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Research Article

Finite Frequency H_∞ Filtering for Time-Delayed Singularly Perturbed Systems

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This paper investigates the problem of finite frequency (FF) H_∞ filtering for time-delayed singularly perturbed systems. Our attention is focused on designing filters guaranteeing asymptotic stability and FF H_∞ disturbance attenuation level of the filtering error system. By the generalized Kalman-Yakubovich-Popov (KYP) lemma, the existence conditions of FF H_∞ filters are obtained in terms of solving an optimization problem, which is delay-independent. The main contribution of this paper is that systematic methods are proposed for designing H_∞ filters for delayed singularly perturbed systems.

1. Introduction

Several physical processes are on one hand of high order and on the other hand complex, what returns their analysis and especially their control, with the aim of certain objectives, very delicate. However, knowing that these systems possess variables evolving in various speeds (temperature, pressure, intensity, voltage...), it could be possible to model these systems by singularly perturbed technique [1, 2]. They arise in many physical systems such as electrical power systems and electrical machines (e.g., an asynchronous generator, a DC motor, and electrical converters), electronic systems (e.g., oscillators), mechanical systems (e.g., fighter aircrafts), biological systems (e.g., bacterial-yeast-virus cultures, heart), and also economic systems with various competing sectors. This class of systems has two time scales, namely, "fast" and "slow" dynamics. This makes their analysis and control more complicated than regular systems. Nevertheless, they have been studied extensively [3–5].

As the dual of control problem, the filtering problems of dynamic systems are of great theoretical and practical meaning in the field of control and signal processing and the filtering problem has always been a concern in the control theory [6–8]. The state estimation of singularly perturbed systems also has attracted considerable attention over the past decades and a great number of results have been proposed in various schemes, such as Kalman filtering [9, 10] and H_∞ filtering [11].

Like all kinds of systems which can contain a time-delay in their dynamic or in their control, the singularly perturbed systems can also contain a delay, which has been studied in many references such as [12–15]. For example, Fridman [12] has considered the effect of small delay on stability of the singularly perturbed systems. In [13, 14], the controllability problem of nonstandard singularly perturbed systems with small state delay and stabilization problem of nonstandard singularly perturbed systems with small delays both in state and control have been studied, respectively. In [15] a composite control law for singularly perturbed bilinear systems via successive Galerkin approximation was presented. However, there is seldom literature dealing with the synthesis design for the delayed singularly perturbed systems, which is the main motivation of this paper.

With the fundamental theory-generalized KYP lemma proposed by Iwasaki and Hara [16], the applications using the GKYP lemma have been sprung up in recent years [17–23]. Actually, if noise belongs to a finite frequency (FF) range, more accurately, low/middle/high frequency (LF/MF/HF) range, design methods for the entire range will be much conservative due to overdesign. Consequently, the FF approach has a wide application range. In future work, we can use this approach to Markovain jump systems [24, 25].

Lately, using FF approach to analyze and design control problems becomes a new interesting in singularly perturbed systems [19–21]. In [20] the author has studied the H_∞ control problem for singularly perturbed systems within the finite frequency. In [21] the positive control problem for singularly perturbed systems has been studied based on the generalized KYP lemma. The idea is that design in the finite frequency is critical to singularly perturbed systems since the transfer function of singularly perturbed systems has two different frequencies, that is, the high frequency and low frequency, which are corresponding to the fast subsystem and the low subsystem separately. Hence the idea based on the generalized KYP lemma actually is constructing the design problem in separate time scale and also separate frequency scale. Obviously, it could be seen that the conservativeness is much less than existing results. As far as the author's knowledge, there is seldom literature referring to the filtering design problem within the separate frequencies for delayed singularly perturbed systems, which is also the main motivation of this paper.

In this paper we are concerned with FF H_∞ filtering for singularly perturbed systems with time-delay. The frequencies of the exogenous noises are assumed to reside in a known rectangular region, which is the most remarkable difference of our results compared with existing ones. Based on the generalized KYP lemma, we first obtained an FF bounded real lemma in the parameter-independent sense. Filter design methods will be derived by a simple procedure. The main contribution of this paper is summarized as follows: the standard H_∞ filtering for singularly perturbed systems has been extended to the FF H_∞ filtering for singularly perturbed systems with time-delay, and systematic filter design methods have been proposed.

The paper is organized as follows. Section 2 gives the problem formulation and preliminaries used in the next sections. The main results are given in Section 3. An illustrative example is given in Section 4. And conclusions are given in the last section.

Notation. For a matrix X, its transpose is denoted by X^T; N_X is an arbitrary matrix whose column forms a basis of the null space of X. $\mathrm{Sym}(X)$ indicates $X^T + X$. $\sigma_{\max}(\cdot)$ denotes maximum singular value of transfer function. $\mathrm{diag}\{\cdots\}$ stands for a block-diagonal matrix.

