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Research Article

Asymptotic Behavior of Weak Solutions to the Generalized Nonlinear Partial Differential Equation Model

Min Wu and Yousheng Wu

School of Mathematics and Statistics, Hubei University of Science and Technology, Xianning 437100, China

Correspondence should be addressed to Min Wu; minwu2000math@163.com

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This paper investigates the asymptotic behavior of weak solutions to the generalized nonlinear partial differential equation model. It is proved that every perturbed weak solution of the perturbed generalized nonlinear partial differential equations asymptotically converges to the solution of the original system under the large perturbation.

1. Introduction

In the past ten years, the study of the fractional order differential equation has attracted more and more attention. In this study, we consider a class of two-dimensional fractional order generalized nonlinear partial differential equation model which is governed by the differential equation

$$\frac{\partial}{\partial t}u + \alpha(-\Delta)^{1/2}u + \beta|u|^2 u = g, \quad \text{in } \mathbb{R}^2 \times (0,\infty) \tag{1}$$

together with the initial condition

$$u(x,0) = u_0(x). \tag{2}$$

Here, $u(x,t)$ is unknown function. α, β are positive constants, $(-\Delta)^{1/2}$ is the fractional power of the Laplacian Δ, and $f(x,t)$ is an external force.

The model is relevant to the theory of the atmosphere and ocean dynamics (refer to [1–4] and references therein). It should be mentioned that there are many results on the stability behaviors of the atmosphere and ocean dynamics in which the derivation is mainly based on the linear and nonlinear stability together with the numerical simulation [5–7]. Recently, Hu [8] investigated the following semilinear parabolic partial differential equation with Laplacian in $\mathbb{R}^3$:

$$\begin{aligned} &\partial_t v - \Delta v + |v|^{p-2}v = 0, \\ &v(x,0) = v_0(x), \end{aligned} \tag{3}$$

and derived the error estimates between the solution of semilinear parabolic partial differential equation (3) and the solution of the linear heat equation. However, it is a challenging problem to consider the fractional order partial differential equation due to some new difficulty. One may also refer to some interesting and important results on the stability of the nonlinear partial differential equations [9–11].

In this study, we will investigate the asymptotic stability for solution of the two-dimensional fractional order partial differential equation (1) under the finite energy initial data u_0.

To do so, we first consider the perturbed fractional order partial differential equation:

$$\begin{aligned} &\frac{\partial}{\partial t}v + \alpha(-\Delta)^{1/2}v + \beta|v|^2 v = g, \\ &v(x,0) = v_0 = u_0 + w_0. \end{aligned} \tag{4}$$

Here, w_0 is any initial perturbation which may be large. We will show that every perturbed solution v of the fractional order partial differential equation (4) asymptotically converges to that of fractional order partial differential equations (1)-(2). That is to say,

$$\|v(t) - u(t)\|_{L^2} \longrightarrow 0, \quad t \longrightarrow \infty. \tag{5}$$

We now give the definition of solution for fractional order partial differential equations (1)-(2) (refer to [12]).

Definition 1. $u(x,t)$ is called a solution of fractional order partial differential equations (1)-(2) with

$$u_0(x) \in L^2(\mathbb{R}^2), \quad g \in L^2_{\text{loc}}(0,\infty; L^2(\mathbb{R}^2)), \tag{6}$$

if the conditions

(i) $u \in L^\infty(0,\infty; L^2(\mathbb{R}^2)) \cap L^2(0,\infty; \dot{H}^{1/2}(\mathbb{R}^2))$;

(ii) for any $\eta \in C_0^\infty(\mathbb{R}^2 \times [0,T))$,

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^2} \left(\alpha(-\Delta)^{1/4} u (-\Delta)^{1/4}\eta + \beta |u|^2 u\eta\right) dx\,dt \\ = \int_0^t \int_{\mathbb{R}^2} u \cdot \partial_t \eta \, dx\,dt + \int_{\mathbb{R}^2} u_0 \eta(0)\,dx;\end{aligned} \tag{7}$$

(iii) energy inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |u|^2 dx + \alpha \int_{\mathbb{R}^2} \left|(-\Delta)^{1/4} u\right|^2 dx \\ + \beta \int_{\mathbb{R}^2} |u|^4 dx \le 0\end{aligned} \tag{8}$$

are valid.

Our result now reads.

