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Research Article

An Optimized Grey GM(2,1) Model and Forecasting of Highway Subgrade Settlement

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Grey prediction technique is a useful tool for few data analysis and short term forecasting. GM(2,1) model is one of the most important grey models. For improving the precision and prediction ability, we proposed a structure optimized GM(2,1) model, namely, SOGM(2,1) model. This study contributes grey prediction theory on three points. First, SOGM(2,1) model utilizes background sequence and inverse accumulating generated sequence to construct new grey equation with optimized structure, and then estimation of parameters is derived based on least errors. Second, reflection equation is constructed and the solving process is derived with the time response function acquired. Third, we put forward a new method for establishing initial values of time response function. After that, the new model is used to predict highway settlement of an engineering assessment. Comparing with other models, the results show that SOGM(2,1) is effective and practicable to forecast.

1. Introduction

Grey prediction is an important theory as part of grey system theory proposed by Professor Deng. It is dedicated to forecasting uncertain system with imperfect information or few data [1, 2]. This theory employs generated data to construct model not directly raw data and utilizes whitenization equation (or reflection equation) to investigate development tendency. Many applications in engineering field have demonstrated its effectiveness and high precision. Due to the simplicity comparing with other models, grey models are more practicable. Over two decades, this prediction theory has caught much attention of scholars all over the world.

GM(2,1) is proposed to change the linear structure of GM(1,1) model and expands application scope of grey prediction theory. It is an important model among the group of grey prediction models. Modeling method of GM(2,1) is derived from GM(1,1) and satisfies a similar mechanism. GM(1,1) model employs accumulating generation operator (AGO) for weakening randomness of raw data and extracting useful information, while GM(2,1) uses the first order AGO sequence (1-AGO) and the first order inverse AGO (1-IAGO) sequence as input. Comparing with AGO generation revealing system's global function, IAGO generation unveils specific information and variation trend. GM(2,1) model combines both and employs a second order reflection equation to restore the uncertain information.

However, the structure of GM(2,1) model and solving process have limited its application. In GM(2,1) model, grey differential equation is constructed with IAGO data as independent variable to simplify solving process. For IAGO operator affects parameter estimation, GM(2,1) is sensitive to random variation of system development.

The present study proposed a structure optimized GM(2,1) model, namely, SOGM(2,1), to improve precision based on traditional GM(2,1) model. The information contained is believed to be useful for forecasting in engineering problems. Structure of the following content is divided into three parts: the first section is a brief literature review; the second section described the approach and modeling program of SOGM(2,1); in the last section, an application is studied to forecast a highway settlement by SOGM(2,1) model, and computational results indicated that the new one has a higher prediction than traditional GM(2,1) and GM(1,1).

2. Literature Review

In recent years, many researchers have promoted the development of grey prediction theory and many optimized algorithms have emerged expanding greatly the application scope of grey prediction theory. A brief summary of literatures is presented as follows.

First, some new methods have been suggested to improve the flexibility of grey prediction model for dealing with more complex system sequences. Professor Deng, originator of grey systems theory, suggested a method with new parameter in the grey equation of GM(1,1) model, and an improved model was constructed to simulate oscillation sequences [3]. Further, based on the optimized GM(1,1) model he proposed another approach taking the periodic characteristic into account. The new model with trigonometric parameters was used to simulate the periodic trend [4]. Improvement of time response function is another way to optimize GM(1,1); Tien studied derivation of time response function of GM(1,1) by matrix tool and put forward a method to optimize the establishment method of initial value in time response function [5]. Besides, background value is reconstructed by some new method to improve the grey model. Li et al. tried to improve the adaptive ability for more complex sequences by reconstructing background formula. In the study, the reflection function was combined with new background formula for extracting development information from raw data, and the new model was used to forecast electricity consumption of Asian countries [6].