2. Problem Formulation and Preliminaries

The following time-delayed singularly perturbed systems will be considered in this paper:

$$\begin{aligned}
\dot{x}_1(t) &= A_{01}x_1(t) + A_{02}x_2(t) + A_{11}x_1(t-d) \\
&\quad + A_{12}x_2(t-d) + B_1\omega(t), \\
\varepsilon\dot{x}_2(t) &= A_{03}x_1(t) + A_{04}x_2(t) + A_{13}x_1(t-d) \\
&\quad + A_{14}x_2(t-d) + B_2\omega(t), \\
y(t) &= C_1x_1(t) + C_2x_2(t) + C_3x_1(t-d) + C_4x_2(t-d), \\
Z(t) &= G_1x_1(t) + G_2x_2(t) + G_3x_1(t-d) + G_4x_2(t-d),
\end{aligned} \tag{1}$$

where $x_1(t) \in R^{n_1}$ is "slow" state and $x_2(t) \in R^{n_2}$ is "fast" state. Denote $n_x = n_1 + n_2$; $x(t) = [x_1(t)^T \;\; x_2(t)^T]^T \in R^{n_x}$ is the state vector; $y(t) \in R^{n_y}$ is the measured output signal, $z(t) \in R^{n_z}$ is the signal to be estimated, and $\omega(t) \in R^{n_\omega}$ is the noise input signal in the $L_2[0,+\infty)$ functional space domain. Time-delay d is known and time-invariant $\Phi(t)$ is the known initial condition in the domain $[-d,0]$; let $E_\varepsilon = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \varepsilon I_{n_2} \end{bmatrix}$, $A = \begin{bmatrix} A_{01} & A_{02} \\ A_{03} & A_{04} \end{bmatrix}$, $A_d = \begin{bmatrix} A_{11} & A_{12} \\ A_{13} & A_{14} \end{bmatrix}$, $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, $C = [C_1 \;\; C_2]$, $C_d = [C_3 \;\; C_4]$, $G = [G_1 \;\; G_2]$, $G_d = [G_3 \;\; G_4]$, A_{ij} $(i = 0,1; j = 1,\ldots,4)$, B_k, C_k, G_k, $k = 1,2$, C_l, G_l, $l = 3,4$, be appropriate constant matrices. Then the above system can be regulated as

$$\begin{aligned}
E_\varepsilon\dot{x} &= Ax(t) + A_dx(t-d) + B\omega(t), \\
y(t) &= Cx(t) + C_dx(t-d), \qquad (2) \\
z(t) &= Gx(t) + G_dx(t-d), \\
x(t) &= \Phi(t) \quad \forall t = [-d,0]. \qquad (3)
\end{aligned}$$

First, we give the following assumption on noise signal $\omega(t)$.

Assumption 1. Noise signal $\omega(t)$ is only defined in the low, medium, and high frequency domains

$$\Omega \triangleq \begin{cases} \{\omega \in R \mid |\omega| \geq \omega_h,\ \omega_h \geq 0\} & \text{(high frequency)} \\ \{\omega \in R \mid \omega_1 \leq |\omega| \leq \omega_2,\ \omega_1 \leq \omega_2\}, & \text{(media frequency)} \\ \{\omega \in R \mid |\omega| \leq \omega_l,\ \omega_l \geq 0\} & \text{(low frequency)}. \end{cases} \tag{4}$$

Remark 2. By the generalized KYP lemma in [16] and by an appropriate choice Φ and Ψ, the set Z can be specialized to define a certain range of the frequency variable ω. For the continuous time setting, we have $\Phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Z = \{j\omega : \omega \in R\}$, where Ω is defined in (4); in this situation, Ψ can be chosen as

$$\Psi \triangleq \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & \omega_h^2 \end{bmatrix}, \\ \quad \text{when } \omega \in \{\omega \in R \mid |\omega| \geq \omega_h,\ \omega_h \geq 0\} \\ \quad \text{(high frequency)} \\ \begin{bmatrix} -1 & j\omega_c \\ -j\omega_c & -\omega_1\omega_2 \end{bmatrix}, \\ \quad \text{when } \omega \in \{\omega \in R \mid \omega_1 \leq |\omega| \leq \omega_2,\ \omega_1 \leq \omega_2\}, \\ \quad \text{(media frequency)} \\ \begin{bmatrix} -1 & 0 \\ 0 & \omega_l^2 \end{bmatrix}, \\ \quad \text{when } \omega \in \{\omega \in R \mid |\omega| \leq \omega_l,\ \omega_l \geq 0\} \\ \quad \text{(low frequency)}, \end{cases} \tag{5}$$

where $\omega_c := (\omega_1 + \omega_2)/2$.

