

By comparing the above problem with (3), we have $a(x) = -x$ and $b(x) = -(8 + 7x + x^3)e^x$. The exact solution for the above problem is $y = x(1 - x)e^x$.

Here we present the numerical solution of the above problem by interpolating and approximating collocation algorithms.

Solution by Interpolating Collocation Algorithm. In this method, by solving the system of linear equations (73) at $N = 10$, we obtain the approximate solution (17) of (90) where $\{z_i\}$, $-8 \leq i \leq 18$, are

$$\begin{aligned} &\{-0.736798818, -0.643536669, -0.554623724, \\ &-0.467529093, -0.379799398, -0.289680649, \\ &-0.196215973, -0.099344276, 0.0000, \\ &0.0997855, 0.196780435, 0.286445566, \\ &0.362823930, 0.4184258296, 0.444109480, \\ &0.428957387, 0.360149377, 0.222833881, \\ &0.000, -0.327646133, -0.781795936, \\ &-1.386637856, -2.168934169, -3.158065904, \\ &-4.386034900, -5.887410047, -7.698656102\}. \end{aligned} \tag{92}$$

Solution by Approximating Collocation Algorithm. In this method, we solve the system of linear equations (82) at $N = 10$ and get solution (23) of (90) where z_i, $-6 \leq i \leq 16$, are

$$\begin{aligned} &\{240576495.97007, 346838.29496, 518.9347067, \\ &-0.27530396, -0.192116155, -0.098802082, \\ &0.0000, 0.099651832, 0.195515154, \\ &0.282951708, 0.357323235, 0.412290577, \\ &0.439106327, 0.426338357, 0.359800241, \\ &0.223138586, 0, -0.325968911, \\ &-0.771121538, -1.351811276, 759.3687002, \\ &508693.855464, 352843031.1188856\}. \end{aligned} \tag{93}$$

Example 2. Consider the following fourth order linear boundary value problem,

$$y^{\text{iv}}(x) = \left(x^4 + 14x^3 + 49x^2 + 32x - 12\right)e^x \qquad 0 \leq x \leq 1 \tag{94}$$

with

$$y(0) = y(1) = y'(0) = y'(1), \tag{95}$$

corresponds to the bending of a thin beam clamped at both ends. The unique solution of (94) is

$$y(x) = x^2(1 - x)^2 e^x. \tag{96}$$

Solution by Interpolating Collocation Algorithm. By using this method, we solve system (73) at $N = 10$ and get solution (17) of (94) where $\{z_i\}$, $-8 \leq i \leq 18$, are

$$\begin{aligned} &\{-1.292188418, 0.538942241, 0.376822099, \\ &0.254979738, 0.157378182, 0.084436111, \\ &0.035355124, 0.008209932, 0, 0.006713642, \\ &0.023450361, 0.044645810, 0.064445941, \\ &0.077282873, 0.078714604, 0.066604028, \\ &0.042729735, 0.014941769, 0, 0.027260994, \\ &0.143411251, 0.418480429, 0.953407669, \\ &1.889477424, 3.419988190, 5.826254985, \\ &6.993262174\}. \end{aligned} \tag{97}$$

Solution by Approximating Collocation Algorithm. In this method, by solving system of linear equations (82) at $N = 10$, we obtain the approximate solution (23) of (94) where $\{z_i\}$, $-6 \le i \le 16$, are

$$\{1.711850528\times 10^{11},\ -2.46996092\times 10^{8},\ 3.5607957013\times 10^{5},\ -5.335463543\times 10^{5},\ 0.06034270857,\ 0.013560666,\ 0,\ 0.010510642930557,\ 0.035942526106682,\ 0.067145581105052,\ 0.094969739502347,\ 0.110264932875244,\ 0.107216514892050,\ 0.086571522615334,\ 0.056424246024606,\ 0.024868975122817,\ 0,\ 0.01008838960215368,\ 0.002698096580563341,\ -3.898592698\times 10^{3},\ 2.597091086\times 10^{6},\ -1.801597056\times 10^{9},\ 1.2486272867\times 10^{12}\}. \quad (98)$$

Example 3. Consider the boundary value problem

$$y^{(\text{iv})} - y = -4\left(2x\cos(x) + 3\sin(x)\right) \quad (99)$$

with boundary conditions

$$y(0) = 0,\quad y(1) = 0,\quad y'(0) = -1,\quad y'(1) = 2\sin(1). \quad (100)$$

The exact solution of this problem is $y = (x^2 - 1)\sin(x)$.

