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Research Article

Dependence of Eigenvalues of a Class of Higher-Order Sturm-Liouville Problems on the Boundary

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We show that the eigenvalues of a class of higher-order Sturm-Liouville problems depend not only continuously but also smoothly on boundary points and that the derivative of the nth eigenvalue as a function of an endpoint satisfies a first order differential equation. In addition, we prove that as the length of the interval shrinks to zero all $2k$th-order Dirichlet eigenvalues march off to plus infinity; this is also true for the first (i.e., lowest) eigenvalue.

1. Introduction

Dauge and Helffer in [1, 2] considered the second-order Sturm-Liouville (SL) problems and obtained the equations for the eigenvalues of self-adjoint separated boundary conditions. In addition, they showed that the lowest Dirichlet eigenvalue is a decreasing function of the endpoints and thus must have a finite or infinite limit as the end-points approach each other but left open the question of whether this limit is finite or infinite. In [3] the authors showed that it is infinite.

Following the above, Ge et al. in [4] considered the fourth-order Sturm-Liouville differential equation

$$\left(p_0 y''\right)'' + p_2 y = \lambda w y \tag{1}$$

with $p_0, p_2, w : I = (A, B) \to \mathbb{R}$, $1/p_0, p_2 \in L_{\text{loc}}(I)$ and $w > 0$ a.e. on I. They showed that its Neumann eigenvalues and Dirichlet eigenvalues, as functions of an endpoint, satisfy the same differential equation form as [1, 2] and the equation for the eigenvalues of self-adjoint separated boundary conditions

$$\lambda' = -\frac{1}{p_0}\left(p_0 u''\right)^2 + \left(p_2 - \lambda w\right) u^2 + 2\left(p_0 u''\right)' u'. \tag{2}$$

In particular, they also proved that the lowest Dirichlet eigenvalue is a decreasing function of the endpoints and thus have infinite limit as the endpoints approach each other.

In this paper, partly motivated by the work of Ge et al. in [4], we continue to consider the dependence of eigenvalues of more general form and higher $2k$th-order Sturm-Liouville problems on the boundary and also show that the eigenvalues depend not only continuously but also smoothly on boundary points and that the $2k$th-order Dirichlet eigenvalues, as functions of the endpoint b, satisfy a differential equation of the form

$$p_0 \lambda' = -\left(u^{[k]}\right)^2. \tag{3}$$

We also find the equation satisfied by the $2k$th-order Neumann eigenvalues

$$\lambda' = \sum_{r=1}^{k-1} p_{k-r}\left(u^{(r)}\right)^2 + \left(p_k - \lambda w\right) u^2, \tag{4}$$

and the equation for the eigenvalues of self-adjoint separated boundary conditions,

$$\lambda' = \sum_{r=1}^{k-1}\left[p_{k-r}\left(u^{(r)}\right)^2 + 2u^{[2k-r]}u^{(r)}\right] + \left(p_k - \lambda w\right) u^2 - \frac{\left(u^{[k]}\right)^2}{p_0}. \tag{5}$$

Furthermore, we prove that as the length of the interval shrinks to zero all higher $2k$th-order Dirichlet eigenvalues march off to plus infinity; this is also true for the first (i.e., lowest) eigenvalue. Although we use the same method of proof as in [4] to get our main results, the specific process of calculation and proof is not completely the same as in [4]. Besides that our conclusions are more concrete and general, theoretical importance, the dependence of the eigenvalues on the interval is fundamental from the numerical point of view (see, e.g., [1–9]).

In Section 2, we summarize some of the basic results needed later and establish the notation. The main results of fourth-order Sturm-Liouville problem are given in Section 3. In Section 4, we consider higher $2k$th-order Sturm-Liouville problems and obtain more important results. The last section involves some interesting description about Sturm-Liouville-type boundary value problems.

2. Notation and Basic Results

Consider the differential equation

$$ly = \sum_{r=0}^{k} (-1)^r \left(p_{k-r} y^{(r)}\right)^{(r)} = \lambda w y \quad \text{on } (A,B),\ -\infty \le A < B \le \infty \quad \text{with } \lambda \in \mathbb{R}, \tag{6}$$

where

$$p_0, p_i, w : I = (A,B) \longrightarrow \mathbb{R}; \qquad \frac{1}{p_0}, p_i \in L_{\text{loc}}(I), \quad w > 0 \quad \text{a.e. on } I, \quad i = 1,2,\ldots,k. \tag{7}$$

We introduce the quasi derivatives of a function y, $y^{[j]}$, $j = 0,1,2,\ldots,2k$ as follows:

$$\begin{aligned} y^{[j]} &= y^{(j)}, \quad j = 0,1,2,\ldots,k-1, \\ y^{[k]} &= p_0 y^{(k)}, \\ y^{[j]} &= \left(y^{[j-1]}\right)' + (-1)^{j-k} p_{j-k} y^{(2k-j)}, \\ &\qquad j = k+1, k+2, \ldots, 2k; \end{aligned} \tag{8}$$

then l in (6) may be simply written by

$$ly = (-1)^k y^{[2k]} = \lambda w y. \tag{9}$$

In this way, the differential expression l on I is defined for all functions y such that $y^{[0]}$, $y^{[1]}, \ldots, y^{[2k-1]}$ exist and are absolutely continuous over compact subintervals of I.

Let

$$J = [a,b], \quad A < a < b < B, \tag{10}$$

and consider boundary conditions (BC)

$$C \begin{pmatrix} y(a) \\ y^{[1]}(a) \\ \vdots \\ y^{[2k-1]}(a) \end{pmatrix} + D \begin{pmatrix} y(b) \\ y^{[1]}(b) \\ \vdots \\ y^{[2k-1]}(b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{11}$$

where the complex $2k \times 2k$ matrices C and D satisfy

$$\text{The } 2k \times 4k \text{ matrices } (C \mid D) \text{ have full rank,} \quad CEC^* = DED^*, \qquad E = \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{12}$$

A SL boundary value problem consists of (6) together with boundary conditions (BC) (11). With conditions (7), (10), and (12) it is well known that problem (6), (11) is a regular $2k$th-order self-adjoint SL problem which has an infinite but countable number of only real eigenvalues.

From [10], these self-adjoint boundary conditions (11)-(12) are divided into three disjoint subclasses: separated, coupled, and mixed. In the separated case, there are many forms for the $2k$th-order problems. In this paper, we only study one form of them.