The main objective of this paper is to design the following full-order linear filtering:

$$\begin{aligned} \dot{x}_F(t) &= A_F x_F(t) + B_F y(t), \quad x_F(0) = 0, \\ z_F(t) &= C_F x_F(t) + D_F y(t), \end{aligned} \tag{6}$$

where $x_F(t) \in R^{n_x}$ is the state vector, $y(t) \in R^{n_y}$ is the measured output signal, $z_F(t) \in R^{n_z}$ is the output for the filtering systems, and A_F, B_F, C_F, D_F are the filtering matrices to be solved. Combine (2) and (6) and let $\overline{x}(t) = [x(t)^T \quad x_F(t)^T]^T$; the following filtering error system is obtained:

$$\begin{aligned} \overline{E}_\varepsilon \dot{\overline{x}}(t) &= \overline{A}\overline{x}(t) + \overline{A}_d \overline{x}(t-d) + \overline{B}\omega(t), \\ e(t) &= \overline{C}\overline{x}(t) + \overline{C}_d \overline{x}(t-d), \end{aligned} \tag{7}$$

$$\overline{x}(t) = \left[\Phi^T(t), 0\right]^T \quad \forall t = [-d, 0], \tag{8}$$

where $e(t) = z(t) - z_F(t)$ is the filtering error and $\overline{E}_\varepsilon = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & \varepsilon I_{n_2} & 0 \\ 0 & 0 & I_{n_x} \end{bmatrix}$; the filtering system matrices are

$$\begin{aligned} \overline{A} &= \begin{bmatrix} A_{01} & A_{02} & 0 \\ A_{03} & A_{04} & 0 \\ B_F C_1 & B_F C_2 & 0 \end{bmatrix}, \\ \overline{A}_d &= \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{13} & A_{14} & 0 \\ B_F C_3 & B_F C_4 & 0 \end{bmatrix}, \qquad \overline{B} = \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix}, \\ \overline{C} &= \left[G_1 - D_F C_1 \quad G_2 - D_F C_2 \quad -C_F\right], \\ \overline{C}_d &= \left[G_3 - D_F C_3 \quad G_4 - D_F C_4 \quad 0\right]. \end{aligned} \tag{9}$$

The transfer function of the filtering error system in (7) from ω to z is given by

$$G_e(s) = \left(\overline{C} + e^{-ds}\overline{C}_d K\right)\left(\overline{E}_\varepsilon s I - \overline{A} - e^{-ds}\overline{A}_d K\right)^{-1}\overline{B}, \tag{10}$$

where "s" is the Laplace operator.

Due to the asymptotic stability of the filtering error system (7) depends on system (2), while delayed system (2) does not include input channel for the input signal, so the following assumption is given.

Assumption 3. Time-delayed singularly perturbed system in (2) is asymptotically stable.

Now the problems to be solved can be summarized as follows.

Problem 4. For the continuous time-delay system in (2), find a full-order linear filtering (6), such that the filtering error system in (7) satisfies the following conditions.

(1) The filtering error system in (7) is asymptotically stable.

(2) Given the appropriate positive real γ, under the zero initial condition, the following finite frequency index is satisfied:

$$\sup \sigma_{\max}\left[G_e(j\omega)\right] < \gamma, \quad \forall \omega \in \Omega. \tag{11}$$

To conclude this section, we give the following technical lemma that plays an instrumental role in deriving our results.

Lemma 5 ((projection lemma) [26]). *Let X, Z, and Σ be given. There exists a matrix Y satisfying*

$$\mathrm{Sym}\left(X^T Y Z\right) + \Sigma < 0 \tag{12}$$

if and only if the following projection inequalities are satisfied:

$$N_X^T \Sigma N_X < 0, \qquad N_Z^T \Sigma N_Z < 0. \tag{13}$$

3. Main Results

3.1. FF H_∞ Filtering Performance Analysis. To ensure the asymptotic stability and FF specification in (11) for the filtering error system, we need to resort to the generalized KYP lemma in [16]. Based on this, the following lemma can be obtained.

Lemma 6. *(i) Given system in (2) and scalars $\gamma > 0$, $\varepsilon > 0$, $d > 0$, the filtering error system in (7) is asymptotically stable for all $\omega \in \Omega$ and satisfies the specifications in (11) if there exist matrices $P_\varepsilon \in R^{2n_x \times 2n_x}$ with the form of (16), $0 < P_{11} \in R^{n_1 \times n_1}$, $0 < P_{13} \in R^{n_2 \times n_2}$, $P_2 \in R^{n_x \times n_x}$, $P_3 \in R^{n_x \times n_x}$, $O_\varepsilon \in R^{2n_x \times 2n_x}$ with the form of (17), $0 < O_{11} \in R^{n_1 \times n_1}$, $0 < O_{13} \in R^{n_2 \times n_2}$, $O_2 \in R^{n_x \times n_x}$, $O_3 \in R^{n_x \times n_x}$ such that $E_\varepsilon P_{1\varepsilon} > 0$, $E_\varepsilon O_{1\varepsilon} > 0$, and matrices $0 < Q \in R^{2n_x \times 2n_x}$, $R_1 \in R^{n_x \times n_x}$, and $0 \le R_2 \in R^{n_x \times n_x}$ that satisfy*

