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Research Article

The $\{P, Q, k+1\}$-Reflexive Solution to System of Matrix Equations $AX = C$, $XB = D$

Chang-Zhou Dong[1,2] and Qing-Wen Wang[1]

[1] *Department of Mathematics, Shanghai University, Shanghai 200444, China*
[2] *School of Mathematics and Science, Shijiazhuang University of Economics, Shijiazhuang 050031, China*

Correspondence should be addressed to Qing-Wen Wang; wqw369@yahoo.com

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Let $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ be Hermitian and $\{k+1\}$-potent matrices; that is, $P^{k+1} = P = P^*$ and $Q^{k+1} = Q = Q^*$, where $(\cdot)^*$ stands for the conjugate transpose of a matrix. A matrix $X \in \mathbb{C}^{m\times n}$ is called $\{P, Q, k+1\}$-reflexive (antireflexive) if $PXQ = X$ ($PXQ = -X$). In this paper, the system of matrix equations $AX = C$ and $XB = D$ subject to $\{P, Q, k+1\}$-reflexive and antireflexive constraints is studied by converting into two simpler cases: $k = 1$ and $k = 2$. We give the solvability conditions and the general solution to this system; in addition, the least squares solution is derived; finally, the associated optimal approximation problem for a given matrix is considered.

1. Introduction

Throughout this paper, $\mathbb{R}^{m\times n}$, $\mathbb{C}^{m\times n}$, and $\mathbb{U}^{n\times n}$ stand for the sets of all $m \times n$ real matrices, the sets of all $m \times n$ complex matrices, and $n\times n$ unitary matrices, respectively. For $A \in \mathbb{C}^{m\times n}$, A^*, $r(A)$, $\operatorname{tr}(A)$, and $\|A\|$ represent the conjugate transpose, the rank, the trace, and the Frobenius norm of a matrix A, respectively. For $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $\langle A, B\rangle = \operatorname{tr}(AB^*)$ denotes the inner product of A and B. Therefore, $\mathbb{C}^{m\times n}$ is a complete inner product space and the norm of a matrix generated by the inner product is the Frobenius norm; that is, $\|A\| = \langle A, A\rangle^{1/2}$. And $A^\dagger$ denotes the Moore-Penrose inverse, namely, the unique matrix X, that satisfies the following four Penrose conditions:

$$\begin{aligned} AXA &= A, \\ XAX &= X, \\ (AX)^* &= AX, \\ (XA)^* &= XA. \end{aligned} \tag{1}$$

Let I_k be identity matrix of size k, and let J_k be cross-identity matrix of size k having the elements 1 along the southwest-northeast diagonal and the remaining elements being zeros. The symbol $O_{m\times n}$ is the $m \times n$ matrix of all zeros entries (if no confusion occurs, we will omit the subscript).

A matrix $P \in \mathbb{C}^{m\times m}$ is called Hermitian and $\{k+1\}$-potent matrix if $P^{k+1} = P = P^*$.

Definition 1. Let $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ be two Hermitian and $\{k+1\}$-potent matrices; we say that matrix $X \in \mathbb{C}^{m\times n}$ is $\{P, Q, k+1\}$-reflexive (antireflexive) if $PXQ = X$ ($PXQ = -X$).

Obviously, the centrosymmetric and centroskew matrices (when $P = Q = J_n$), reflexive and antireflexive matrices (when $P = Q$ is Hermitian involution), are the natural extensions of the $\{P, Q, k+1\}$-reflexive and the antireflexive matrices, respectively. These kinds of matrices play important roles in information theory, linear system theory, linear estimation theory, numerical analysis theory, and so forth (see [1–3]). If P and Q are involution, then the $\{P, Q, k+1\}$-reflexive and antireflexive matrices are called (P, Q)-symmetric and (P, Q)-skew symmetric matrices [4]. Moreover, the $\{P, k+1\}$-reflexive and antireflexive matrices mentioned in [5] are

also the particular case when $P = Q$ in Definition 1. More generally, [6] has obtained the $\{P, Q, k+1\}$-reflexive and antireflexive solutions to the matrix equation $AXB = C$.