Theorem 2. *Assume $u(x,t)$ a solution of fractional order differential equations (1)-(2) with $u_0(x) \in L^2(\mathbb{R}^2), g \in L^2_{\text{loc}}(0,\infty; L^2(\mathbb{R}^2))$; then, for any large initial perturbation $w_0 \in L^2(\mathbb{R}^2)$, the solution $v(x,t)$ of the perturbed fractional order partial differential equation (4) asymptotically converges to the global solution $u(x,t)$ as*

$$\|v(t) - u(t)\|_{L^2} \longrightarrow 0, \quad t \longrightarrow \infty. \tag{9}$$

Remark 3. An important feature is that our result here has not small assumption on the initial perturbation w_0.

Remark 4. Our methods are mainly based on the generalized Fourier splitting methods which are first used by Schonbek [13] (see also [14–16]) on the time decay issue of the classic Navier-Stokes equations and related partial differential equations [17].

2. Preliminaries

In this study, we denote by C's the abstractly positive constants which may be different from line to line. We denoted by $L^p(\mathbb{R}^2)$ the usual Lebesgue space with the norm

$$\|\varphi\|_{L^p} = \begin{cases} \left(\int_{\mathbb{R}^2} |\varphi(x)|^p dx\right)^{1/p}, & 1 \le p < \infty, \\ \operatorname{ess\,sup}_{x\in\mathbb{R}^2} |\varphi(x)|, & p = \infty. \end{cases} \tag{10}$$

We also denoted by $\dot{H}^s(\mathbb{R}^2)$ the homogeneous fractional Sobolev space:

$$\|\varphi\|_{\dot{H}^s} = \left(\int_{\mathbb{R}^2} |\xi|^{2s} |\widehat{\varphi}|^2 d\xi\right)^{1/2}. \tag{11}$$

Here, $\widehat{\varphi}$ is Fourier transformation:

$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}^2} e^{-ix\cdot\xi}\varphi(x)\,dx. \tag{12}$$

In order to prove our main result, we now give some important lemmas which play a central role in the argument of the next section.

Lemma 5 (Gagliardo-Nirenberg inequality [18]). *Assume $f \in W^{m,p}(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$; then*

$$\left\|D^j f\right\|_{L^r(\mathbb{R}^2)} \le C \|f\|^{1-\lambda}_{L^q(\mathbb{R}^2)} \left\|D^m f\right\|^{\lambda}_{L^p(\mathbb{R}^2)}. \tag{13}$$

Here, $p, q, r, n, m, j, \lambda$ satisfy

$$\frac{1}{r} - \frac{j}{n} = (1-\lambda)\frac{1}{q} + \lambda\left(\frac{1}{p} - \frac{m}{n}\right) \tag{14}$$

with

$$1 < p, q, r \le \infty, \quad 0 \le j < m, \ \frac{j}{m} \le \lambda < 1. \tag{15}$$

Lemma 6. *Assume $f \in L^\infty(0,\infty; L^2(\mathbb{R}^2)) \cap L^2(0,\infty; \dot{H}^{1/2}(\mathbb{R}^2))$; then*

$$\left\{\int_0^\infty \left(\int_{\mathbb{R}^2} |f|^q dx\right)^{p/q} dt\right\}^{1/p} \le C \tag{16}$$

with

$$\frac{1}{p} + \frac{2}{q} = 1 \tag{17}$$

is valid.

Proof of Lemma 6. Since f satisfies

$$\begin{aligned}\operatorname*{ess\,sup}_{0<t<\infty} \left(\int_{\mathbb{R}^2} |f|^2 dx\right) \\ + \left\{\int_0^\infty \int_{\mathbb{R}^2} \left|(-\Delta)^{1/4} f\right|^2 dx\,dt\right\}^{1/2} := C,\end{aligned} \tag{18}$$

applying Gagliardo-Nirenberg inequality in Lemma 6, the direct computation becomes

$$\begin{aligned}\left\{\int_0^\infty \left(\int_{\mathbb{R}^2} |f|^q dx\right)^{p/q} dt\right\}^{1/p} \\ \le \left\{\int_0^\infty \|f\|^{p\theta}_{L^2} \|f\|^{p(1-\theta)}_{L^4} ds\right\}^{1/p},\end{aligned} \tag{19}$$