Consider the following boundary conditions (BC):

$$\cos\alpha y(a) - \sin\alpha y^{[2k-1]}(a) = 0, \tag{13a}$$

$$\cos\alpha y^{[1]}(a) - \sin\alpha y^{[2k-2]}(a) = 0, \quad \vdots \tag{13b}$$

$$\cos\alpha y^{[k-1]}(a) - \sin\alpha y^{[k]}(a) = 0, \quad 0 \le \alpha < \pi, \tag{13c}$$

$$\cos\beta y(b) - \sin\beta y^{[2k-1]}(b) = 0, \tag{14a}$$

$$\cos\beta y^{[1]}(b) - \sin\beta y^{[2k-2]}(b) = 0, \quad \vdots \tag{14b}$$

$$\cos\beta y^{[k-1]}(b) - \sin\beta y^{[k]}(b) = 0, \quad 0 < \beta \le \pi. \tag{14c}$$

Here we fix p_i $(i = 0,1,2,\ldots,k)$, w and the boundary condition (constants), and one endpoint and study the dependence of the eigenvalues and eigenfunctions on the other endpoint.

By a solution of (6) on I we mean a function $y^{[0]}$, $y^{[1]}, \ldots, y^{[2k-1]} \in AC_{\text{loc}}(I)$ and (6) is satisfied a.e. on I. Here $AC_{\text{loc}}(I)$ denotes the set of functions which are absolutely continuous on all compact subintervals of I.

It is well known that the $2k$th-order SL boundary value problem consisting of (6) together with boundary conditions (BC) (13a)–(13c), (14a)–(14c) is a regular $2k$th-order self-adjoint boundary value problem which has an infinite but countable number of only real eigenvalues. If $p_0 \ge 0$, a.e. on $J = (a,b)$, then the eigenvalues are bounded below and can be ordered to satisfy

$$-\infty < \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots, \qquad \lambda_n \longrightarrow +\infty \quad \text{as } n \longrightarrow +\infty. \tag{15}$$

Notation. Let $N_0 = \{0, 1, 2, \ldots\}$; for the fourth-order or higher-order Dirichlet and Neumann eigenvalues we use the special notation

$$\lambda_n^D = \lambda_n (0, \pi; a, b), \qquad \lambda_n^N = \lambda_n \left(\frac{\pi}{2}, \frac{\pi}{2}; a, b\right). \tag{16}$$

By a normalized eigenfunction u of the BVP (6), (13a)–(14c), we mean an eigenfunction u that satisfies

$$\int_a^b |u|^2 w = 1. \tag{17}$$

For fixed a and fixed boundary condition constants α, β we abbreviate the notation to $\lambda_n(b)$ and study $\lambda_n(b)$ as a function of b for fixed $n \in N_0$, as b varies in the interval (a, B).

In the following, we present a continuity result for the eigenvalues and eigenfunctions.

Lemma 1. *Let self-adjoint boundary value problems be described as (6), (13a)–(14c). Fix the BC and the endpoint a or b. Fix $n \in N_0$. Let $\lambda_n = \lambda_n(b)$ for $b \in (a, B)$. Then*

(1) *$\lambda_n(b)$ is a continuous function of b for $b \in (a, B)$.*

(2) *If $\lambda_n(b)$ is simple for some $b \in (a, B)$ then $\lambda_n(b)$ is simple for every $b \in (a, B)$.*

(3) *There exists a normalized eigenfunction $u_n(\cdot, b)$ of $\lambda_n(b)$ for $b \in (a, B)$ such that, $(u_n^{[j]})(\cdot, b)$ $(j = 0, 1, 2, \ldots, 2k-1)$ are uniformly convergent in b on any compact subinterval of (a, B); that is,*

$$u_n^{[j]} (\cdot, b + h) \longrightarrow u_n^{[j]} (\cdot, b), \quad j = 0, 1, 2, \ldots, 2k-1, \tag{18}$$

and this convergence is uniform on any compact subinterval of (a, B).

Proof. See the proof of Theorem 3 in [3]. □

Lemma 2. *Assume u and v are solutions of (6) with $\lambda = \mu$ and $\lambda = \nu$, respectively. Then*

$$\begin{aligned} [u, v]_a^b &= [u, v] (b) - [u, v] (a) \\ &= (-1)^k \sum_{r=0}^{2k-1} (-1)^{2k+1-r} u^{[r]} \overline{v}^{[2k-r-1]} \\ &= (\mu - \nu) \int_a^b u \overline{v} w. \end{aligned} \tag{19}$$

Proof. This follows from integration by parts. □

Lemma 3. *Assume a real valued function $f \in L_{\text{loc}}(A, B)$. Then*

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} f = f(t) \quad \textit{a.e. in } (A, B). \tag{20}$$

Proof. See the proof given in [3]. □

3. Eigenvalues of Fourth-Order Sturm-Liouville Problem

In this section, we obtain the differentiability of the eigenvalues of the fourth-order boundary value problem, establish differential equations satisfied by them, and discuss the behavior of the Dirichlet eigenvalues as functions of the endpoint b.

Consider the differential equation

$$\begin{gathered} \left(p_0 y''\right)'' - \left(p_1 y'\right)' + p_2 y = \lambda w y \\ \text{on } (A, B), \ -\infty \le A < B \le \infty \quad \text{with } \lambda \in \mathbb{R}, \end{gathered} \tag{21}$$

where

$$\begin{gathered} p_0, p_1, p_2, w : I = (A, B) \longrightarrow \mathbb{R}; \\ \frac{1}{p_0}, p_1, p_2 \in L_{\text{loc}} (I), \quad w > 0 \text{ a.e. on } I. \end{gathered} \tag{22}$$

Let $J = [a, b]$, $A < a < b < B$ and consider the following boundary conditions (BC)

$$\cos \alpha y (a) - \sin \alpha y^{[3]} (a) = 0, \tag{23a}$$

$$\cos \alpha y^{[1]} (a) - \sin \alpha y^{[2]} (a) = 0, \quad 0 \le \alpha < \pi, \tag{23b}$$

$$\cos \beta y (b) - \sin \beta y^{[3]} (b) = 0, \tag{24a}$$

$$\cos \beta y^{[1]} (b) - \sin \beta y^{[2]} (b) = 0, \quad 0 < \beta \le \pi, \tag{24b}$$

where $y^{[1]} = y'$, $y^{[2]} = p_0 y''$, $y^{[3]} = (p_0 y'')' - p_1 y'$ are quasiderivative. Fix p_0, p_1, p_2, w and the boundary condition (constants) and one endpoint and study the dependence of the eigenvalues and eigenfunctions on the other endpoint.