Investigating the classical system of matrix equations

$$\begin{aligned} AX &= C, \\ XB &= D \end{aligned} \tag{2}$$

has attracted many authors' attention. For instance, [7] gave the necessary and sufficient conditions for the consistency of (2) and [8, 9] derived an expression for the general solution by using singular value decomposition of a matrix and generalized inverses of matrices, respectively. Moreover, many results have been obtained about system (2) with various constraints, such as bisymmetric, Hermitian, positive semidefinite, reflexive, and generalized reflexive solutions (see [10–27]). To our knowledge, so far, there has been little investigation of the $\{P, Q, k+1\}$-reflexive and antireflexive solutions to (2).

Motivated by the work mentioned above, we investigate the $\{P, Q, k+1\}$-reflexive and antireflexive solutions to (2). We also consider the optimal approximation problem, which can be described as follows: let E be a given matrix in $\mathbb{C}^{m\times n}$ and let S_X be the set of all $\{P, Q, k+1\}$-reflexive (antireflexive) solutions to (2); find $\widehat{X}$ such that

$$\left\|\widehat{X} - E\right\| = \min_{X\in S_X} \|X - E\|. \tag{3}$$

In many cases, the system of matrix equations (2) has no $\{P, Q, k+1\}$-reflexive (antireflexive) solution; hence, we need to further study its least squares solution, which can be described as follows: let $\mathbb{PQC}^{m\times n}$ ($\mathbb{APQC}^{m\times n}$) denote the set of all $\{P, Q, k+1\}$-reflexive (antireflexive) matrices in $\mathbb{C}^{m\times n}$; find X such that

$$\begin{aligned} &\min_{X\in\mathbb{PQC}^{m\times n}} \left(\|AX - C\|^2 + \|XB - D\|^2\right), \\ &\min_{X\in\mathbb{APQC}^{m\times n}} \left(\|AX - C\|^2 + \|XB - D\|^2\right). \end{aligned} \tag{4}$$

In Section 2, we present necessary and sufficient conditions for the existence of the $\{P, Q, k+1\}$-reflexive (antireflexive) solution to (2) and give an expression of this solution when the solvability conditions are met. In Section 3, we derive an optimal approximation solution to (3). In Section 4, we provide the least squares $\{P, Q, k+1\}$-reflexive (antireflexive) solution to (2). In Section 5, we give an algorithm and a numerical example to illustrate our results.

2. The $\{P, Q, k+1\}$-Reflexive (Antireflexive) Solution to (2)

The following lemma derived from [5, 6] characterizes the Hermitian and $\{k+1\}$-potent matrices.

Lemma 2. *Let $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ be Hermitian; then, P and Q are $\{k+1\}$-potent matrices if and only if P and Q are idempotent (i.e., $P^2 = P$, $Q^2 = Q$) when k is odd, or tripotent (i.e., $P^3 = P$, $Q^3 = Q$) when k is even. Moreover, there exist $U \in \mathbb{U}^{m\times m}$ and $V \in \mathbb{U}^{n\times n}$ such that*

$$\begin{aligned} P &= U \begin{bmatrix} I_p & \\ & O \end{bmatrix} U^*, \\ Q &= V \begin{bmatrix} I_q & \\ & O \end{bmatrix} V^*, \end{aligned} \tag{5}$$

if k is odd,

$$\begin{aligned} P &= U \begin{bmatrix} I_r & & \\ & -I_{p-r} & \\ & & O \end{bmatrix} U^*, \\ Q &= V \begin{bmatrix} I_s & & \\ & -I_{q-s} & \\ & & O \end{bmatrix} V^*, \end{aligned} \tag{6}$$

if k is even,

where $p = r(P)$ and $q = r(Q)$.

In the sequel, we always think that the Hermitian and $\{k+1\}$-potent matrices P and Q are fixed and have taken the form of (5) or (6). From Lemma 2, we can see that the general $\{k+1\}$-potent matrices P and Q can be reduced to only two simpler cases: $P^2 = P$, $Q^2 = Q$ (i.e., $k = 1$) and $P^3 = P$, $Q^3 = Q$ (i.e., $k = 2$). Consequently, we will discuss our problems through $\{P, Q, 2\}$-reflexive (antireflexive) and $\{P, Q, 3\}$-reflexive (antireflexive) constraints.

Lemma 3. *Let $X \in \mathbb{C}^{m\times n}$ be $\{P, Q, 2\}$-reflexive if and only if*

$$X = U \begin{bmatrix} X_{11} & \\ & O \end{bmatrix} V^*, \tag{7}$$

or $\{P, Q, 3\}$-reflexive if and only if

$$X = U \begin{bmatrix} X_1 & & \\ & X_2 & \\ & & O \end{bmatrix} V^*, \tag{8}$$

where $X_{11} \in \mathbb{C}^{p\times q}$, $X_1 \in \mathbb{C}^{r\times s}$, and $X_2 \in \mathbb{C}^{(p-r)\times(q-s)}$.