$$F_0^T \Xi_0 F_0 + F_1^T \Xi_1 F_1 + F_2^T \left(\Phi \otimes P_\varepsilon + \Psi \otimes Q\right) F_2 < 0, \tag{14}$$

$$F_3^T \Xi_2 F_3 + F_4^T \left(\Phi \otimes O_\varepsilon\right) F_4 < 0, \tag{15}$$

where

$$P_\varepsilon = \begin{bmatrix} P_{1\varepsilon} & 0 \\ P_2 & P_3 \end{bmatrix}, \quad P_{1\varepsilon} = \begin{bmatrix} P_{11} & \varepsilon P_{12}^T \\ P_{12} & P_{13} \end{bmatrix}, \tag{16}$$

$$O_\varepsilon = \begin{bmatrix} O_{1\varepsilon} & 0 \\ O_2 & O_3 \end{bmatrix}, \quad O_{1\varepsilon} = \begin{bmatrix} O_{11} & \varepsilon O_{12}^T \\ O_{12} & O_{13} \end{bmatrix}, \tag{17}$$

$$\begin{aligned} F_0 &= \begin{bmatrix} \overline{C} & \overline{C}_d & 0 \\ 0 & 0 & I \end{bmatrix}, \qquad F_1 = \begin{bmatrix} \overline{A} & \overline{A}_d & \overline{B} \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}, \\ F_2 &= \begin{bmatrix} \overline{A} & \overline{A}_d & \overline{B} \\ I & 0 & 0 \end{bmatrix}, \qquad F_3 = \begin{bmatrix} \overline{A} & \overline{A}_d \\ I & 0 \\ 0 & I \end{bmatrix}, \\ F_4 &= \begin{bmatrix} \overline{A} & \overline{A}_d \\ I & 0 \end{bmatrix}, \qquad \Xi_0 \cong \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}, \\ \Xi_i &= \mathrm{diag}\left\{0 \quad R_i \quad -R_i\right\} \quad (i = 1, 2). \end{aligned} \tag{18}$$

(ii) Given $d > 0$, if there exist matrices $P_0 \in R^{2n_x \times 2n_x}$ with the form of (16) when $\varepsilon = 0$, $0 < P_{11} \in R^{n_1 \times n_1}$, $0 < P_{13} \in R^{n_2 \times n_2}$, $P_2 \in R^{n_x \times n_x}$, $P_3 \in R^{n_x \times n_x}$, $O_0 \in R^{2n_x \times 2n_x}$ with the form of (17) when $\varepsilon = 0$, $0 < O_{11} \in R^{n_1 \times n_1}$, $0 < O_{13} \in R^{n_2 \times n_2}$, $O_2 \in R^{n_x \times n_x}$, $O_3 \in R^{n_x \times n_x}$, $0 < Q \in R^{2n_x \times 2n_x}$, $R_1 \in R^{n_x \times n_x}$, and $0 \le R_2 \in R^{n_x \times n_x}$ such that (14), (15) are feasible for $\varepsilon = 0$ then filtering error system (7) is asymptotic stable and satisfies the specifications in (11) for all small enough $\varepsilon > 0$ and $0 \le \overline{d} \le d$.

Proof. (i) Let $\xi(t) \widehat{=} \begin{bmatrix} \overline{x}(t) \\ \overline{x}(t-d) \\ \omega(t) \end{bmatrix}$.

Give the following Lyapunov-Krasovskii functional

$$V_1(t) \widehat{=} V_{1,1}(t) + V_{1,2}(t), \tag{19}$$

where

$$\begin{aligned} V_{1,1}(t) &= \overline{x}^T(t)\overline{E}_\varepsilon P_\varepsilon \overline{x}(t), \\ V_{1,2}(t) &= \int_{t-d}^{t} \overline{x}^T(\eta) R_1 \overline{x}(\eta)\, d\eta, \\ \dot{V}_{1,1} &= \dot{\overline{x}}^T \overline{E}_\varepsilon P_\varepsilon \overline{x}(t) + \overline{x}^T(t) P_\varepsilon^T \overline{E}_\varepsilon \dot{\overline{x}}(t) \\ &= \left(\overline{A}\overline{x}(t) + \overline{A}_d K \overline{x}(t-d) + \overline{B}\omega(t)\right)^T \\ &\quad \cdot P_\varepsilon \overline{x}(t) + \overline{x}^T(t) \\ &\quad \cdot P_\varepsilon^T \left(\overline{A}\overline{x}(t) + \overline{A}_d K \overline{x}(t-d) + \overline{B}\omega(t)\right), \\ \dot{V}_{1,2} &= \overline{x}^T(t) R_1 x(t) - \overline{x}^T(t-d) R_1 \overline{x}(t-d). \end{aligned} \tag{20}$$