Theorem 4 (fourth-order Dirichlet eigenvalue-eigenfunction differential equation). *Let (22) hold. Consider the BVP (21), (23a)–(24b), with $0 \le \alpha < \pi$ and $\beta = \pi$, that is, arbitrary separated conditions at a and the fourth-order Dirichlet conditions at b. Using the notation of Section 2 and letting $\lambda = \lambda_n$, $u = u_n$, we have the following differential equation:*

$$\left(p_0 \lambda'\right) (b) = -\left(u^{[2]}\right)^2 (b, b) \quad \textit{a.e. in } (a, B). \tag{25}$$

In particular, if p_0 is continuous at $b \in [a, B)$ and $p_0(b) \neq 0$, then (25) holds at b.

Proof. For small h, in (19), choose $k = 2$, $\mu = \lambda(b)$, $\nu = \lambda(b + h)$, and $u = u(\cdot, b)$, $v = u(\cdot, b+h)$. From (19) and the boundary conditions (23a)–(24b), noting that $[u, v](a) = 0$, $u(b, b) = 0$ and $u'(b, b) = 0$, we have

$$\begin{aligned} &u^{[3]} (b, b) u (b, b + h) - u^{[2]} (b, b) u' (b, b + h) \\ &\quad = [\lambda (b) - \lambda (b + h)] \int_a^b u (s, b) u (s, b + h) w (s) \, ds. \end{aligned} \tag{26}$$

Since

$$u(b,b+h) = u(b,b+h) - u(b+h,b+h) = -\int_b^{b+h} u'(s,b+h)\,ds. \tag{27}$$

By Lemmas 1 and 3, we have

$$\lim_{h\to 0}\frac{u(b,b+h)}{h} = -u'(b,b). \tag{28}$$

Also from

$$\begin{aligned} u'(b,b+h) &= u'(b,b+h) - u'(b+h,b+h) \\ &= -\int_b^{b+h} u''(s,b+h)\,ds \\ &= -\int_b^{b+h} \frac{\left(p_0u''\right)(s,b+h)}{p_0(s)}ds, \end{aligned} \tag{29}$$

we have

$$\lim_{h\to 0}\frac{u'(b,b+h)}{h} = -\frac{1}{p_0(b)}\left(p_0u''\right)(b,b). \tag{30}$$

And we can obtain

$$\int_a^b u(s,b)\,u(s,b+h)\,w(s)\,ds \longrightarrow \int_a^b u^2(s,b)\,w(s)\,ds = 1, \quad \text{as } h \longrightarrow 0. \tag{31}$$

Plugging (28), (30), and (31) into (26) divided by h and taking the limit as $h \to 0$, we get (25). The second part of the theorem follows from above. □

Theorem 5 (fourth-order Neumann eigenvalue-eigenfunction differential equation). *Let (22) hold. Consider the BVP (21), (23a)–(24b), with $0 \le \alpha < \pi$ and $\beta = \pi/2$, that is, arbitrary separated conditions at a and the fourth-order Neumann conditions at b. Using the notation of Section 2 and letting $\lambda = \lambda_n$, $u = u_n$, we have the following differential equation:*

$$\begin{aligned} \lambda'(b) = {} & p_1(b)\left(u'(b,b)\right)^2 \\ & + \left(p_2(b) - \lambda(b)\,w(b)\right)u^2(b,b) \quad \textit{a.e. in } (a,B). \end{aligned} \tag{32}$$

In particular, if p_1, p_2 and w are continuous at $b \in [a,B)$, then (32) holds at b.

Proof. The proof is similar to Theorem 4. For small h, we choose $\mu = \lambda(b)$, $\nu = \lambda(b+h)$, and $u = u(\cdot,b)$, $v = u(\cdot,b+h)$. From (19) and the boundary conditions (23a)–(24b), noting that $[u,v](a) = 0$, $u^{[2]}(b,b) = 0$ and $u^{[3]}(b,b) = 0$, we have

$$\begin{aligned} & u'(b,b)\,u^{[2]}(b,b+h) - u(b,b)\,u^{[3]}(b,b+h) \\ & \quad = [\lambda(b) - \lambda(b+h)]\int_a^b u(s,b)\,u(s,b+h)\,w(s)\,ds. \end{aligned} \tag{33}$$

$$\begin{aligned} & u^{[3]}(b,b+h) \\ & = u^{[3]}(b,b+h) - u^{[3]}(b+h,b+h) \\ & = -\int_b^{b+h}\left(u^{[3]}\right)'(s,b+h)\,ds \\ & = \int_b^{b+h}\left[p_2(s)\,u(s,b+h) - \lambda(b+h)\,u(s,b+h)\,w(s)\right]ds \\ & = \int_b^{b+h} p_2(s)\,u(s,b)\,ds \\ & \quad + \int_b^{b+h} p_2(s)\,u(s,b+h) - p_2(s)\,u(s,b)\,ds \\ & \quad - \lambda(b+h)\int_b^{b+h} u(s,b)\,w(s)\,ds \\ & \quad + \lambda(b+h)\int_b^{b+h}[u(s,b) - u(s,b+h)]\,w(s)\,ds \\ & = \int_b^{b+h} p_2(s)\,u(s,b)\,ds - \lambda(b+h)\int_b^{b+h} u(s,b)\,w(s)\,ds. \end{aligned} \tag{34}$$

By Lemmas 1 and 3 we have

$$\lim_{h\to 0}\frac{u^{[3]}(b,b+h)}{h} = \left(p_2(b) - \lambda(b)\,w(b)\right)u(b,b). \tag{35}$$