Assume that P and Q are idempotent (i.e., $k = 1$) and $PXQ = -X$; then, by premultiplying and postmultiplying it by P and Q, respectively, we get $PXQ = -PXQ$, which implies that $PXQ = O$, and, thus, $X = O$. In this case, the system of matrix equations (2) is inconsistent when $C, D \neq O$. Consequently, for the antireflexive case, we only consider the $\{P, Q, 3\}$-antireflexive solution.

From (6), $X \in \mathbb{C}^{m\times n}$ is $\{P, Q, 3\}$-antireflexive if and only if it can be written as

$$X = U \begin{bmatrix} O & X_{12} & O \\ X_{21} & O & O \\ O & O & O \end{bmatrix} V^*, \qquad X_{12} \in \mathbb{C}^{r\times(q-s)},\ X_{21} \in \mathbb{C}^{(p-r)\times s}. \tag{9}$$

In addition, the following lemma is well known (see, e.g., [17]).

Lemma 4. *The system of matrix equations (2) is consistent for unknown matrix X if and only if*

$$\begin{aligned} AA^{\dagger}C &= C, \\ DB^{\dagger}B &= D, \\ AD &= CB. \end{aligned} \tag{10}$$

In this case, the general solution is

$$A^{\dagger}C + DB^{\dagger} - A^{\dagger}ADB^{\dagger} + \left(I - A^{\dagger}A\right)U\left(I - BB^{\dagger}\right), \tag{11}$$

where U is arbitrary.

2.1. The $\{P, Q, 2\}$-Reflexive Solution to (2). Let $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ be Hermitian and $\{k+1\}$-potent matrices as given in (5), and partition

$$\begin{aligned} AU &= [A_1, A_2], \\ CV &= [C_1, C_2], \\ V^*B &= \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \\ U^*D &= \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}, \end{aligned} \tag{12}$$

where $A_1 \in \mathbb{C}^{l\times p}$, $C_1 \in \mathbb{C}^{l\times q}$, $B_1 \in \mathbb{C}^{q\times k}$, and $D_1 \in \mathbb{C}^{p\times k}$. Then, we have the following result.

Theorem 5. *Given $A \in \mathbb{C}^{l\times m}$, $C \in \mathbb{C}^{l\times n}$, $B \in \mathbb{C}^{n\times k}$, and $D \in \mathbb{C}^{m\times k}$, let A_1, C_1, B_1, D_1 be defined in (12); then, the system of matrix equations (2) is consistent for $\{P, Q, 2\}$-reflexive X if and only if*

$$\begin{aligned} A_1A_1^{\dagger}C_1 &= C_1, \\ D_1B_1^{\dagger}B_1 &= D_1, \\ A_1D_1 &= C_1B_1. \end{aligned} \tag{13}$$

In this case, the general solution is

$$X = U \begin{bmatrix} A_1^{\dagger}C_1 + D_1B_1^{\dagger} - A_1^{\dagger}A_1D_1B_1^{\dagger} + \left(I - A_1^{\dagger}A_1\right)U_1\left(I - B_1B_1^{\dagger}\right) & O \\ O & O \end{bmatrix} V^*, \tag{14}$$

where $U_1 \in \mathbb{C}^{p\times q}$ is arbitrary.

Proof. From formula (7) in Lemma 3, we deduce that (2) is consistent for $\{P, Q, 2\}$-reflexive X can be equivalently converted into solving the following system of matrix equations:

$$\begin{aligned} A_1X_{11} &= C_1, \\ X_{11}B_1 &= D_1, \\ X_{11} &\in \mathbb{C}^{p\times q}. \end{aligned} \tag{15}$$

It follows from Lemma 4 that (15) is consistent if and only if all equalities in (13) hold, and the expression of the general solution is (14). □

2.2. The $\{P, Q, 3\}$-Reflexive Solution to (2). Let $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ be Hermitian and $\{k+1\}$-potent matrices as given in (6), and partition

$$\begin{aligned} AU &= [A_1, A_2, A_3], \\ CV &= [C_1, C_2, C_3], \\ V^*B &= \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}, \\ U^*D &= \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix}, \end{aligned} \tag{16}$$

where $A_1 \in \mathbb{C}^{l\times r}$, $A_2 \in \mathbb{C}^{l\times(p-r)}$, $C_1 \in \mathbb{C}^{l\times s}$, $C_2 \in \mathbb{C}^{l\times(q-s)}$, $B_1 \in \mathbb{C}^{s\times k}$, $B_2 \in \mathbb{C}^{(q-s)\times k}$, $D_1 \in \mathbb{C}^{r\times k}$, and $D_2 \in \mathbb{C}^{(p-r)\times k}$.