Then

$$\begin{aligned} \dot{V}_1 &= \dot{V}_{1,1}(t) + \dot{V}_{1,2}(t) \\ &= \xi(t)^T \left[F_2^T (\Phi \otimes P_\varepsilon) F_2 + F_1^T \Xi_1 F_1\right] \xi(t). \end{aligned} \tag{21}$$

Define the following performance index:

$$J = \int_0^\infty \left[e^T(t) e(t) - \gamma^2 \omega^T(t) \omega(t)\right] dt. \tag{22}$$

Using the zero initial conditions, we could get

$$\begin{aligned} J &\le \int_0^\infty \left[e^T(t) e(t) - \gamma^2 \omega^T(t) \omega(t)\right] dt + V_1(\infty) - V_1(0) \\ &= \int_0^\infty \xi^T(t) \left[F_0^T \Xi_0 F_0 + F_1^T \Xi_1 F_1 + F_2^T (\Phi \otimes P_\varepsilon) F_2\right] \xi(t). \end{aligned} \tag{23}$$

Let $\Theta = F_0^T \Xi_0 F_0 + F_1^T \Xi_1 F_1 + F_2^T (\Phi \otimes P_\varepsilon) F_2$ and use the Parseval equality to get

$$\int_0^\infty \xi^T(t) \Theta \xi(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \xi_s^T \Theta \xi_s\, d\omega. \tag{24}$$

Considering frequency $\omega \in \Omega$ and combining (23) and (24), we know that $\xi_s^T \Theta \xi_s < 0$ is a sufficient condition of $J < 0$, for all $\omega \in \Omega$. Additionally, due to $\Gamma_\lambda F_2 \xi_s = 0$, we know that ξ_s is a zero space of $\Gamma_\lambda F_2$.

By the zero space theory, we know that

$$N_{\Gamma_\lambda F_2}^T \Theta N_{\Gamma_\lambda F_2} < 0 \Longrightarrow \xi_s^T \Theta \xi_s < 0. \tag{25}$$

Then by generalized KYP lemma in [16], the sufficient condition for $N_{\Gamma_\lambda F_2}^T \Theta N_{\Gamma_\lambda F_2} < 0$ is existing $P \in R^{2n_x \times 2n_x}$, $0 < Q \in R^{2n_x \times 2n_x}$, such that the following inequality is satisfied:

$$\Theta + F_2^T (\Phi \otimes P + \Psi \otimes Q) F_2 < 0. \tag{26}$$

By redefining $P_\varepsilon + P$ as P_ε, we obtain inequality (14) that completes the first part of (i).

As for the asymptotic stability, the Lyapunov-Krasovskii functional can be reselected as follows:

$$V_2(t) \widehat{=} V_{2,1}(t) + V_{2,2}(t), \tag{27}$$

where $V_{2,1}(t) = \overline{x}^T(t)\overline{E}_\varepsilon O_\varepsilon \overline{x}(t)$ and $V_{2,2}(t) = \int_{t-d}^{t} \overline{x}^T(\eta) R_2 \overline{x}(\eta) d\eta$.

Then similar to the proof of the first part of (i), it could be shown that inequality (15) can guarantee the asymptotic stability of (7).

(ii) If (14), (15) are feasible for $\varepsilon = 0$, then they are feasible for all small enough $\varepsilon > 0$ and thus, due to (i), filtering error system in (7) is asymptotic stable and satisfies the specifications in (11) for these values $\varepsilon > 0$. Furthermore, linear matrix inequalities (14), (15) are convex with respect to d; hence they are feasible for some d; then they are feasible for all $0 \le \overline{d} \le d$. □

Remark 7. Though the essence of the proof of Lemma 6 follows from that of Theorem 1 in [6], there are also some differences in Lemma 6. First, in Lemma 6, time-delayed singularly perturbed systems are considered, which are totally different from the regular systems. Since the singularly perturbed systems have perturbed parameter ε, the existence of ε can lead to the ill-conditioned numerical problems, so here comes part (ii) of Lemma 6. Furthermore, considering the special structure of singularly perturbed systems, note that, in the proof of part (i), the selected Lyapunov-Krasovskii functionals are different from the regular systems.

To facilitate and reduce the conservatism of the filter design using the projection lemma, we present an alternative of Lemma 6.