In a similar way, we have

$$\lim_{h\to 0}\frac{p_0u''(b,b+h)}{h} = -\left(p_0u''\right)'(b,b). \tag{36}$$

Combining $u^{[3]}(b,b) = [(p_0u'')' - p_1u'](b,b) = 0$, we also can get

$$\lim_{h\to 0}\frac{p_0u''(b,b+h)}{h} = -p_1u'(b,b). \tag{37}$$

When $h \to 0$, noting that

$$\int_a^b u(s,b)\,u(s,b+h)\,w(s)\,ds \longrightarrow \int_a^b u^2(s,b)\,w(s)\,ds = 1, \tag{38}$$

and plugging (35)–(38) into (33), then we obtain (32). The second part of the theorem follows from the above. □

Theorem 6 (eigenvalue-eigenfunction differential equation for separated BVPs). *Let (22) hold. Consider the BVP (21), (23a)–(24b), with $0 \le \alpha < \pi$, $0 < \beta \le \pi$, that is, arbitrary separated conditions at a and b. Using the notation of Section 2 and letting $\lambda = \lambda_n$, $u = u_n$, we have the following differential equations:*

$$\lambda'(b) = p_1(b)\left(u'(b,b)\right)^2 + 2u^{[3]}(b,b)\,u'(b,b) + \left(p_2(b) - \lambda(b)w(b)\right)u^2(b,b) - \frac{\left(u^{[2]}\right)^2(b,b)}{p_0(b)} \quad \text{a.e. in } (a,B). \tag{39}$$

Furthermore, if $\beta \ne \pi$, then

$$\lambda'(b) = p_1(b)\left(u'(b,b)\right)^2 + 2\cot\beta \mathrm{u}(\mathrm{b},\mathrm{b})\,\mathrm{u}'(\mathrm{b},\mathrm{b}) + \left(p_2(b) - \lambda(b)w(b)\right)u^2(b,b) - \frac{\left(u^{[2]}\right)^2(b,b)}{p_0(b)}. \tag{40}$$

If $\beta \ne \pi/2$, then

$$\lambda'(b) = p_1(b)\left(u'(b,b)\right)^2 + 2\tan\beta u^{[2]}(b,b)\,u^{[3]}(b,b) + \left(p_2(b) - \lambda(b)w(b)\right)u^2(b,b) - \frac{\left(u^{[2]}\right)^2(b,b)}{p_0(b)}. \tag{41}$$

In particular, if p_0, p_1, and p_2 and w are continuous at b and $p_0(b) \ne 0$, then (39)–(41) hold at b.

Proof. The proof is more complicated but consists basically of combining the techniques in the proofs of Theorems 4 and 5. For small h, we choose $\mu = \lambda(b)$, $\nu = \lambda(b+h)$, and $u = u(\cdot,b)$, $v = u(\cdot,b+h)$. From (19) and the boundary conditions (23a)–(24b), noting that $[u,v](a) = 0$, we have

$$\begin{aligned}[\lambda(b) - \lambda(b+h)]\int_a^b u(s,b)\,u(s,b+h)\,w(s)\,ds &= u^{[3]}(b,b)\,u(b,b+h) - u^{[2]}(b,b)\,u'(b,b+h) \\ &\quad + u'(b,b)\,u^{[2]}(b,b+h) - u(b,b)\,u^{[3]}(b,b+h).\end{aligned} \tag{42}$$

Now dividing (42) by h and taking the limit as $h \to 0$, plugging (28), (30), (35), and (36) into (42), and using the continuity of λ at b, the uniform convergence of $u(\cdot,b+h)$ to $u(\cdot,b)$, and Lemma 3, we obtain (39). In addition, from the boundary conditions (24a)–(24b) we note that if $\beta \ne \pi$ then $y^{[3]}(b) = \cot\beta \mathrm{y}(\mathrm{b})$ and if $\beta \ne \pi/2$ then $y^{[1]}(b) = \tan\beta y^{[2]}(b)$; plugging them into (39) we obtain (40) and (41). □

It is easy to see that Theorem 6 includes Theorems 4 and 5.

Theorem 7. *Let (22) hold. Fix a and consider the fourth-order Dirichlet eigenvalues $\lambda_n^D(b) = \lambda_n(b)(0,\pi;a,b)$ for b in (a,B) defined as in (16). If*

$$p_0 \ge 0 \text{ a.e.}, \qquad \frac{p_2^2}{w} \in L_{\mathrm{loc}}(A,B), \tag{43}$$

then, for $n \in N_0$, $\lambda_n(b)$ is strictly decreasing on (a,B) and

$$\lambda_n^D(b) \longrightarrow +\infty \quad \text{as } b \longrightarrow a^+. \tag{44}$$

Proof. The decreasing property of λ_n^D as a function of b follows directly from Theorem 4. Assume (44) is false, and then by Theorem 4 $\lambda(b) = \lambda_0^D(b)$ has a finite limit, say $\lambda^+(a)$, as $b \to a^+$ and hence is bounded on (a,B_1) for $B_1 < B$. Let $u = u_0(\cdot,b)$ be an eigenfunction of $\lambda(b)$ normalized to satisfy

$$\int_a^b u^2 w = 1. \tag{45}$$

Next we show that

$$\left(p_0 u''\right)'(a,b) \longrightarrow 0 \quad \text{as } b \longrightarrow a^+. \tag{46}$$

To see this, we first show there exists at least one point $c \in (a,b)$ such that $(p_0u'')'(c,b) = 0$. Noting that $u(a,b) = u(b,b) = 0$ and according to the Rolle's theorem we know, there exists at least one point $\xi_0 \in (a,b)$ such that $u'(\xi_0,b) = 0$. Similarly, noting that $u'(a,b) = u'(b,b) = 0$, hence there exist at least two points $\xi_1 \in (a,\xi_0)$ and $\xi_2 \in (\xi_0,b)$, such that $(p_0u'')(\xi_1,b) = (p_0u'')(\xi_2,b)$. Therefor there exists at least one point $c \in (\xi_1,\xi_2)$, such that $(p_0u'')'(c,b) = 0$. Using $(p_0u'')'(c,b) = 0$, the boundedness of λ and the Schwarz inequality, we get

$$\begin{aligned}\left[\left(p_0u''\right)'(a,b)\right]^2 &= \left[\left(p_0u''\right)'(c,b) - \left(p_0u''\right)'(a,b)\right]^2 \\ &= \left[\int_a^c \left(p_0u''\right)''\right]^2 \\ &= \left[\int_a^c \left(p_1u'\right)' - \left(p_2 - \lambda w\right)u\right]^2 \\ &= \left[\left(p_1u'\right)(c,b) - \int_a^c \left(p_2 - \lambda w\right)u\right]^2 \\ &\le 2\left[\left(p_1u'\right)(c,b)\right]^2 + 2\left[\int_a^c \left(p_2 - \lambda w\right)u\right]^2,\end{aligned}$$

$$\begin{aligned}\left[\int_a^c \left(p_2 - \lambda w\right)u\right]^2 &= \left[\int_a^c \left(p_2w^{-1/2} - \lambda w^{1/2}\right)w^{1/2}u\right]^2 \\ &\le \int_a^c \left(p_2w^{-1/2} - \lambda w^{1/2}\right)^2 \int_a^c u^2w \\ &\le \int_a^b \left(\frac{p_2^2}{w} - 2\lambda p_2 + \lambda^2 w\right)\int_a^b u^2w \longrightarrow 0 \quad \text{as } b \longrightarrow a^+.\end{aligned} \tag{47}$$