Solution by Interpolating Collocation Algorithm. Here, we solve system (73) at $N = 10$ and get solution (17) of (99) where $\{z_i\}$, $-8 \le i \le 18$, are

$$\{0.0600152940,\ 0.18160335240,\ 0.25833493990,\ 0.29216208360,\ 0.2870204837,\ 0.1824477485,\ 0.09700293,\ 0.000000000,\ -0.1001836945,\ -0.1951709144,\ -0.2768975252,\ -0.3379193763,\ -0.3717065517,\ -0.372918743,\ -0.3376558661,\ -0.26367852,\ -0.1505934691,\ 0.000000000,\ 0.1844062122,\ 0.3967742453,\ 0.6290856272,\ 0.87118909250,\ 1.11088175850,\ 1.33403484280,\ 1.52476071733,\ 1.6661398769\}. \quad (101)$$

Solution by Approximating Collocation Algorithm. By solving system of linear equations (82) at $N = 10$, we obtain the approximate solution (23) of (99) where $\{z_i\}$, $-6 \le i \le 16$, are

$$\{115585876195.8271,\ 166975210.2723412,\ 240725.36249556,\ 360.797052040,\ 0.1912504195,\ 0.0989171622,\ 0.000000,\ -0.0988737231,\ -0.1910766633,\ -0.2699814765,\ -0.328960819,\ -0.361387346,\ -0.3620271786,\ -0.3276693903,\ -0.2568529513,\ -0.1481168313,\ 0.00000,\ 0.1889585727,\ 0.4202199172,\ 648.2401207,\ 432617.012,\ 300071307.86,\ 207719711144.92\}. \quad (102)$$

4.2. Comparison. The numerical results of Examples 1, 2, and 3 by interpolating and approximating collocation algorithms are presented in Tables 6, 7, and 8, respectively. The maximum absolute errors in the solution of Examples 1, 2, and 3 obtained by interpolating and approximating collocation algorithms are given in Table 9. Graphical representation of these results is shown in Figures 1, 2, and 3. In these figures solid curve represents the exact solutions, dashed lines represent approximate solutions obtained by (73), and dotted lines represent the approximate solutions obtained by (82). Following is the comparison of the numerical solutions obtained by proposed algorithms and other approaches of this type of boundary value problems:

(i) From the above results we see that approximating schemes based collocation algorithms give better results than interpolating schemes based collocation algorithms.

(ii) Example 1 is also solved by [15]. He solved this problem by second order finite difference method and obtained the maximum absolute errors at different step sizes h. We observe that the maximum absolute error at the step size $h = 1/10$ by the proposed approximating collocation algorithm is better than the maximum absolute error obtained by [15] at step size $h = 1/16$. The comparison of proposed methods with second order finite difference method of [15] at difference step sizes is shown in Table 10.

(iii) Example 2 is also solved by [18] by quintic spline based collocation methods. We observe that order of error approximation obtained by [18] and proposed approximating collocation algorithm is the same.

5. Conclusion

In this paper, we have presented interpolating and approximating collocation algorithms based on interpolating and

Table 6: Numerical results of Example 1.

x_i	Analytic solution Y	Approximate solution Z_1 by interpolating collocation algorithm	Approximate solution Z_2 by approximating collocation algorithm	$\|err_1(x_i)\|_\infty$	$\|err_2(x_i)\|_\infty$
0.0	0	0	0	0	0
0.1	0.09946538	0.0997855152	0.0996518317	0.00032013	0.00018645
0.2	0.19542444	0.1967804015	0.1955151540	0.00135596	0.00009071
0.3	0.28347035	0.2864455108	0.2829517080	0.00297516	0.00051864
0.4	0.35803793	0.3628238578	0.3573232345	0.00478593	0.00071469
0.5	0.41218032	0.4184257512	0.4122905765	0.00624543	0.00011025
0.6	0.43730851	0.4441094075	0.4391063274	0.00680090	0.00179781
0.7	0.42288807	0.4289573314	0.4263383569	0.00606926	0.00345029
0.8	0.35608655	0.3601493430	0.3598002408	0.00406279	0.00371369
0.9	0.22136428	0.2228338692	0.2231385861	0.00146959	0.00177431
1.0	0	0	0	0	0

Table 7: Numerical results of Example 2.

x_i	Analytic solution Y	Approximate solution Z_1 by interpolating collocation algorithm	Approximate solution Z_2 by approximating collocation algorithm	$\|err_1(x_i)\|_\infty$	$\|err_2(x_i)\|_\infty$
0.0	0	0	0	0	0
0.1	0.008951884	0.006713642	0.010510643	0.002238243	0.001558758
0.2	0.0312679106	0.023450361	0.035942526	0.007817550	0.004674615
0.3	0.059528773	0.044645810	0.067145581	0.0148829636	0.007616808
0.4	0.085929102	0.064445941	0.094969740	0.021483162	0.009040637
0.5	0.103045079	0.077282873	0.110264933	0.025762206	0.007219853
0.6	0.104954043	0.078714604	0.107216515	0.026239439	0.002262472
0.7	0.088806494	0.066604028	0.086571523	0.022202467	0.002234972
0.8	0.056973848	0.042729735	0.056424256	0.014244112	0.000549602
0.9	0.019922785	0.014941769	0.024868975	0.004981016	0.0049461810
1.0	0	0	0	0	0

Table 8: Numerical results of Example 3.