Then, we have the following result.

Theorem 6. *Given $A \in \mathbb{C}^{l\times m}$, $C \in \mathbb{C}^{l\times n}$, $B \in \mathbb{C}^{n\times k}$, and $D \in \mathbb{C}^{m\times k}$, let A_1, A_2, C_1, C_2, B_1, B_2, D_1, D_2 be defined in (16); then, the system of matrix equations (2) is consistent for $\{P, Q, 3\}$-reflexive X if and only if*

$$\begin{aligned} A_1A_1^{\dagger}C_1 &= C_1, \\ D_1B_1^{\dagger}B_1 &= D_1, \end{aligned}$$

$$\begin{aligned} A_1D_1 &= C_1B_1, \\ A_2A_2^\dagger C_2 &= C_2, \\ D_2B_2^\dagger B_2 &= D_2, \\ A_2D_2 &= C_2B_2. \end{aligned} \tag{17}$$

In this case, the general solution is

$$X = U\begin{bmatrix} X_1 & O & O \\ O & X_2 & O \\ O & O & O \end{bmatrix}V^*, \tag{18}$$

where $X_1 = A_1^\dagger C_1 + D_1B_1^\dagger - A_1^\dagger A_1D_1B_1^\dagger + (I - A_1^\dagger A_1)U_{11}(I - B_1B_1^\dagger)$, $X_2 = A_2^\dagger C_2 + D_2B_2^\dagger - A_2^\dagger A_2D_2B_2^\dagger + (I - A_2^\dagger A_2)U_{22}(I - B_2B_2^\dagger)$, *and* U_{11}, U_{22} *are arbitrary.*

Proof. From formula (8) in Lemma 3, we deduce that (2) is consistent for $\{P, Q, 3\}$-reflexive X can be equivalently converted into solving the following matrix equations:

$$\begin{aligned} A_1X_1 &= C_1, \\ X_1B_1 &= D_1, \\ A_2X_2 &= C_2, \\ X_2B_2 &= D_2, \\ X_1 \in \mathbb{C}^{r\times s},\ X_2 &\in \mathbb{C}^{(p-r)\times(q-s)}. \end{aligned} \tag{19}$$

It follows from Lemma 4 that (19) is consistent if and only if all equalities in (17) hold, and the expression of the general solution is (18). □

Remark 7. If P and Q are tripotent and $PXQ = -X$, then $PX(-Q) = X$ and $(-Q)$ is tripotent. Hence, the $\{P, Q, 3\}$-antireflexive constraint can be reduced to the $\{P, Q, 3\}$-reflexive case. Similar to Theorem 6, the conclusion is omitted.

3. Solution to the Optimal Approximation Problem

This section solves the optimal approximation problem; that is, suppose that the solvability conditions of the system of matrix equations (2) in Theorem 5 and Theorem 6 hold; then, we derive the following results.

Lemma 8 (see [28]). *Suppose that* $A \in \mathbb{C}^{q\times m}$, $\Delta \in \mathbb{C}^{q\times q}$ *and* $\Gamma \in \mathbb{C}^{m\times m}$, *where* $\Delta^2 = \Delta = \Delta^T$ *and* $\Gamma^2 = \Gamma = \Gamma^T$. *Then,*

$$\|A - \Delta D\Gamma\| = \min_{E\in\mathbb{C}^{q\times m}} \|A - \Delta E\Gamma\| \tag{20}$$

if and only if $\Delta(A - D)\Gamma = 0$, *in which case*

$$\|A - \Delta D\Gamma\| = \|A - \Delta A\Gamma\|. \tag{21}$$

By Lemma 8, let $A \in \mathbb{C}^{l\times m}$, $B \in \mathbb{C}^{n\times p}$, $C \in \mathbb{C}^{m\times n}$, and then the Procrustes problem