Based on the improvement method for original GM(1,1), many new models were derived and constructed. Qian et al. expanded the grey function variable in basic grey differential equation, adding time power items to simulate a certain kind of systems [7]. Grey power models are derived from GM(1,1) model, which is used to simulate growth process of the uncertain systems; Wang analyzed the property of grey power GM(1,1) model and suggested an optimization method based on error vector's linear programming with unbiased constrain [8]. Evans constructed a combining model consisting of grey Verhulst model and statistic model, and the new Verhulst model was used to study the tendency of steel production intensity of UK [9].

Second order grey model is an important expansion from GM(1,1) model group. As for the second order grey model, Zeng and Xiao considered the mobility of GM(2,1) with matrix tool and put forward an algorithm by accumulating product for original sequences; the new method improved prediction accuracy and decreased the mobility of parameters evaluation process [10]. Wang et al. proposed a computing formula of window length for GM(2,1) modeling and simulated the deformation in landslide process by improved GM(2,1) model [11]. Zhao and Chen indicated that the GM(2,1) equation setting was unreasonable, and his further study analyzed the cause of systemic error in GM(2,1) [12]. Liu and Zhang studied the optimization method of GM(2,1), in which a linear combination parameter λ and an accumulation adjust parameter ρ were used to improve grey equation of GM(2,1) model [13]. Shen and Zhao tried to simulate oscillatory sequence with optimized GM(2,1), and its computational results showed that the optimized GM(2,1) model could fit a certain scope of oscillatory systems action with obvious trend.

3. Methodology

In this section, we suggest an optimized method for GM(2,1) and propose a new algorithm to achieve high prediction ability. A new grey equation is constructed, and the original sequence is set as independent variable. By the grey equation, we derived the parameter estimation, solving process and time response function. Besides, the latest two points are employed to establish the initial terms of simulation function and keep the simulation function with a high precision and a similar trend as actual data series.

Operators AGO and IAGO are used to generate new sequences $X^{(1)}$ and $X^{(-1)}$ from original data $X^{(0)}$, and the background sequence is constructed for smoothing the increasing rates. $X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n))$ denotes nonnegative sequence of system action. $X^{(1)}$ is the sequence by AGO operator, defined as $X^{(1)} = (x^{(1)}(1), x^{(1)}(2), \ldots, x^{(1)}(n))$, where $x^{(1)}(k) = \sum_{i=1}^{k} x^{(0)}(i)$, $k = 1, 2, \ldots, n$. Similarly, generate $X^{(-1)}$ by IAGO and denote it as $X^{(-1)} = (x^{(-1)}(1), x^{(-1)}(2), \ldots, x^{(-1)}(n))$, where $x^{(-1)}(k) = x^{(0)}(k) - x^{(0)}(k-1)$, $k = 1, 2, \ldots, n$. Besides, construct the background sequence $Z^{(1)} = (z^{(1)}(2), z^{(1)}(3), \ldots, z^{(1)}(n))$, which is generated by mean generation of adjoining neighbors based on $X^{(1)}$ as $z^{(1)}(k) = (x^{(1)}(k-1) + x^{(1)}(k))/2$, $k = 1, 2, \ldots, n$.

The new model is stated stepwise as follows. Construct the grey differential equation of SOGM(2,1) model:

$$\alpha_1 x^{(-1)}(k) + x^{(0)}(k) + \alpha_2 z^{(1)}(k) = b, \tag{1}$$

where α_1, α_2 are the low order parameter and high order parameter, respectively. b denotes the control variable and $Z^{(1)}$ is the background sequence as mentioned earlier.

Structure of grey differential equation of SOGM(1,1) is different from that of traditional GM(2,1) model. Comparing with traditional GM(2,1) model, original sequence is an explained variable in (1). And background value and IAGO data are used to construct the grey differential equation.