So

$$\left[\left(p_0 u''\right)'(a,b)\right]^2 \longrightarrow 0 \quad \text{as } b \longrightarrow a^+. \tag{48}$$

For $p_0 u''(a,b)$, we have

$$\begin{aligned}
&\left[\left(p_0 u''\right)(a,b)\right]^2 \\
&= \left[\left(p_0 u''\right)(\xi_1,b) - \left(p_0 u''\right)(a,b)\right]^2 \\
&= \left[\int_a^{\xi_1} \left(p_0 u''\right)'\right]^2 \\
&= \left[\int_a^{\xi_1} \int_t^c \left(p_1 u'\right)' - (p_2 - \lambda w)\, u d\xi\, dt\right]^2 \\
&\le 2\left[\int_a^{\xi_1} \int_t^c \left(p_1 u'\right)' d\xi\, dt\right]^2 + 2\left[\int_a^{\xi_1} \int_t^c (p_2 - \lambda w)\, u d\xi\, dt\right]^2 \\
&= 2\left[\int_a^{\xi_1} \left(p_1 u'\right)(c) - \left(p_1 u'\right)(t)\, dt\right]^2 \\
&\quad + 2\left[\int_a^{\xi_1} \int_t^c (p_2 - \lambda w)\, u d\xi\, dt\right]^2, \\
&\left[\int_a^{\xi_1} \int_t^c (p_2 - \lambda w)\, u d\xi\, dt\right]^2 \\
&\le (b-a) \int_a^{\xi_1} \left[\int_t^c \left(p_2 w^{-1/2} - \lambda w^{1/2}\right)^2 d\xi \int_t^c u^2 w\, d\xi\right] dt \\
&\le (b-a)^2 \int_a^b \left(\frac{p_2^2}{w} - 2\lambda p_2 + \lambda^2 w\right) \\
&\quad \times \int_a^b u^2 w \longrightarrow 0 \quad \text{as } b \longrightarrow a^+.
\end{aligned} \tag{49}$$

Thus

$$\left[\left(p_0 u''\right)(a,b)\right]^2 \longrightarrow 0 \quad \text{as } b \longrightarrow a^+. \tag{50}$$

Noting that $\lambda(b) \to \lambda^+(a)$ as $b \to a^+$, by (46) and the continuous dependence of solutions (21) on initial conditions and on the parameter we conclude that $u(\cdot,b) \to 0$ uniformly on any compact subinterval of $[a,B)$. Therefore, for $\varepsilon > 0$, there exists a $b_0 \in (a,B)$, such that

$$|u(t,b)| < \varepsilon, \quad t \in [a,b], \; a < b < b_0. \tag{51}$$

This implies that

$$\int_a^b u^2 w < \varepsilon^2 \int_a^b w \tag{52}$$

for ε sufficiently small. This contradicts the normalization (45), which completes the proof. □

4. Eigenvalues of Higher-Order Sturm-Liouville Problem

In this section, we obtain the differentiability of the eigenvalues of the $2k$th-order boundary value problem and establish differential equations satisfied by them and discuss the behavior of $2k$th-order Dirichlet eigenvalueas functions of the endpoint b.

Theorem 8 ($2k$th-order Dirichlet eigenvalue-eigenfunction differential equation). *Let (7) hold. Consider the BVP (6), (13a)–(14c), with $0 \le \alpha < \pi$ and $\beta = \pi$, that is, arbitrary separated conditions at a and the 2kth-order Dirichlet conditions at b. Using the notation of Section 2 and letting $\lambda = \lambda_n$, $u = u_n$, we have the following differential equation:*

$$\left(p_0 \lambda'\right)(b) = -\left(u^{[k]}\right)^2 (b,b) \quad \text{a.e. in } (a,B). \tag{53}$$

In particular, if p_0 is continuous at $b \in [a,B)$ and $p_0(b) \ne 0$, then (53) holds at b.

Proof. For small h, in (19), choose $\mu = \lambda(b)$, $\nu = \lambda(b+h)$, and $u = u(\cdot,b)$, $v = u(\cdot,b+h)$. From (19) and the boundary conditions (13a)–(14c), noting that $[u,v](a) = 0$, $u^{[j]}(b,b) = 0$ $(j = 0,1,2,\ldots,k-1)$, we have

$$\begin{aligned}
&(-1)^k \sum_{r=k}^{2k-1} (-1)^{2k+1-r} u^{[r]}(b,b)\, u^{[2k-r-1]}(b,b+h) \\
&\quad = [\lambda(b) - \lambda(b+h)] \int_a^b u(s,b)\, u(s,b+h)\, w(s)\, ds.
\end{aligned} \tag{54}$$

Hence

$$\begin{aligned}
&u^{[r]}(b,b+h) \\
&\quad = u^{(r)}(b,b+h) - u^{(r)}(b+h,b+h) \\
&\quad = -\int_b^{b+h} u^{(r+1)}(s,b+h)\, ds, \quad r = 0,1,\ldots,k-2.
\end{aligned} \tag{55}$$

So by Lemmas 1 and 3, we have

$$\lim_{h \to 0} \frac{u^{(r)}(b,b+h)}{h} = -u^{(r+1)}(b,b), \quad r = 0,1,\ldots,k-2. \tag{56}$$