$$\begin{aligned} &\left\|\left(I - A^\dagger A\right)\widehat{X}\left(I - BB^\dagger\right) - C\right\| \\ &\quad = \min_{X\in\mathbb{C}^{m\times n}} \left\|\left(I - A^\dagger A\right)X\left(I - BB^\dagger\right) - C\right\| \end{aligned} \tag{22}$$

has a solution if and only if

$$\left(I - A^\dagger A\right)\left(\widehat{X} - C\right)\left(I - BB^\dagger\right) = 0; \tag{23}$$

that is, the solution can be expressed as

$$\widehat{X} = C + A^\dagger G_1 + G_2B^\dagger, \tag{24}$$

where $G_1 \in \mathbb{C}^{l\times n}$, $G_2 \in \mathbb{C}^{m\times p}$ are arbitrary matrices.

Theorem 9. *Let* S_X *be the set of all* $\{P, Q, 2\}$*-reflexive solutions to the system of matrix equations (2) as in Theorem 5 and let* E *be a given matrix in* $\mathbb{C}^{m\times n}$. *Partition*

$$U^*EV = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \quad \text{with } E_{11} \in \mathbb{C}^{p\times q}. \tag{25}$$

Then,

$$\left\|\widehat{X} - E\right\| = \min_{X\in S_X} \|X - E\| \tag{26}$$

has an only solution $\widehat{X}$ *which can be expressed as*

$$\widehat{X} = U\begin{bmatrix} A_1^\dagger C_1 + D_1B_1^\dagger - A_1^\dagger A_1D_1B_1^\dagger + \left(I - A_1^\dagger A_1\right)E_{11}\left(I - B_1B_1^\dagger\right) & O \\ O & O \end{bmatrix}V^*. \tag{27}$$

Proof. It is easy to verify that the solution set S_X is a closed and convex set in the matrix space $\mathbb{C}^{m\times n}$ under Frobenius norm, so the solution to the approximation problem is unique. Note that, from (14), (25), and the unitary invariance

of the Frobenius norm, we get

$$\begin{aligned}\|X-E\|^2 &= \left\| U\begin{bmatrix} A_1^\dagger C_1 + D_1B_1^\dagger - A_1^\dagger A_1 D_1 B_1^\dagger + \left(I - A_1^\dagger A_1\right)U_1\left(I - B_1B_1^\dagger\right) & O \\ O & O \end{bmatrix} V^* - E\right\|^2 \\ &= \left\| \begin{bmatrix} A_1^\dagger C_1 + D_1B_1^\dagger - A_1^\dagger A_1 D_1 B_1^\dagger + \left(I - A_1^\dagger A_1\right)U_1\left(I - B_1B_1^\dagger\right) & O \\ O & O \end{bmatrix} - U^*EV\right\|^2 \\ &= \left\| A_1^\dagger C_1 + D_1B_1^\dagger - A_1^\dagger A_1 D_1 B_1^\dagger + \left(I - A_1^\dagger A_1\right)U_1\left(I - B_1B_1^\dagger\right) - E_{11}\right\|^2 + \|E_{12}\|^2 + \|E_{21}\|^2 + \|E_{22}\|^2 .\end{aligned} \tag{28}$$

By Lemma 8, we have

$$\begin{aligned}\min_{U_1\in\mathbb{C}^{p\times q}} &\left\| A_1^\dagger C_1 + D_1B_1^\dagger - A_1^\dagger A_1 D_1 B_1^\dagger \right. \\ &\left. + \left(I - A_1^\dagger A_1\right)U_1\left(I - B_1B_1^\dagger\right) - E_{11}\right\|\end{aligned} \tag{29}$$

if and only if

$$\begin{aligned}U_1 &= E_{11} - A_1^\dagger C_1 - D_1B_1^\dagger + A_1^\dagger A_1 D_1 B_1^\dagger + A_1^\dagger G_1 \\ &\quad + G_2 B_1^\dagger\end{aligned} \tag{30}$$

with $G_1 \in \mathbb{C}^{l\times q}$, $G_2 \in \mathbb{C}^{p\times k}$ being arbitrary. Substituting U_1 into (14) deduces that (27) holds. □