where

$$\frac{1}{q} = \frac{\theta}{2} + (1-\theta)\frac{1}{4} \tag{20}$$

for

$$0 \leq \theta \leq 1. \tag{21}$$

Since

$$\|f\|_{L^4} \leq C \left\|(-\Delta)^{1/4} f\right\|_{L^2}, \tag{22}$$

thus, we have

$$\begin{aligned}
&\left\{\int_0^\infty \left(\int_{\mathbb{R}^2} |f|^q\,dx\right)^{p/q} dt\right\}^{1/p} \\
&\leq C\left\{\int_0^\infty \|f\|_{L^2}^{p\theta} \left\|(-\Delta)^{1/4} f\right\|_{L^2}^{p(1-\theta)} ds\right\}^{1/p} \\
&\leq C\left\{\operatorname*{ess\,sup}_{0<t<\infty}\left(\int_{\mathbb{R}^2} |f|^2\,dx\right)\right\}^{\theta} \\
&\cdot \left\{\int_0^\infty \left\|(-\Delta)^{1/4} f\right\|_{L^2}^{p(1-\theta)} ds\right\}^{1/p} \\
&\leq C\left\{\operatorname*{ess\,sup}_{0<t<\infty}\left(\int_{\mathbb{R}^2} |f|^2\,dx\right)\right\}^{\theta} \\
&\cdot \left\{\int_0^\infty \left\|(-\Delta)^{1/4} f\right\|_{L^2}^{p(1-\theta)} ds\right\}^{(1/(p(1-\theta)))(1-\theta)} \leq C \\
&\cdot \operatorname*{ess\,sup}_{0<t<\infty}\left(\int_{\mathbb{R}^2} |f|^2\,dx\right) \\
&+ C\left\{\int_0^\infty \int_{\mathbb{R}^2} \left|(-\Delta)^{1/4} f\right|^2 dx\,dt\right\}^{1/2} \leq C,
\end{aligned} \tag{23}$$

where we used the following relation:

$$p(1-\theta) = 2. \tag{24}$$

That is to say,

$$\frac{1}{p} + \frac{2}{q} = 1 \quad 2 \leq q \leq 4. \tag{25}$$

In particular, we take $p = q$; then

$$p = q = 3. \tag{26}$$

Then,

$$\int_0^\infty \int_{\mathbb{R}^2} |f|^3\,dx\,dt \leq C. \tag{27}$$

□

Hence, we complete the proof of Lemma 6.

Lemma 7. *Suppose $u(x,t), v(x,t)$ are two solutions of fractional order differential equations (1)-(2) and the perturbed fractional order partial differential equation (4) with*

$$\begin{aligned}
u_0(x), w_0(x) &\in L^2\left(\mathbb{R}^2\right), \\
g &\in L^2_{\mathrm{loc}}\left(0,\infty; L^2\left(\mathbb{R}^2\right)\right).
\end{aligned} \tag{28}$$

Then,

$$\left|\widehat{(u-v)}(\xi,t)\right| \leq \left|e^{-\alpha|\xi|t}\widehat{w}_0\right| + C. \tag{29}$$

Proof of Lemma 7. Since $u(x,t), v(x,t)$ are two solutions of fractional order partial differential equations (1)-(2) and the perturbed fractional order differential equation (4), then we take $w = u - v$ which satisfies the following equations formally:

$$\begin{aligned}
\frac{\partial}{\partial t} w + \alpha(-\Delta)^{1/2} w + \left(\beta |u|^2 u - \beta |v|^2 v\right) &= 0, \\
w(x,0) &= w_0.
\end{aligned} \tag{30}$$

Taking Fourier transformation to both sides of the equation (30), one shows that

$$\begin{aligned}
\widehat{w}_t + \alpha|\xi|\widehat{w} &= -\left(\widehat{\beta |u|^2 u - \beta} |v|^2 v\right), \\
\widehat{w}(t,0) &= \widehat{w}_0.
\end{aligned} \tag{31}$$