Then, estimate parameters of the grey differential equation. Let

$$Y = \begin{bmatrix} x^{(0)}(2) \\ x^{(0)}(3) \\ \vdots \\ x^{(0)}(n) \end{bmatrix}, \qquad B = \begin{bmatrix} -x^{(-1)}(2), -z^{(1)}(2), 1 \\ -x^{(-1)}(3), -z^{(1)}(3), 1 \\ \vdots \\ -x^{(-1)}(n), -z^{(1)}(n), 1 \end{bmatrix}. \tag{2}$$

And $P = \left[\begin{smallmatrix}\alpha_1\\ \alpha_2\\ b\end{smallmatrix}\right]$ is parameter vector. After that, estimate the parameter vector by least error square method, and computation formula is $P = \left[\begin{smallmatrix}\alpha_1\\ \alpha_2\\ b\end{smallmatrix}\right] = (B^T B)^{-1} B^T Y$.

Construct the grey reflection equation (or grey whitenization equation):

$$\widehat{\alpha}_1 \frac{d^2 x^{(-1)}(t)}{dt^2} + \frac{dx^{(0)}(t)}{dt} + \widehat{\alpha}_2 x^{(1)}(t) = \widehat{b}. \tag{3}$$

Then, solve the grey reflection equation and obtain time response function. The homogenous differential equation of (3) is constructed as follows:

$$\widehat{\alpha}_1 \frac{d^2 x^{(1)}(t)}{dt^2} + \frac{dx^{(1)}(t)}{dt} + \widehat{\alpha}_2 x^{(1)}(t) = 0. \tag{4}$$

And the characteristic equation is

$$\widehat{\alpha}_1 r^2 + r + \widehat{\alpha}_2 = 0. \tag{5}$$

For analyzing the simulation function, let $\Lambda = \widehat{\alpha}_1\widehat{\alpha}_2$ and Λ is used to check the pattern of system development. For $\Lambda > 1/4$, solution of SOGM(2,1) has a pair of conjugate complex roots $r_1, r_2 = \gamma \pm \beta i$; for $\Lambda = 1/4$, SOGM(2,1) model has a pair of equal real roots, $r_1, r_2 = -1/2\alpha_1$; for $\Lambda < 1/4$, SOGM(2,1) model has a pair of unequal real roots.

According to the roots of (5), the form of time response function can be determined. First, while $\Lambda > 1/4$, the time response function will be

$$\widehat{x}^{(0)}(t) = f(t) = e^{\gamma t}\left[\left(\gamma c_1 + \beta c_2\right)\cos\beta t + \left(\gamma c_2 - \beta c_1\right)\sin\beta t\right]. \tag{6}$$

Second, while $\Lambda = 1/4$, the time response function will be

$$\widehat{x}^{(0)}(t) = f(t) = \left(rc_1 + rc_2 t + c_2\right)e^{rt}. \tag{7}$$

Third, while $\Lambda < 1/4$, the function will be

$$\widehat{x}^{(0)}(t) = f(t) = c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t}. \tag{8}$$

The SOGM(2,1) model can simulate a certain pattern of actual system running. If $\Lambda = 1/4$, the model reveals a nonlinear development combining linear growth and exponential growth; if $\Lambda > 1/4$, it reveals a periodic trend combining a with power growth; if $\Lambda < 1/4$, SOGM(2,1) model contains the scope of quasiexponential law with a certain deviation.

The constants of TRF could be established by the latest actual data according to priority of new information.

After the algorithm program, an analysis of the TRF is presented, and the initial terms will be established, which is used to acquire discrete simulation values. Unlike GM(1,1) model, the TRF needs two different initial terms. Grey systems theory suggests a priority of the latest information in few data modeling.