Theorem 10. *Let S_X be the set of all $\{P, Q, 3\}$-reflexive solutions to the system of matrix equations (2) as in Theorem 6 and let E be a given matrix in $\mathbb{C}^{m\times n}$. Partition*

$$U^*EV = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} \quad \text{with } E_{11} \in \mathbb{C}^{r\times s},\ E_{22} \in \mathbb{C}^{(p-r)\times(q-s)}. \tag{31}$$

Then,

$$\left\|\widehat{X} - E\right\| = \min_{X\in S_X} \|X - E\| \tag{32}$$

has an only solution $\widehat{X}$ which can be expressed as

$$\widehat{X} = U\begin{bmatrix} X_1 & 0 & O \\ O & X_2 & O \\ O & O & O \end{bmatrix} V^*, \tag{33}$$

where $X_1 = A_1^\dagger C_1 + D_1B_1^\dagger - A_1^\dagger A_1 D_1 B_1^\dagger + (I - A_1^\dagger A_1)E_{11}(I - B_1B_1^\dagger)$ and $X_2 = A_2^\dagger C_2 + D_2B_2^\dagger - A_2^\dagger A_2 D_2 B_2^\dagger + (I - A_2^\dagger A_2)E_{22}(I - B_2B_2^\dagger)$.

Proof. Similar to Theorem 9, the proof is omitted. □

4. The Least Squares $\{P, Q, k+1\}$-Reflexive (Antireflexive) Solution to (2)

Lemma 11 (see [11]). *Given $E, F \in \mathbb{C}^{m\times n}$, $\Omega_1 = \operatorname{diag}(a_1, \ldots, a_m)$, $\Omega_2 = \operatorname{diag}(b_1, \ldots, b_n)$, and $a_i > 0$ $(i = 1, \ldots, m)$, $b_j > 0$ $(j = 1, \ldots, n)$. Then, there exists a unique matrix $\overline{S} \in \mathbb{C}^{m\times n}$ such that*

$$\left\|\Omega_1 S - E\right\|^2 + \left\|S\Omega_2 - F\right\|^2 = \min, \tag{34}$$

and $\overline{S}$ can be expressed as

$$\overline{S} = \emptyset * \left(\Omega_1 E + F\Omega_2\right), \tag{35}$$

where $\emptyset = (1/(a_i^2 + b_j^2)) \in \mathbb{R}^{m\times n}$.

Theorem 12. *Let $A \in \mathbb{C}^{l\times m}$, $C \in \mathbb{C}^{l\times n}$, $B \in \mathbb{C}^{n\times k}$, $D \in \mathbb{C}^{m\times k}$ and $AU = [A_1, A_2]$, $CV = [C_1, C_2]$, $V^*B = \left[\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right]$, $U^*D = \left[\begin{smallmatrix} D_1 \\ D_2 \end{smallmatrix}\right]$, where $A_1 \in \mathbb{C}^{l\times p}$, $C_1 \in \mathbb{C}^{l\times q}$, $B_1 \in \mathbb{C}^{q\times k}$, $D_1 \in \mathbb{C}^{p\times k}$. Assume that the singular value decomposition of A_1, B_1 is as follows:*

$$\begin{aligned}A_1 &= W\begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix} Z^*, \\ B_1 &= P\begin{bmatrix} N_1 & 0 \\ 0 & 0 \end{bmatrix} Q^*,\end{aligned} \tag{36}$$

where $W = [W_1, W_2] \in \mathbb{C}^{l\times l}$, $Z = [Z_1, Z_2] \in \mathbb{C}^{p\times p}$, $P = [P_1, P_2] \in \mathbb{C}^{q\times q}$, and $Q = [Q_1, Q_2] \in \mathbb{C}^{k\times k}$ are unitary matrices, $M_1 = \operatorname{diag}(\sigma_1, \ldots, \sigma_{r_1})$, $\sigma_i > 0$ $(i = 1, \ldots, r_1)$, $r_1 = \operatorname{rank}(M_1)$, $W_1 \in \mathbb{C}^{l\times r_1}$, $Z_1 \in \mathbb{C}^{p\times r_1}$, $N_1 = \operatorname{diag}(\rho_1, \ldots, \rho_{r_2})$, $\rho_j > 0$ $(j = 1, \ldots, r_2)$, $r_2 = \operatorname{rank}(N_1)$,

$P_1 \in \mathbb{C}^{q\times r_2}$, $Q_1 \in \mathbb{C}^{k\times r_2}$. Then, $X \in S_L$ can be expressed as

$$X = U\begin{bmatrix} Z\begin{bmatrix} \emptyset * (M_1W_1^*C_1P_1 + Z_1^*D_1Q_1N_1) & M_1^{-1}W_1^*C_1P_2 \\ Z_2^*D_1Q_1N_1^{-1} & Y_4 \end{bmatrix} P^* & O \\ O & O \end{bmatrix} V^*, \tag{37}$$

where $\emptyset = (1/(\sigma_i^2 + \rho_j^2)) \in \mathbb{R}^{r_1\times r_2}$ and $Y_4 \in \mathbb{C}^{(p-r_1)\times(q-r_2)}$ is an arbitrary matrix.