By solving this ordinary partial differential equation ,

$$\begin{aligned}
&|\widehat{w}(\xi,t)| \\
&= \left|e^{-\alpha|\xi|t}\widehat{w}_0 + \int_0^t e^{-\alpha|\xi|t}\left\{-\left(\widehat{\beta |u|^2 u - \beta} |v|^2 v\right)\right\} ds\right| \\
&\leq \left|e^{-\alpha|\xi|t}\widehat{w}_0\right| + C\int_0^t \left|\widehat{|u|^2 u}\right| ds + C\int_0^t \left|\widehat{|v|^2 v}\right| ds \\
&\leq \left|e^{-\alpha|\xi|t}\widehat{w}_0\right| + C\int_0^t \left(\int_{\mathbb{R}^2} |u|^3\,dx\right) ds \\
&\quad + C\int_0^t \left(\int_{\mathbb{R}^2} |u|^3\,dx\right) ds.
\end{aligned} \tag{32}$$

According to the definitions of u, v, that is,

$$u, v \in L^\infty\left(0,\infty; L^2\left(\mathbb{R}^2\right)\right) \cap L^2\left(0,\infty; \dot{H}^{1/2}\left(\mathbb{R}^2\right)\right), \tag{33}$$

then applying Lemma 6, we obtain

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}^2} |u|^3\,dx\,dt &\leq C, \\
\int_0^\infty \int_{\mathbb{R}^2} |v|^3\,dx\,dt &\leq C.
\end{aligned} \tag{34}$$

Thus,

$$|\widehat{w}(\xi,t)| \leq \left|e^{-\alpha|\xi|t}\widehat{w}_0\right| + C, \tag{35}$$

which completes the proof of Lemma 7. □

3. Stability of the Solution

We now prove Theorem 2. Firstly, as stated in the proof of Lemma 7, since $u(x,t)$, $v(x,t)$ are two solutions of (1)-(2) and (4), respectively, we take $w = u - v$; then

$$\frac{\partial}{\partial t} w + \alpha(-\Delta)^{1/2} w + \left(\beta|u|^2 u - \beta|v|^2 v\right) = 0, \qquad w(x,0) = w_0. \tag{36}$$

Taking the L^2 inner product of (36), it follows that

$$\frac{d}{dt}\int_{\mathbb{R}^2} |w(t)|^2 dx + 2\alpha \int_{\mathbb{R}^2} \left|(-\Delta)^{1/4} w\right|^2 dx = -2\beta \int_{\mathbb{R}^2} \left(|u|^2 u - |v|^2 v\right) w\, dx. \tag{37}$$

For the right hand side of above equation,

$$\int_{\mathbb{R}^2} \left(|u|^2 u - |v|^2 v\right) w\, dx = \int_{\mathbb{R}^2} |u|^4 dx - \int_{\mathbb{R}^2} |u|^2 uv\, dx - \int_{\mathbb{R}^2} |v|^2 uv\, dx + \int_{\mathbb{R}^2} |v|^4 dx; \tag{38}$$

since

$$\int_{\mathbb{R}^2} |u|^2 uv\, dx \le \left(\int_{\mathbb{R}^2} |u|^4 dx\right)^{3/4} \left(\int_{\mathbb{R}^2} |v|^4 dx\right)^{1/4}, \quad \int_{\mathbb{R}^2} |v|^2 uv\, dx \le \left(\int_{\mathbb{R}^2} |v|^4 dx\right)^{3/4} \left(\int_{\mathbb{R}^2} |u|^4 dx\right)^{1/4}, \tag{39}$$

thus,

$$\int_{\mathbb{R}^2} \left(|u|^2 u - |v|^2 v\right) w\, dx \ge \left\{\left(\int_{\mathbb{R}^2} |u|^4 dx\right)^{3/4} - \left(\int_{\mathbb{R}^2} |v|^4 dx\right)^{3/4}\right\} \cdot \left\{\left(\int_{\mathbb{R}^2} |u|^4 dx\right)^{1/4} - \left(\int_{\mathbb{R}^2} |v|^4 dx\right)^{1/4}\right\} \ge 0. \tag{40}$$

Inserting the above inequality into the right hand side of (37), it follows that

$$\frac{d}{dt}\int_{\mathbb{R}^2} |w(t)|^2 dx + 2\alpha \int_{\mathbb{R}^2} \left|(-\Delta)^{1/4} w\right|^2 dx \le 0. \tag{41}$$

Applying Parseval inequality,

$$\frac{d}{dt}\int_{\mathbb{R}^2} \left|\widehat{w}(\xi,t)\right|^2 d\xi + 2\alpha \int_{\mathbb{R}^2} |\xi| \left|\widehat{w}(\xi,t)\right|^2 d\xi \le 0. \tag{42}$$