Criteria 1 (priority of latest information). Suppose time response function could be described as $\widehat{x}^{(1)}(t) = f(t, c_1, \ldots, c_p)$, and $1 \le p \le n$. Restraint conditions for the function satisfying priority of new information should be

$$\begin{aligned} &f\left(n, c_1, \ldots, c_p\right) - x^{(0)}(n) = 0,\\ &\vdots\\ &f\left(n-p+1, c_1, \ldots, c_p\right) - x^{(0)}(n-p+1) = 0. \end{aligned} \tag{9}$$

For $p = 2$, the conditions are

$$\begin{aligned} &f\left(n, c_1, c_2\right) - x^{(0)}(n) = 0,\\ &f\left(n-1, c_1, c_2\right) - x^{(0)}(n-1) = 0. \end{aligned} \tag{10}$$

Theorem 1. *For $\Lambda < 1/4$, values of c_1, c_2 satisfying Criteria 1 should be*

$$\begin{aligned} c_1 &= \frac{r_2 e^{r_2(n-1)} x^{(0)}(n) - r_1 e^{r_1 n} x^{(0)}(n-1)}{r_1 r_2 e^{r_1 n + r_2(n-1)} - r_1 r_2 e^{r_2 n + r_1(n-1)}},\\ c_2 &= \frac{r_1 e^{r_1 n} x^{(0)}(n-1) - r_1 e^{r_1(n-1)} x^{(0)}(n)}{r_1 r_2 e^{r_1 n + r_2(n-1)} - r_1 r_2 e^{r_2 n + r_1(n-1)}}. \end{aligned} \tag{11}$$

Proof. Using Criteria 1 as constraint condition, equations system (10) with substitution of original sequence can be presented as matrix form as follows:

$$\begin{bmatrix} r_1 e^{r_1 n} & r_2 e^{r_2 n} \\ r_1 e^{r_1(n-1)} & r_2 e^{r_2(n-1)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x^{(0)}(n) \\ x^{(0)}(n-1) \end{bmatrix}. \tag{12}$$

According to the Cramer rule, solution of the equation is presented as

$$\begin{aligned} c_1 &= \frac{r_2 e^{r_2(n-1)} x^{(0)}(n) - r_1 e^{r_1 n} x^{(0)}(n-1)}{r_1 r_2 e^{r_1 n + r_2(n-1)} - r_1 r_2 e^{r_2 n + r_1(n-1)}},\\ c_2 &= \frac{r_1 e^{r_1 n} x^{(0)}(n-1) - r_1 e^{r_1(n-1)} x^{(0)}(n)}{r_1 r_2 e^{r_1 n + r_2(n-1)} - r_1 r_2 e^{r_2 n + r_1(n-1)}}. \end{aligned} \tag{13}$$

As for the integral modeling algorithm, Figure 1 shows the stepwise programs. □

4. Case Study

4.1. Model Test. The modeling precision could be tested by the following criteria, and we investigate the forecasting performance by these methods.

(1) Relative percentage error (PRE) compares the original data and the simulation values to evaluate the precision of a specific point; $e(k)$ is defined as

$$e(k) = \frac{\left|\widehat{x}^{(0)}(k) - x^{(0)}(k)\right|}{x^{(0)}(k)} \times 100\%. \tag{14}$$

(2) Average relative percentage error (APRE) can evaluate the model's total precision, and computation method can be shown as follows:

$$\text{ARPE} = \frac{1}{n}\sum_{k=1}^{n} e(k). \tag{15}$$

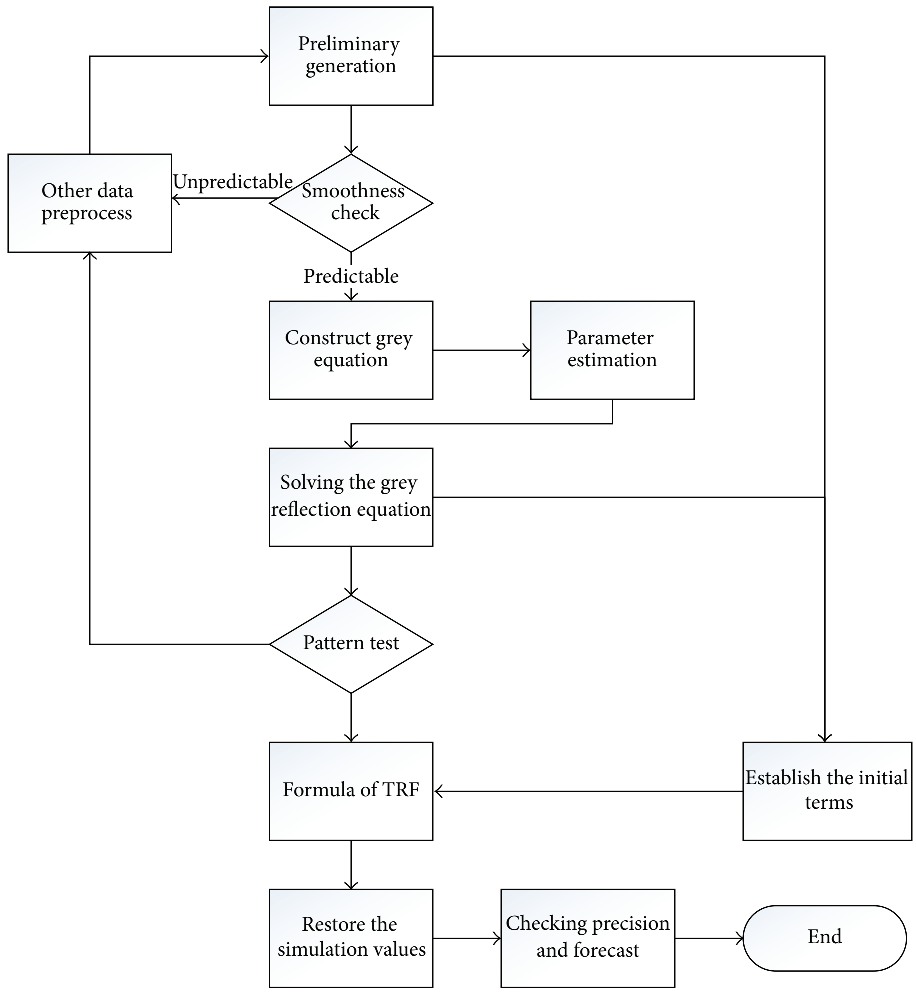

Figure 1: Modeling steps of SOGM(2,1).

Table 1: Evaluation for model precision.

APRE (%)	Forecasting ability
<5	Highly accurate predictability
5–10	Good predictability
10–20	Reasonable predictability
>20	Weak and inaccurate predictability

(3) Forecasting ability can be evaluated mainly by ARPE. Table 1 shows the levels for testing a model forecasting ability.

4.2. Modeling Process of the Numerical Case. Grey prediction technique is useful in engineering. The new model SOGM(2,1) is used to forecast a highway subgrade settlement as one part of a project postassessment. The data is measured in a highway construction project in Jiangsu, one of China's provinces. Measurements have equal time intervals. Measurements are listed in Table 2 and the development trend is presented in Figure 2.

Table 2: Measurements of highway settlement (unit: mm).

Time	1	2	3	4	5
Measurement	1.9892	2.1702	2.3266	2.4332	2.4525

Analyze the trend of the measurements; Figure 1 indicates that the highway subgrade settlement has a saturating tendency, in which the increase rate decreases. Since the data size is few and cannot reach statistical requirement, grey model is a reasonable alternative.

Brief description presents the modeling process. Let actual data in Table 1 be the original sequence $X^{(0)}$. $X^{(1)}$ is the AGO sequence and $X^{(-1)}$ the IAGO sequence as introduced earlier. Construct SOGM(2,1) model and calculate the values of parameters by LS method.

Construct the grey reflection equation of SOGM(2,1) and estimated values of parameter are substituted in

$$-2.4881\frac{d^2x^{(1)}(t)}{dt^2}+\frac{dx^{(1)}(t)}{dt}-0.0968x^{(1)}(t)=1.4221. \quad (16)$$

Table 3: Calculations and comparison of SOGM(2,1) and GM(2,1).