Lemma 8. *Given delayed systems in (2) and scalars $\gamma > 0$, $d > 0$, filter in (6) exists such that the filtering error system in (7) is asymptotically stable and satisfies the specifications in (11) if there exist matrices $P_0 \in R^{2n_x \times 2n_x}$ with the form of (16), $O_0 \in R^{2n_x \times 2n_x}$ with the form of (17), $0 < Q \in R^{2n_x \times 2n_x}$, $R_1 \in R^{n_x \times n_x}$, and $0 \le R_2 \in R^{n_x \times n_x}$ and $Y_i \in R^{4n_x \times 2n_x}$ $(i = 1, 2)$ satisfying*

$$\begin{bmatrix} -I_{n_z} & \left[0_{n_z \times 2n_x} \;\; \overline{C} \;\; \overline{C}_d \;\; 0\right] \\ * & \operatorname{diag}\left\{\Xi_1 + \Xi(P_0) + \Xi(Q), -\gamma^2 I_{n_\omega}\right\} + \operatorname{Sym}\left(X^T Y_1 Z_1\right) \end{bmatrix} < 0, \tag{28}$$

$$\Xi_2 + \Xi(O_0) + \operatorname{Sym}\left(X_2^T Y_2 Z_2\right) < 0, \tag{29}$$

with

$$\Xi(P_0) = \begin{bmatrix} \Phi_c \otimes P_0 & 0_{4n_x \times n_x} \\ 0_{n_x \times 4n_x} & 0_{n_x \times n_x} \end{bmatrix},$$
$$\Xi(O_0) = \begin{bmatrix} \Phi_c \otimes O_0 & 0_{4n_x \times n_x} \\ 0_{n_x \times 4n_x} & 0_{n_x \times n_x} \end{bmatrix},$$
$$\Xi(Q) = \begin{bmatrix} \Psi_c \otimes Q & 0_{4n_x \times n_x} \\ 0_{n_x \times 4n_x} & 0_{n_x \times n_x} \end{bmatrix},$$
$$X_1 = \lfloor I_{4n_x} \quad 0_{4n_x \times (n_x + n_\omega)} \rfloor, \qquad Z_1 = \left[-I_{2n_x} \quad \overline{A} \quad \overline{A}_d \quad \overline{B}\right],$$
$$X_2 = \lfloor I_{4n_x} \quad 0_{4n_x \times n_x} \rfloor, \qquad Z_2 = \left[-I_{2n_x} \quad \overline{A} \quad \overline{A}_d\right]; \tag{30}$$

the other notations are defined in (14) and (15).

Proof. Define $M = \left[0_{n_x \times 2n_x} \quad \overline{C} \quad \overline{C}_d \quad 0\right]$. By the Schur complement, (28) is equivalent to

$$\begin{aligned} &\operatorname{diag}\left\{\Xi_1 + \Xi(P_0) + \Xi(Q), -\gamma^2 I_{n_\omega}\right\} \\ &\quad + M^T M + \operatorname{Sym}\left(X_1^T Y_1 Z_1\right) < 0. \end{aligned} \tag{31}$$

By the definition of matrices X_1 and Z_1, one can choose $N_{X_1} = 0$ and $N_{Z_1} = \begin{bmatrix} \overline{A} & \overline{A}_d & \overline{B} \\ I_{2n_x} & 0 & 0 \\ 0 & I_{n_x} & 0 \\ 0 & 0 & I_{n_\omega} \end{bmatrix}$; thus the first inequality in (13) vanishes and then by Lemma 5, (31) is equivalent to

$$N_{Z_1}^* \left(\operatorname{diag}\left\{\Xi_1 + \Xi(P_0) + \Xi(Q), -\gamma^2 I_{n_\omega}\right\} + M^* M\right) N_{Z_1} < 0. \tag{32}$$

By calculation, we can obtain (32) is equivalent to (14) when $\varepsilon = 0$.

Similarly, by introducing the following null space,

$$N_{X_2} = 0, \qquad N_{Z_2} = \begin{bmatrix} \overline{A} & \overline{A}_d \\ I_{2n_x} & 0 \\ 0 & I_{n_x} \end{bmatrix}. \tag{33}$$

Using projection lemma, (29) is equivalent to the following inequality:

$$N_{Z_2}^T \left\{\Xi_2 + \Xi(O_0) + \operatorname{Sym}\left(X_2^T Y_2 Z_2\right)\right\} N_{Z_2} < 0. \tag{34}$$

By calculation, (34) is equivalent to (15) when $\varepsilon = 0$.

Thus, Lemma 8 is equivalent to Lemma 6. □

3.2. Design of FF H_∞ Filters. Lemma 8 does not give a solution to filter realization explicitly. Based on the result in the following section we focus on developing methods for designing FF H_∞ filters. The following result can be derived via specifying the structure of the slack matrices Y_1 and Y_2 in Lemma 8.