Proof. It yields from (36) that

$$\begin{aligned}
&\|AX - C\|^2 + \|XB - D\|^2 \\
&= \left\| AU\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} V^* - C \right\|^2 \\
&\quad + \left\| U\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} V^*B - D \right\|^2 \\
&= \left\| [A_1, A_2]\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} - [C_1, C_2] \right\|^2 \\
&\quad + \left\| \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} - \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \right\|^2 \\
&= \|A_1X_1 - C_1\|^2 + \|X_1B_1 - D_1\|^2 + \|C_2\|^2 \\
&\quad + \|D_2\|^2 \\
&= \left\| W\begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix} Z^*X_1 - C_1 \right\|^2 \\
&\quad + \left\| X_1P\begin{bmatrix} N_1 & 0 \\ 0 & 0 \end{bmatrix} Q^* - D_1 \right\|^2 + \|C_2\|^2 + \|D_2\|^2 \\
&= \left\| \begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix} Z^*X_1P - W^*C_1P \right\|^2 \\
&\quad + \left\| Z^*X_1P\begin{bmatrix} N_1 & 0 \\ 0 & 0 \end{bmatrix} - Z^*D_1Q \right\|^2 + \|C_2\|^2 \\
&\quad + \|D_2\|^2 .
\end{aligned} \tag{38}$$

Assume that

$$Z^*X_1P = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}, \qquad Y_1 \in \mathbb{C}^{r_1\times r_2},\ Y_4 \in \mathbb{C}^{(p-r_1)\times(q-r_2)}. \tag{39}$$

Then, we have

$$\begin{aligned}
&\|AX - C\|^2 + \|XB - D\|^2 \\
&\quad = \|M_1Y_1 - W_1^*C_1P_1\|^2 + \|Y_1N_1 - Z_1^*D_1Q_1\|^2 \\
&\qquad + \|M_1Y_2 - W_1^*C_1P_2\|^2 + \|Y_3N_1 - Z_2^*D_1Q_1\|^2 \\
&\qquad + \|W_2^*C_1P_1\|^2 + \|W_2^*C_1P_2\|^2 + \|Z_1^*D_1Q_2\|^2 \\
&\qquad + \|Z_2^*D_1Q_2\|^2 + \|C_2\|^2 + \|D_2\|^2 .
\end{aligned} \tag{40}$$

Hence,

$$\min_{X\in\mathbb{PQC}^{m\times n}} \|AX - C\|^2 + \|XB - D\|^2 \tag{41}$$

is solvable if and only if there exist Y_1, Y_2, Y_3 such that

$$\begin{aligned}
\|M_1Y_1 - W_1^*C_1P_1\|^2 + \|Y_1N_1 - Z_1^*D_1Q_1\|^2 &= \min, \\
\|M_1Y_2 - W_1^*C_1P_2\|^2 &= \min, \\
\|Y_3N_1 - Z_2^*D_1Q_1\|^2 &= \min.
\end{aligned} \tag{42}$$

It follows from (42) that

$$\begin{aligned}
Y_1 &= \emptyset * (M_1W_1^*C_1P_1 + Z_1^*D_1Q_1N_1), \\
Y_2 &= M_1^{-1}W_1^*C_1P_2, \\
Y_3 &= Z_2^*D_1Q_1N_1^{-1},
\end{aligned} \tag{43}$$

where $\emptyset = (1/(\sigma_i^2 + \rho_j^2)) \in \mathbb{R}^{r_1\times r_2}$. Substituting (43) into (39), we can get that the form of elements in S_L is (37). □

Remark 13. The $\{P, Q, 3\}$-reflexive and antireflexive constraint least squares problem can be reduced similar to Theorem 9; the conclusion is omitted here.