Time	Original data	SOGM(2,1)		GM(2,1)		GM(1,1)	
		Fitting value	RPE	Fitting value	RPE	Fitting value	RPE
1	1.9892	2.0460	2.85%	1.9892	0.00%	1.9892	0
2	2.1702	2.2071	1.70%	2.2504	1.93%	2.2060	1.65%
3	2.3266	2.3426	0.69%	3.1339	13.68%	2.2964	1.30%
4	2.4332	2.4332	0.00%	4.3049	30.94%	2.3905	1.75%
5	2.4525	2.4525	0.00%	5.8496	54.14%	2.4885	1.47%
ARPE			1.05%		20.14%		1.23%

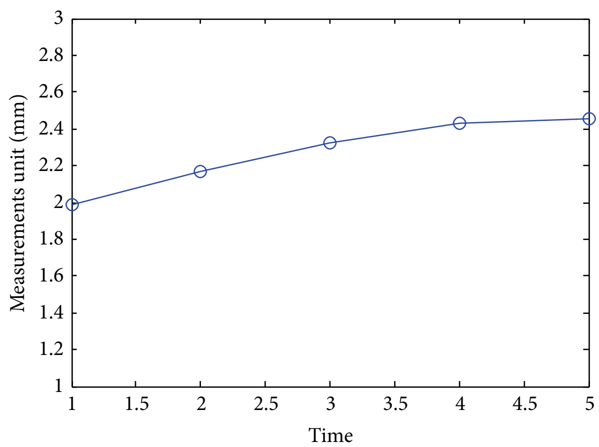

Figure 2: Scattering of original data.

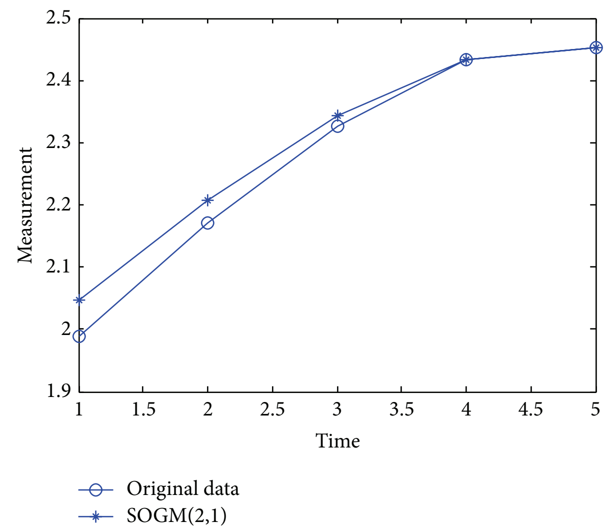

Figure 3: Simulation of SOGM(2,1) model.

Characteristic equation is $-2.4881r^2 + r - 0.0968 = 0$. Solve the equation and acquire the solution as $r_1 = 0.1626, r_2 = 0.2393$. After that, we could get $\Lambda = 0.0389$. Since $\Lambda < 1/4$, time response function should be $\widehat{x}^{(0)}(t) = c_1 0.1626e^{0.1626t} + c_2 0.2393e^{0.2393t}$.

Establish the initial constants c_1, c_2. Calculated as Theorem 1, the solution is $c_1 = 21.8565, c_2 = -7.0294$. Then, the time response function can be achieved as

$$\widehat{x}^{(0)}(k) = 3.5539 \cdot e^{0.1626k} - 1.6821 \cdot e^{0.2393k}. \tag{17}$$

The simulation values can be calculated from (17). For $k > n$, calculated value will be treated as prediction one. Table 3 shows the results of SOGM(2,1) model, GM(2,1) model, and GM(1,1) model as comparison.

From the accuracy point of view, APRE of SOGM(2,1) model is 1.05% and is lower than that of GM(2,1) 20.14% and GM(1,1) 1.23%. The error series of SOGM(2,1) has a convergence trend, which is indicated in Figure 1. The result shows that SOGM(2,1) is better than traditional GM(2,1) model.