Similarly, from

$$\begin{aligned}
u^{(k-1)}(b,b+h) &= u^{(k-1)}(b,b+h) - u^{(k-1)}(b+h,b+h) \\
&= -\int_b^{b+h} u^{(k)}(s,b+h)\, ds \\
&= -\int_b^{b+h} \frac{\left(p_0 u^{(k)}\right)(s,b+h)}{p_0(s)} ds,
\end{aligned} \tag{57}$$

we can have

$$\lim_{h\to 0}\frac{u^{(k-1)}(b,b+h)}{h}=-\frac{1}{p_0(b)}\left(p_0u^{(k)}\right)(b,b). \qquad (58)$$

In addition, noting that

$$\int_a^b u(s,b)\,u(s,b+h)\,w(s)\,ds \longrightarrow \int_a^b u^2(s,b)\,w(s)\,ds=1, \quad \text{as } h\longrightarrow 0. \qquad (59)$$

Plugging (56), (58), and (59) into (54) divided by h and taking the limit as $h \to 0$, we get (53). The second part of the theorem follows from above. □

Theorem 9 (2kth-order Neumann eigenvalue-eigenfunction differential equation). *Let (7) hold. Consider the BVP (6), (13a)–(14c), with $0 \le \alpha < \pi$ and $\beta = \pi/2$, that is, arbitrary separated conditions at a and the 2kth-order Neumann conditions at b. Using the notation of Section 2 and letting $\lambda=\lambda_n$, $u=u_n$, we have the following differential equation:*

$$\lambda'(b)=\sum_{r=1}^{k-1}p_{k-r}(b)\left(u^{(r)}(b,b)\right)^2+\left(p_k(b)-\lambda(b)\,w(b)\right)u^2(b,b) \quad \text{a.e. in } (a,B). \qquad (60)$$

In particular, if $p_1,p_2,\dots,p_k$ and w are continuous at $b\in[a,B)$, then (60) holds at b.

Proof. The proof is similar to Theorem 8. For small h, we choose $\mu=\lambda(b)$, $\nu=\lambda(b+h)$, and $u=u(\cdot,b)$, $v=u(\cdot,b+h)$. From (19) and the boundary conditions (13a)–(14c), noting that $[u,v](a)=0$, $u^{[j]}(b,b)=0$, $j=k,k+1,\dots,2k-1$, we have

$$(-1)^k\sum_{r=0}^{k-1}(-1)^{2k+1-r}u^{[r]}(b,b)\,u^{[2k-r-1]}(b,b+h)=[\lambda(b)-\lambda(b+h)]\int_a^b u(s,b)\,u(s,b+h)\,w(s)\,ds, \qquad (61)$$

$$\begin{aligned}
(-1)^k&\left(u^{[2k-1]}\right)(b,b+h)\\
&=(-1)^k u^{[2k-1]}(b,b+h)-(-1)^k u^{[2k-1]}(b+h,b+h)\\
&=-\int_b^{b+h}(-1)^k\left(u^{[2k-1]}\right)'(s,b+h)\,ds\\
&=-\int_b^{b+h}(-1)^k\left(u^{[2k]}-(-1)^k p_k u\right)(s,b+h)\,ds\\
&=\int_b^{b+h}\left[p_k(s)\,u(s,b+h)-\lambda(b+h)\,u(s,b+h)\,w(s)\right]ds\\
&=\int_b^{b+h}p_k(s)\,u(s,b)\,ds\\
&\quad+\int_b^{b+h}p_k(s)\,u(s,b+h)-p_k(s)\,u(s,b)\,ds\\
&\quad-\lambda(b+h)\int_b^{b+h}u(s,b)\,w(s)\,ds\\
&\quad+\lambda(b+h)\int_b^{b+h}[u(s,b)-u(s,b+h)]\,w(s)\,ds\\
&=\int_b^{b+h}p_k(s)\,u(s,b)\,ds-\lambda(b+h)\int_b^{b+h}u(s,b)\,w(s)\,ds.
\end{aligned} \qquad (62)$$

By Lemmas 1 and 3 we have

$$\lim_{h\to 0}\frac{(-1)^k\left(u^{[2k-1]}\right)(b,b+h)}{h}=\left(p_k(b)-\lambda(b)\,w(b)\right)u(b,b). \qquad (63)$$

In a similar way, we have

$$\lim_{h\to 0}\frac{(-1)^k\left(u^{[2k-r-1]}\right)(b,b+h)}{h}=(-1)^{k+1}\left(u^{[2k-r-1]}\right)'(b,b). \qquad (64)$$

Combining $(u^{[2k-r-1]})'=u^{[2k-r]}-(-1)^{k-r}p_{k-r}y^{(r)}$, we also can get

$$\begin{aligned}
\lim_{h\to 0}&\frac{(-1)^k\left(u^{[2k-r-1]}\right)(b,b+h)}{h}\\
&=(-1)^{k+1}u^{[2k-r]}(b,b)-(-1)^{2k-r+1}p_{k-r}u^{(r)}(b,b)\\
&=-(-1)^{2k-r+1}p_{k-r}u^{(r)}(b,b), \quad r=1,2,\dots,k-1.
\end{aligned} \qquad (65)$$

When $h \to 0$, noting that

$$\int_a^b u(s,b)\,u(s,b+h)\,w(s)\,ds \longrightarrow \int_a^b u^2(s,b)\,w(s)\,ds=1, \qquad (66)$$

and plugging (63)–(66) into (61), then we obtain (60). The second part of the theorem follows from the above. □

Theorem 10 (eigenvalue-eigenfunction differential equation for separated BVPs). *Let (7) hold. Consider the BVP (6), (13a)–(14c), with $0 \le \alpha < \pi$, $0 < \beta \le \pi$, that is, arbitrary separated conditions at a and b. Using the notation of Section 2* *and letting $\lambda = \lambda_n$, $u = u_n$, we have the following differential equations:*

$$\lambda'(b) = \sum_{r=1}^{k-1}\left[p_{k-r}(b)\left(u^{[r]}(b,b)\right)^2 + 2u^{[2k-r]}(b,b)\,u^{[r]}(b,b)\right] + \left(p_k(b) - \lambda(b)\,w(b)\right)u^2(b,b) - \frac{\left(u^{[k]}\right)^2(b,b)}{p_0(b)} \quad \text{a.e. in } (a,B). \tag{67}$$