Theorem 9. *Given time-delayed systems in (2) and scalars $\gamma > 0$, $d > 0$, filter in (6) exists such that the filtering error system in (7) is asymptotically stable and satisfies the specifications in (11) if there exist matrices $P_0 \in R^{2n_x \times 2n_x}$ with the form of (16), $O_0 \in R^{2n_x \times 2n_x}$ with the form of (17), $0 < Q \in R^{2n_x \times 2n_x}$, $R_1 \in R^{n_x \times n_x}$, $0 \le R_2 \in R^{n_x \times n_x}$, $\Gamma_{i,j} \in R^{n_x \times n_x}$ $(i = 1, 2,\ j = 1, \ldots, 4)$, $\Gamma_5 \in R^{n_x \times n_x}$, $\Delta_1 \in R^{n_x \times n_x}$, $\Delta_2 \in R^{n_x \times n_y}$, $\Delta_3 \in R^{n_z \times n_x}$, $\Delta_4 \in R^{n_z \times n_y}$ such that the following matrices are satisfied for the given scalars κ_i $(i = 1, \ldots, 4)$:*

$$\begin{aligned} &\begin{bmatrix} -I_{n_z} & \Sigma \\ * & \operatorname{diag}\left\{\Xi_1 + \Xi(P_0) + \Xi(Q), -\gamma^2 I_{n_\omega}\right\} + \operatorname{Sym}(\Lambda_1) \end{bmatrix} \\ &\quad < 0, \\ &\quad \Xi_2 + \Xi(O_0) + \operatorname{Sym}(\Lambda_2) < 0, \end{aligned} \tag{35}$$

where

$$\Sigma \mathrel{\hat{=}} \lfloor 0_{n_z \times 2n_x} \quad G - \Delta_4 C \quad -\Delta_3 \quad G_d - \Delta_4 C_d \quad 0 \rfloor,$$
$$\Lambda_1 \mathrel{\hat{=}} \begin{bmatrix} -\Gamma_{1,1} & -\Gamma_5 & \Gamma_{1,1}A + \Delta_2 C & \Delta_1 & \Gamma_{1,1}A_d + \Delta_2 C_d & \Gamma_{1,1}B \\ -\Gamma_{1,2} & -\Gamma_5 & \Gamma_{1,2}A + \Delta_2 C & \Delta_1 & \Gamma_{1,2}A_d + \Delta_2 C_d & \Gamma_{1,2}B \\ -\Gamma_{1,3} & -\kappa_1\Gamma_5 & \Gamma_{1,3}A + \kappa_1\Delta_2 C & \kappa_1\Delta_1 & \Gamma_{1,3}A_d + \kappa_1\Delta_2 C_d & \Gamma_{1,3}B \\ -\Gamma_{1,4} & -\kappa_2\Gamma_5 & \Gamma_{1,4}A + \kappa_2\Delta_2 C & \kappa_2\Delta_1 & \Gamma_{1,4}A_d + \kappa_2\Delta_2 C_d & \Gamma_{1,4}B \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{36}$$
$$\Lambda_2 \mathrel{\hat{=}} \begin{bmatrix} -\Gamma_{2,1} & -\Gamma_5 & \Gamma_{2,1}A + \Delta_2 C & \Delta_1 & \Gamma_{2,1}A_d + \Delta_2 C_d \\ -\Gamma_{2,2} & -\Gamma_5 & \Gamma_{2,2}A + \Delta_2 C & \Delta_1 & \Gamma_{2,2}A_d + \Delta_2 C_d \\ -\Gamma_{2,3} & -\kappa_3\Gamma_5 & \Gamma_{2,3}A + \kappa_3\Delta_2 C & \kappa_3\Delta_1 & \Gamma_{2,3}A_d + \kappa_3\Delta_2 C_d \\ -\Gamma_{2,4} & -\kappa_4\Gamma_5 & \Gamma_{2,4}A + \kappa_4\Delta_2 C & \kappa_4\Delta_1 & \Gamma_{2,4}A_d + \kappa_4\Delta_2 C_d \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Ξ_i, $\Xi(P_0)$, $\Xi(Q)$, and $\Xi(O_0)$ are defined in (28) and (29). Moreover, if the previous conditions are satisfied, an acceptable state-space realization of the filter in (6) is given by

$$\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} = \begin{bmatrix} \Gamma_5^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Delta_1 & \Delta_2 \\ \Delta_3 & \Delta_4 \end{bmatrix}. \tag{37}$$

Proof. First, in order to prove Theorem 9, we just need to prove that (28) and (29) in Lemma 8 can be deduced from (35) in Theorem 9. It is noted that the slack matrix Y_2 has the following form:

$$Y_2 = \begin{bmatrix} \Gamma_{2,1} & \Gamma_5 \\ \Gamma_{2,2} & \Gamma_6 \\ \Gamma_{2,3} & \Gamma_7 \\ \Gamma_{2,4} & \Gamma_8 \end{bmatrix}. \tag{38}$$