x_i	Analytic solution Y	Approximate solution Z_1 by interpolating collocation algorithm	Approximate solution Z_2 by approximating collocation algorithm	$\|err_1(x_i)\|_\infty$	$\|err_2(x_i)\|_\infty$
0.0	0	0	0	0	0
0.1	−0.09883508	−0.10018369	−0.098873723	0.0013486	0.000038641
0.2	−0.19072256	−0.19517091	−0.1910766633	0.0044484	0.00035411
0.3	−0.26892339	−0.27689752	−0.2699814765	0.0079741	0.001058
0.4	−0.32711141	−0.33791938	−0.328960819	0.010808	0.0018494
0.5	−0.35956915	−0.37170655	−0.361387346	0.012137	0.0018182
0.6	−0.36137118	−0.37291874	−0.3620271786	0.011548	0.00065600
0.7	−0.32855102	−0.33765587	−0.3276693903	0.0091048	0.00088163
0.8	−0.25824819	−0.26367852	−0.2568529513	0.0054303	0.0013952
0.9	−0.14883211	−0.15059347	−0.1481168313	0.0017614	0.00071528
1.0	0	0	0	0	0

Table 9: Maximum absolute errors of Examples 1, 2, and 3.

Example	Max. absolute errors by interpolating collocation algorithm	Max. absolute errors by approximating collocation algorithm
1	6.8010×10^{-3}	3.7137×10^{-3}
2	2.6239×10^{-2}	0.9041×10^{-2}
3	1.2137×10^{-2}	0.18494×10^{-2}

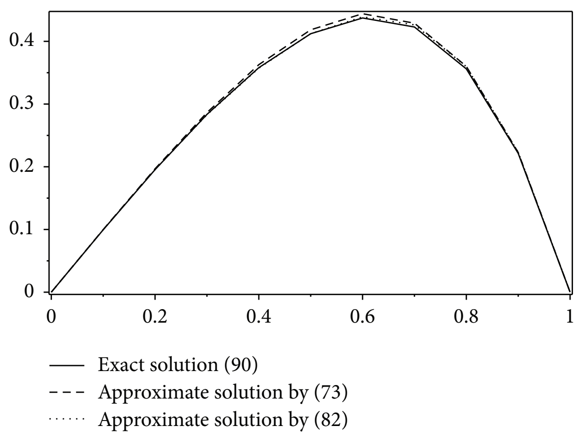

Figure 1: Comparison between exact and approximate solutions of Example 1 obtained by interpolating and approximating collocation algorithms.

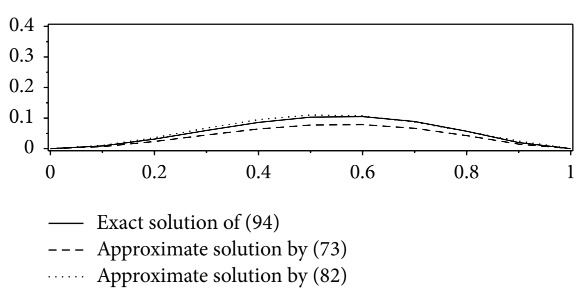

Figure 2: Comparison between exact and approximate solutions of Example 2 obtained by interpolating and approximating collocation algorithms.

approximating subdivision schemes for the solution of linear fourth order boundary value problems. The proposed algorithms have been applied on different linear fourth order boundary value problems. Results show that the approximating collocation algorithm gives better results comparative to interpolating collocation algorithm. We have also observed that the accuracy of the solution can be improved by choosing different subdivision schemes with the proper adjustment of boundary conditions.

Approximating subdivision scheme based collocation algorithm gives better results comparative to second order finite difference method. However, approximating subdivision scheme based collocation algorithm and quintic spline based collocation algorithm have the same order of approximation.

Table 10: Comparison of Example 1 with different methods.

h	Max. absolute errors by interpolating collocation algorithm	Max. absolute errors by approximating collocation algorithm	Second order finite difference method [15]
$\frac{1}{4}$	⋯	⋯	8.50×10^{-2}
$\frac{1}{8}$	⋯	1.4098×10^{-2}	2.09×10^{-2}
$\frac{1}{10}$	6.8010×10^{-3}	3.7137×10^{-3}	⋯
$\frac{1}{16}$	6.7469×10^{3}	4.4572×10^{-3}	5.27×10^{-3}

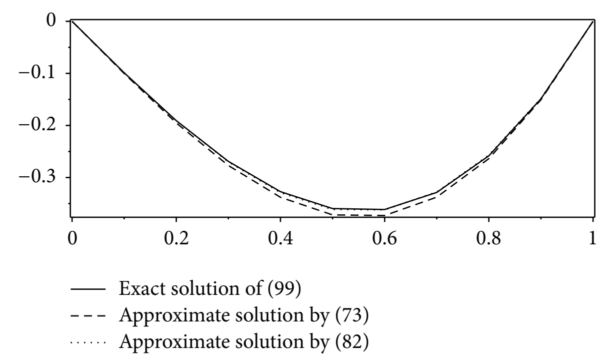

Figure 3: Comparison between exact and approximate solutions of Example 3 obtained by interpolating and approximating collocation algorithms.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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