Let

$$\begin{aligned} r(t) &= \left\{\xi \in \mathbb{R}^2 : |\xi| \le \frac{4}{\alpha(1+t)}\right\}, \\ 2\alpha(1+t)^4 \int_{\mathbb{R}^2} |\xi| \left|\widehat{w}(\xi,t)\right|^2 d\xi &= 2\alpha(1+t)^4 \int_{r(t)^c} |\xi| \left|\widehat{w}(\xi,t)\right|^2 d\xi + 2\alpha(1+t)^4 \int_{r(t)} |\xi| \left|\widehat{w}(\xi,t)\right|^2 d\xi \\ &\ge 2\alpha(1+t)^4 \int_{r(t)^c} |\xi| \left|\widehat{w}(\xi,t)\right|^2 d\xi \\ &\ge 4(1+t)^3 \int_{\mathbb{R}^2} \left|\widehat{w}(\xi,t)\right|^2 d\xi - 4(1+t)^3 \int_{r(t)} \left|\widehat{w}(\xi,t)\right|^2 d\xi. \end{aligned} \tag{43}$$

Thus, we have

$$\frac{d}{dt}\int_{\mathbb{R}^2} \left|\widehat{w}(\xi,t)\right|^2 d\xi + 4(1+t)^{-1} \int_{\mathbb{R}^2} \left|\widehat{w}(\xi,t)\right|^2 d\xi \le -4(1+t)^{-1} \int_{r(t)} \left|\widehat{w}(\xi,t)\right|^2 d\xi. \tag{44}$$

Multiplying both sides of (44) by $(1+t)^4$ and taking the direct computation, one shows that

$$\frac{d}{dt}\left((1+t)^4 \int_{\mathbb{R}^2} \left|\widehat{w}(\xi,t)\right|^2 d\xi\right) \le C(1+t)^3 \int_{r(t)} \left|\widehat{w}(\xi,t)\right|^2 d\xi. \tag{45}$$

Applying Lemma 7, we have

$$\begin{aligned} C(1+t)^3 \int_{r(t)} \left|\widehat{w}(\xi,t)\right|^2 d\xi &\le C(1+t)^3 \int_{r(t)} \left\{\left|e^{-\alpha|\xi|t}\widehat{w}_0\right| + C\right\}^2 d\xi \\ &\le C(1+t)^3 \int_{\mathbb{R}^2} \left|e^{-\alpha|\xi|t}\widehat{w}_0\right|^2 d\xi + C(1+t)^3 \int_{r(t)} 1 d\xi \\ &\le C(1+t)^3 \int_{\mathbb{R}^2} \left|e^{-\alpha|\xi|t}\widehat{w}_0\right|^2 d\xi + C(1+t). \end{aligned} \tag{46}$$

Then, integrating in time,

$$\begin{aligned}
\int_{\mathbb{R}^2} \left|\widehat{w}(\xi,t)\right|^2 d\xi
&\le C(1+t)^{-4}\left\|\widehat{w}_0\right\|_{L^2} \\
&\quad + C(1+t)^{-4}\left\{(1+t)^3\int_{\mathbb{R}^2}\left|e^{-\alpha|\xi|t}\widehat{w}_0\right|^2 d\xi\right\} \\
&\quad + C(1+t)^{-2} \\
&\le C(1+t)^{-2} \\
&\quad + C(1+t)^{-4}\left\{(1+t)^3\int_{\mathbb{R}^2}\left|e^{-\alpha|\xi|t}\widehat{w}_0\right|^2 d\xi\right\}
\end{aligned} \tag{47}$$

since

$$(1+t)^{-4}\left\{(1+t)^3\int_{\mathbb{R}^2}\left|e^{-\alpha|\xi|t}\widehat{w}_0\right|^2 d\xi\right\} \longrightarrow 0 \tag{48}$$

as $t \to \infty$.

Hence,

$$\int_{\mathbb{R}^2} \left|\widehat{w}(\xi,t)\right|^2 d\xi \longrightarrow 0 \tag{49}$$

as $t \to \infty$.

That is,

$$\|v(t)-u(t)\|_{L^2} \longrightarrow 0, \quad t \longrightarrow \infty, \tag{50}$$

which implies that the proof of Theorem 2 is completed.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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