Here, $\Gamma_{2,i}$ $(i = 1, \ldots, 4)$, Γ_j $(j = 5, \ldots, 8)$ are complex matrices with dimension $n_x \times n_x$. In fact, Γ_6 is nonsingular and can be implied from (24); then by multiplying Y_2 from the left side and the right side, respectively, with $I_1 = \operatorname{diag}\{I \;\; \Gamma_5\Gamma_6^{-1} \;\; I \;\; I\}$ and $I_2 = \operatorname{diag}\left\{I \;\; (\Gamma_5\Gamma_6^{-1})^T\right\}$, we could get

$$I_1 Y_2 I_2 = \begin{bmatrix} \Gamma_{2,1} & \Gamma_5\left(\Gamma_5\Gamma_6^{-1}\right)^T \\ \Gamma_5\Gamma_6^{-1}\Gamma_{2,2} & \Gamma_6\left(\Gamma_5\Gamma_6^{-1}\right)^T \\ \Gamma_{2,3} & \Gamma_7\left(\Gamma_5\Gamma_6^{-1}\right)^T \\ \Gamma_{2,4} & \Gamma_8\left(\Gamma_5\Gamma_6^{-1}\right)^T \end{bmatrix}. \tag{39}$$

Without loss of generality, we could restrict $\Gamma_5 \equiv \Gamma_6$.

On the other hand, to overcome the difficulty of filtering design, more work should be done. Next, we will consider Γ_7 and Γ_8 to be linearly Γ_5-dependent; that is, $\Gamma_7 = \kappa_3\Gamma_5$, $\Gamma_8 = \kappa_4\Gamma_5$, κ_3 and κ_4 are scalars, respectively. Similarly, the slack matrix Y_1 has the same structure restriction; that is,

$$Y_1 = \begin{bmatrix} \Gamma_{1,1} & \Gamma_5 \\ \Gamma_{1,2} & \Gamma_5 \\ \Gamma_{1,3} & \kappa_1\Gamma_5 \\ \Gamma_{1,4} & \kappa_2\Gamma_5 \end{bmatrix}, \qquad Y_2 = \begin{bmatrix} \Gamma_{2,1} & \Gamma_5 \\ \Gamma_{2,2} & \Gamma_5 \\ \Gamma_{2,3} & \kappa_3\Gamma_5 \\ \Gamma_{2,4} & \kappa_4\Gamma_5 \end{bmatrix}. \tag{40}$$

Define

$$\begin{bmatrix} \Delta_1 & \Delta_2 \\ \Delta_3 & \Delta_4 \end{bmatrix} = \begin{bmatrix} \Gamma_5 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_f & B_f \\ C_f & D_f \end{bmatrix}. \tag{41}$$

By substituting X_i, Z_i in (28) and (29) and Y_i of (40) into $X_i^T Y_i Z_i$, one can obtain $\Lambda_i = X_i^T Y_i Z_i$ $(i = 1, 2)$. Moreover, by using (41), one can also obtain $\Sigma \equiv M$. The proof is complete. □

4. Numerical Example

In this section, we use an example to illustrate the effectiveness and advantages of the design methods developed in this paper. Consider the singularly perturbed system with time-invariant delay in (2) with matrices given by

$$\begin{gathered} A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad A_d = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} -0.5 \\ 2 \end{bmatrix}, \\ C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \qquad C_d = \begin{bmatrix} 1 & 2 \end{bmatrix}, \qquad G = \begin{bmatrix} 2 & 1 \end{bmatrix}, \\ G_d = \begin{bmatrix} 0 & 0 \end{bmatrix}, \qquad \varepsilon = 0.1, \qquad d = 0.1. \end{gathered} \tag{42}$$

Suppose the frequencies $\omega_l = 1$, $\omega_h = 100$, we calculate the achieved minimum performance γ^* by using Theorem 9 in this paper. For brevity, the scalar parameters in Theorem 9 are given by $\kappa_1 = \kappa_2 = 5$, $\kappa_3 = \kappa_4 = 1$. The obtained minimum performance is $\gamma^* = 0.9976$, when $Q = 0$; the problem becomes a standard H_∞ filtering problem and the minimum performance of the nominal H_∞ filtering is $\gamma^* = 1.4533$.

And the obtained state-space matrices of H_∞ filtering are

$$\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} = \left[\begin{array}{cc|c} -1.5861 & -0.0552 & 0 \\ -0.2522 & -1.1106 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]. \tag{43}$$

5. Conclusions

This paper has studied the problem of FF H_∞ filtering for time-delayed singularly perturbed systems. The frequencies of the exogenous noise are assumed to reside in a known rectangular region and the standard H_∞ filtering for singularly perturbed systems has been extended to the FF H_∞ case. The generalized KYP lemma for singularly perturbed systems has been further developed to derive conditions that are more suitable for FF H_∞ performance synthesis with time-delay. Via structural restriction for the slack matrices, systematic methods have been proposed for the design of the filters that guarantee the asymptotic stability and FF H_∞ disturbance attenuation level of the filtering error system.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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