Furthermore, if $\beta \neq \pi$, then

$$\lambda'(b) = \sum_{r=1}^{k-1}\left[p_{k-r}(b)\left(u^{[r]}(b,b)\right)^2 + 2\cot\beta\,u^{[r-1]}(b,b)\,u^{[r]}(b,b)\right] + \left(p_k(b) - \lambda(b)\,w(b)\right)u^2(b,b) - \frac{\left(u^{[k]}\right)^2(b,b)}{p_0(b)}. \tag{68}$$

If $\beta \neq \pi/2$, then

$$\lambda'(b) = \sum_{r=1}^{k-1}\left[p_{k-r}(b)\left(u^{[r]}(b,b)\right)^2 + 2\tan\beta\,u^{[2k-r]}(b,b)\,u^{[2k-r+1]}(b,b)\right] + \left(p_k(b) - \lambda(b)\,w(b)\right)u^2(b,b) - \frac{\left(u^{[k]}\right)^2(b,b)}{p_0(b)}. \tag{69}$$

In particular, if $p_0, p_1, \ldots, p_k$ and w are continuous at b and $p_0(b) \neq 0$, then (67)–(69) hold at b.

Proof. The proof is more complicated but consists basically of combining the techniques in the proofs of Theorems 8 and 9. The concrete process is omitted. □

It is easy to see that Theorem 10 includes Theorems 8 and 9.

Theorem 11. *Let (7) hold. Fix a and consider the 2kth-order Dirichlet eigenvalues $\lambda_n^D(b) = \lambda_n(b)(0,\pi;a,b)$ for b in (a,B) defined as in (16). If*

$$p_0 \geq 0 \text{ a.e.}, \qquad \frac{p_k^2}{w} \in L_{\text{loc}}(A,B), \tag{70}$$

then, for $n \in N_0$, $\lambda_n(b)$ is strictly decreasing on (a,B) and

$$\lambda_n^D(b) \longrightarrow +\infty \quad \text{as } b \longrightarrow a^+. \tag{71}$$

Proof. The decreasing property of λ_n^D as a function of b follows directly from Theorem 8. Assume (71) is false, and then by Theorem 8 $\lambda(b) = \lambda_0^D$ has a finite limit, say $\lambda^+(a)$, as $b \to a^+$, and hence is bounded on (a,B_1) for $B_1 < B$. Let $u = u_0(\cdot,b)$ be an eigenfunction of $\lambda(b)$ normalized to satisfy

$$\int_a^b u^2 w = 1. \tag{72}$$

First we show that

$$\left(p_0 u^{(k)}\right)^{(k-1)}(a,b) \longrightarrow 0 \quad \text{as } b \longrightarrow a^+. \tag{73}$$

To see this, we first show there exists at least one point $c \in (a,b)$ such that $(p_0u^{(k)})^{(k-1)}(c,b) = 0$. Noting that $u^{(r)}(a,b) = u^{(r)}(b,b) = 0$, $r = 0,1,\ldots,k-1$, and according to the Rolle's theorem we know, there exists at least one point $\xi_0 \in (a,b)$ such that $u'(\xi_0,b) = 0$. In the same way, there exist at least r points $\xi_i \in (a,b)$, $\xi_1 < \xi_2 < \cdots < \xi_r$, such that $u^{(r)}(\xi_i,b) = 0$, $i = 1,2,\ldots,r$, $r = 1,2,\ldots,k-1$. So there exist at least k points $\eta_s \in (\xi_{r-1},\xi_r)$, $r = 1,2,\ldots,k$, $\xi_0 = a$, $\xi_k = b$, such that $(p_0u^{(k)})(\eta_s,b) = 0$. Thus by the Rolle's theorem we can get the conclusion that there exist at least $k-r$ points ζ_{k-r}, such that $(p_0u^{(k)})^r(\zeta_{k-r},b) = 0$, $r = 0,1,\ldots,k-1$. In addition using the boundedness of λ and the Schwarz inequality, we get

$$\begin{aligned}
\left[\left(p_0u^{(k)}\right)^{(k-1)}(a,b)\right]^2
&= \left[\left(p_0u^{(k)}\right)^{(k-1)}(c,b) - \left(p_0u^{(k)}\right)^{(k-1)}(a,b)\right]^2 \\
&= \left[\int_a^c \left(p_0u^{(k)}\right)^{(k)}\right]^2 \\
&= \left[\int_a^c \sum_{r=1}^{k-1}\left(p_{k-r}u^{(r)}\right)^{(r)} - \left(p_k - \lambda w\right)u\right]^2 \\
&\leq 2\left[\int_a^c \sum_{r=1}^{k-1}\left(p_{k-r}u^{(r)}\right)^{(r)}\right]^2 + 2\left[\int_a^c \left(p_k - \lambda w\right)u\right]^2 \\
&\leq 2\left[\sum_{r=1}^{k-1}(b-a)^{r-1}\left(p_{k-r}u^{(r)}\right)(t,b)\right]^2 + 2\left[\int_a^c \left(p_k - \lambda w\right)u\right]^2, \\
\left[\int_a^c \left(p_k - \lambda w\right)u\right]^2
&= \left[\int_a^c \left(p_k w^{-1/2} - \lambda w^{1/2}\right)w^{1/2}u\right]^2 \\
&\leq \int_a^c \left(p_k w^{-1/2} - \lambda w^{1/2}\right)^2 \int_a^c u^2 w \\
&\leq \int_a^b \left(\frac{p_k^2}{w} - 2\lambda p_k + \lambda^2 w\right)\int_a^b u^2 w \longrightarrow 0 \quad \text{as } b \longrightarrow a^+.
\end{aligned} \tag{74}$$

So

$$\left[\left(p_0 u^{(k)}\right)^{(k-1)}(a,b)\right]^2 \longrightarrow 0 \quad \text{as } b \longrightarrow a^+. \tag{75}$$

Next we show that

$$\left(p_0 u^{(k)}\right)^{(k-r)}(a,b) \longrightarrow 0, \quad r = 2,3,\ldots,k, \text{ as } b \longrightarrow a^+. \tag{76}$$