From Figures 3, 4, and 5, we could conclude that SOGM(2,1) model has a highly accurate predictability. The APRE of SOGM(2,1) and GM(1,1) is 1.05% and 1.23%, respectively. However, SOGM(2,1) simulates a similar trend of actual data but the simulation of GM(1,1) has an ascending trend.

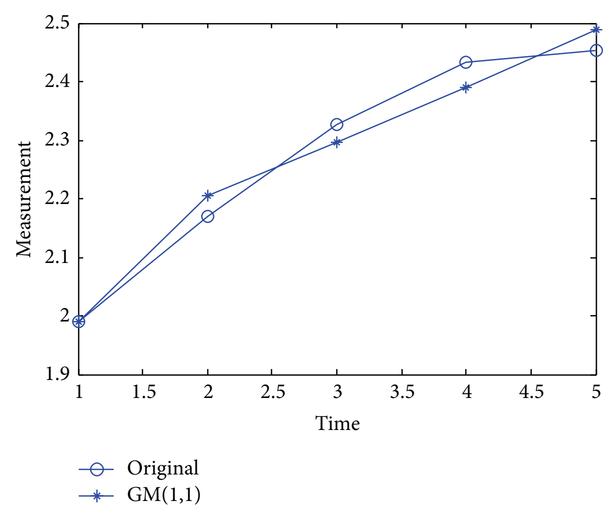

Figure 4: Simulation of GM(2,1) model.

The original sequence describes a system tending to saturation, and SOGM(2,1) has extracted the tendency information from AGO sequence and IAGO sequence, simulating precisely. However, time response function of GM(2,1) has a disparity from original sequence. So the simulation indicates

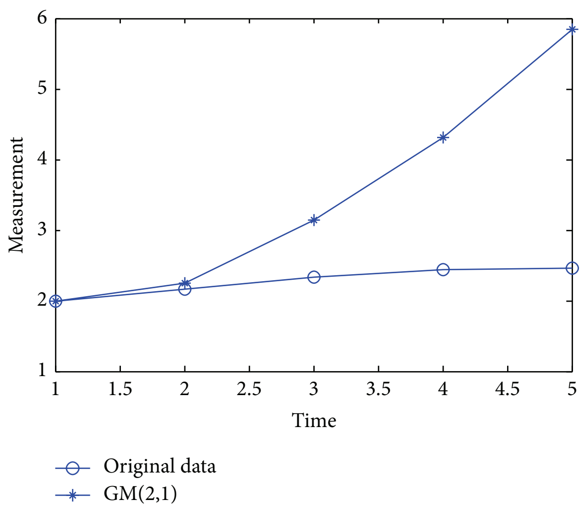

Figure 5: Simulation of GM(1,1) model.

that the new model has a better information processing ability and is more flexible to fit larger scope of system characteristic than traditional grey model method.

5. Conclusion

We proposed the algorithm of SOGM(2,1) model for few data problems; its main objective is to forecast in short terms of engineering project. The new model can achieve high precision comparing with traditional GM(1,1) and GM(2,1) models. And it is believed to be adaptive to many uncertain system problems.

Nomenclature

SOGM(2,1):	Structure optimized GM(2,1) model
AGO:	Accumulated generating operation
IAGO:	Inverse accumulated generating operation
α_1, α_2:	Development coefficient
b:	Grey control variable
n:	Length of sequence in model
h:	Length of forecasting horizon
$X^{(0)}$:	Original sequence
$X^{(1)}$:	First-order generated sequence by AGO
$X^{(-1)}$:	First-order generated sequence by IAGO
c_1, c_2:	Constants of the time response function
$Z^{(1)}$:	Background value sequence
$e(k)$:	Relative error of k point
TRF:	Time response function
PRE:	Relative percentage error
APRE:	Average of relative percentage error.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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