5. An Algorithm and Numerical Example

The algorithm below constructs the $\{P, Q, 2\}$-reflexive solution, the optimal approximation $\{P, Q, 2\}$-reflexive solution, and the least squares $\{P, Q, 2\}$-reflexive solution for the problem stated in Sections 2, 3, and 4, respectively.

Algorithm 14. (1) Input $A \in \mathbb{C}^{l\times m}$, $C \in \mathbb{C}^{l\times m}$, $B \in \mathbb{C}^{n\times k}$, $D \in \mathbb{C}^{n\times k}$.

(2) Compute A_1, C_1, B_1, D_1 by (12).

(3) If (13) holds, compute the $\{P, Q, 2\}$-reflexive solution of (2) by (14); input $E \in \mathbb{C}^{m\times n}$ and compute E_{11} by (25); then, compute the solution $\widehat{X}$ of problem (3) according to (27).

(4) If (13) does not hold, compute the singular value decomposition of A_1, B_1 by (36); then, compute the least squares $\{P, Q, 2\}$-reflexive solution of (2) by (37).

Similar algorithms for the $\{P, Q, 2\}$-antireflexive and $\{P, Q, 3\}$-reflexive cases can be developed. Next, we illustrate the obtained results with two examples.

Let $P \in \mathbb{C}^{4\times 4}$, $Q \in \mathbb{C}^{3\times 3}$ and

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \tag{44}$$

we obtain

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -0.7071 & -0.7071 \\ 0 & 0 & -0.7071 & 0.7071 \end{bmatrix}, \quad V = \begin{bmatrix} -0.7071 & 0 & -0.7071 \\ -0.7071 & 0 & 0.7071 \\ 0 & 1 & 0 \end{bmatrix}. \tag{45}$$

Example 1. Suppose $A \in \mathbb{C}^{2\times 4}$, $C \in \mathbb{C}^{2\times 3}$, $B \in \mathbb{C}^{3\times 2}$, $D \in \mathbb{C}^{4\times 2}$ and

$$A = \begin{bmatrix} 1 & 2 & -0.7071 & 2.1213 \\ 0 & 0 & -3.5355 & -0.7071 \end{bmatrix}, \quad C = \begin{bmatrix} -5.6568 & 4.2426 & 2 \\ -1.4142 & 1.4142 & 0 \end{bmatrix},$$
$$B = \begin{bmatrix} 0.3535 & -0.7071 \\ -1.0606 & 0.7071 \\ -0.3 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0.5 & 0 \\ -0.3 & 0 \\ -0.7071 & 0.7071 \\ -2.1213 & 2.1213 \end{bmatrix}. \tag{46}$$

We can verify that (13) holds. Hence, system (2) has the $\{P, Q, 2\}$-reflexive solution, which can be expressed as

$$X = U \begin{bmatrix} \begin{bmatrix} 0.9647 & -0.0588 \\ 0.0176 & 1.0294 \end{bmatrix} + \begin{bmatrix} 0.8 & -0.4 \\ -0.4 & 0.2 \end{bmatrix} U_1 \begin{bmatrix} 0.2647 & 0.4412 \\ 0.4412 & 0.7353 \end{bmatrix} & O \\ O & O \end{bmatrix} V^*. \tag{47}$$

Given

$$E = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{48}$$

Applying Algorithm 14, we obtain the following:

$$E_{11} = \begin{bmatrix} -1.4142 & 1 \\ -1.4142 & 1 \end{bmatrix},$$

$$\widehat{X} = \begin{bmatrix} -0.7010 & -0.7010 & -0.0143 \\ -0.0030 & -0.0030 & 1.0071 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{49}$$

Example 2. Let A, B, C be the same as Example 1, and let D in Example 1 be changed into

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -0.7071 & 0.7071 \\ -2.1213 & 2.1213 \end{bmatrix}. \tag{50}$$

We can verify that (13) does not hold. By Algorithm 14, the least squares $\{P, Q, 2\}$-reflexive solution of (2) can be expressed as

$$X = U \begin{bmatrix} \begin{bmatrix} 0.4472 & -0.8944 \\ -0.8944 & 0.4472 \end{bmatrix} \begin{bmatrix} 0.1227 & 0.9970 \\ 1.5339 & Y_4 \end{bmatrix} \begin{bmatrix} -0.8575 & 0.5145 \\ 0.5145 & 0.8575 \end{bmatrix} & O \\ O & O \end{bmatrix} V^*. \tag{51}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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