For $i = 1,2,\ldots,k-1$, $i \geq j$, also according to the Rolle's theorem we know, there exist at least $i-j$ zero points such that $(p_{k-i}u^{(i)})^{(j)}(\cdot,b) = 0$. Let c_r be the first zero point of $(p_0 u^{(k)})^{(k-r)}(\cdot,b) = 0$. For $(p_0 u^{(k)})^{(k-r)}(a,b)$, we have

$$\begin{aligned}
&[(p_0 u^{(k)})^{(k-r)}(a,b)]^2 \\
&\quad = \left[\left(p_0 u^{(k)}\right)^{(k-r)}(c_r,b) - \left(p_0 u^{(k)}\right)^{(k-r)}(a,b)\right]^2 \\
&\quad = \left[\int_a^{c_r}\left(p_0 u^{(k)}\right)^{(k-r+1)}\right]^2 \\
&\quad = \left[\int_a^{c_r}\left(p_0 u^{(k)}\right)^{(k-r+1)}(c_{r+1},b) - \left(p_0 u^{(k)}\right)^{(k-r+1)}(t_{r+1},b)\right]^2 \\
&\quad = \left[\int_a^{c_r}\int_{t_{r+1}}^{c_{r+1}}\left(p_0 u^{(k)}\right)^{(k-r+2)}\right]^2 \\
&\quad = \cdots = \left[\int_a^{c_r}\int_{t_{r+1}}^{c_{r+1}}\cdots\int_{t_k}^{c_k}\left(p_0 u^{(k)}\right)^{(k)}\right]^2 \\
&\quad = \left[\int_a^{c_r}\int_{t_{r+1}}^{c_{r+1}}\cdots\int_{t_k}^{c_k}\sum_{i=1}^{k-1}\left(p_{k-i}u^{(i)}\right)^{(i)}\right. \\
&\qquad\qquad \left. -\left(p_k - \lambda w\right)u\,d\xi_k\cdots d\xi_{r+1}dt\right]^2 \\
&\quad \leq 2\left[\int_a^{c_r}\int_{t_{r+1}}^{c_{r+1}}\cdots\int_{t_k}^{c_k}\sum_{i=1}^{k-1}\left(p_{k-i}u^{(i)}\right)^{(i)}d\xi_k\cdots d\xi_{r+1}dt\right]^2 \\
&\qquad + 2\left[\int_a^{c_r}\int_{t_{r+1}}^{c_{r+1}}\cdots\int_{t_k}^{c_k}\left(p_k - \lambda w\right)u\,d\xi_k\cdots d\xi_{r+1}dt\right]^2 \\
&\quad \leq 2(b-a)^{2(k-r)}\left[\sum_{i=1}^{k-1}\int_{t_k}^{c_k}\left(p_{k-i}u^{(i)}\right)^{(i)}\right]^2 \\
&\qquad + 2(b-a)^{2(k-r)}\left[\int_{t_k}^{c_k}\left(p_k - \lambda w\right)u\,d\xi_k\right]^2 \\
&\quad \leq 2\sum_{i=1}^{k-1}(b-a)^{2(k-r+i)}\left[\int_{t_k}^{c_k}p_{k-i}u^{(i)}\right]^2 \\
&\qquad + 2(b-a)^{2(k-r)}\left[\int_{t_k}^{c_k}\left(p_k - \lambda w\right)u\,d\xi_k\right]^2,
\end{aligned}$$

$$\begin{aligned}
&\left[\int_{t_k}^{c_k}p_{k-i}u^{(i)}\right]^2 \\
&\quad \leq (b-a)^2\left|p_{k-i}u^{(i)}(t)\right|^2 \\
&\quad \longrightarrow 0 \quad \text{as } b \longrightarrow a^+,\ t \in (a,b),\ i = 1,2,\ldots,k-1, \\
&\left[\int_{t_k}^{c_k}\left(p_k - \lambda w\right)u\,d\xi\right]^2 \\
&\quad \leq \left[\int_{t_k}^{c_k}\left(p_k w^{-1/2} - \lambda w^{1/2}\right)^2 d\xi \int_t^c u^2 w\,d\xi\right] \\
&\quad \leq \int_a^b\left(\frac{p_k^2}{w} - 2\lambda p_k + \lambda^2 w\right)\int_a^b u^2 w \longrightarrow 0 \quad \text{as } b \longrightarrow a^+.
\end{aligned} \tag{77}$$

Thus

$$\left[\left(p_0 u^{(k)}\right)^{(k-r)}(a,b)\right]^2 \longrightarrow 0, \quad r = 2,3,\ldots,k, \text{ as } b \longrightarrow a^+. \tag{78}$$

Noting that $\lambda(b) \to \lambda^+(a)$ as $b \to a^+$, by (73), (78) and the continuous dependence of solutions (6) on initial conditions and on the parameter we conclude that $u(\cdot,b) \to 0$ uniformly on any compact subinterval of $[a,B)$. Therefore, for $\varepsilon > 0$, there exists a $b_0 \in [a,B)$, such that

$$|u(t,b)| < \varepsilon, \quad t \in [a,b],\ a < b < b_0. \tag{79}$$

This implies that

$$\int_a^b u^2 w < \varepsilon^2 \int_a^b w \tag{80}$$

for ε sufficiently small. This contradicts the normalization (72), which completes the proof. □

5. Conclusion

With a simple analysis, we showed that the eigenvalues of a class of $2k$th-order Sturm-Liouville problems depend not only continuously but also smoothly on boundary points and that the derivative of the nth eigenvalue as a function of an endpoint satisfies a first order differential equation. It is satisfying that these equations are established without any smoothness assumptions on the coefficients and also for the case that the leading coefficient p_0 is not assumed to be bounded away from zero and is even allowed to change sign. More importantly, we show that the lowest Dirichlet eigenvalue is a decreasing function of the endpoints and has an infinite limit as the endpoints approach each other.

In recent years, the various physics applications of this kind Sturm-Liouville problem are found in much literature (see, e.g., [11–15]). Many topics in mathematical physics require the investigation of the eigenvalues and eigenfunctions of Sturm-Liouville-type boundary value problems. Our results contain all the cases when k is equal to certain special positive integer. In particular, for $k = 2$, Theorem 11 explains that natural frequency of the rod will increase with the shortening of its length.

Furthermore, highly important results in this field have been obtained for the case when the eigenparameter appears not only in the differential equation with transmission conditions but also in the boundary conditions. Particularly, on computing eigenvalues of these types Sturm-Liouville problems, we can refer to [16–18]. Therefore, our proof methods and results will be useful to resolve eigenvalue problem of discontinuous Sturm-Liouville operators and differential operators with eigenparameter boundary conditions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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