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Research Article

Properties of the Set of Hadamardized Hurwitz Polynomials

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We say that a Hurwitz polynomial $p(t)$ is a Hadamardized polynomial if there are two Hurwitz polynomials $f(t)$ and $g(t)$ such that $f * g = p$, where $f * g$ is the Hadamard product of f and g. In this paper, we prove that the set of all Hadamardized Hurwitz polynomials is an open, unbounded, nonconvex, and arc-connected set. Furthermore, we give a result so that a fourth-degree Hurwitz interval polynomial is a Hadamardized polynomial family and we discuss an approach of differential topology in the study of the set of Hadamardized Hurwitz polynomials.

1. Introduction

The study of Hurwitz polynomials is important for the following: if the characteristic polynomial of a matrix A is a Hurwitz polynomial, then the system $\dot{x} = Ax$ is stable. Such study began when Maxwell posed the problem of finding necessary and sufficient conditions for a polynomial has all of its roots with negative real part. Since then, a lot information has been generated and part of it can be found in [1–4]. On the other hand, in the study of Hurwitz polynomials, topological approaches have recently been reported in [5–9]. Now, in this paper we present some topological and geometric results about the set of Hurwitz polynomials that admit a Hadamard factorization; such set will be called the set of Hadamardized polynomials. First, we give some definitions. Given two polynomials $f(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0$ and $g(t) = b_n t^n + b_{n-1} t^{n-1} + \cdots + b_0$, we say that

$$(f * g)(t) = a_n b_n t^n + a_{n-1} b_{n-1} t^{n-1} + \cdots + a_0 b_0 \tag{1}$$

is the Hadamard product of f and g. Garloff and Wagner have shown that the set of Hurwitz polynomials is closed under the Hadamard product (see [10]) and preserves total nonnegativity (see [11]). And, conversely, we say that a Hurwitz polynomial $p(t)$ admits a Hadamard factorization if there are two Hurwitz polynomials $f(t)$ and $g(t)$ such that $(f * g)(t) = p(t)$. In this case, we say that $p(t)$ is a Hadamardized Hurwitz polynomial, we denote by $\mathscr{H}(\mathrm{Had})$ the set of all Hadamardized Hurwitz polynomials, and we denote by $\mathscr{H}(\mathrm{Had})_n$ the set of all n-degree Hadamardized Hurwitz polynomials. It is shown in [12] that there are Hurwitz polynomials of degree equal to four that do not have a Hadamard factorization. In [13] necessary conditions for a Hurwitz polynomial to have a Hadamard factorization were obtained, and in [14, 15] necessary and sufficient conditions for a Hurwitz polynomial of degree equal to four to admit a Hadamard factorization were obtained. Now in this paper we prove that $\mathscr{H}(\mathrm{Had})$ is an unbounded, nonconvex open set and arc-connected. Additionally, in Section 3 we give a result about Hadamard factorization of a fourth-degree Hurwitz interval polynomials. Finally, in Section 4 we present an approach of differential topology for studying the set of Hadamardized Hurwitz polynomials.

2. Main Results

2.1. The Set of Hadamardized Hurwitz Polynomials Is an Open Set. The fact that the set of Hadamardized Hurwitz polynomials of degree equal to four is an open set is an immediate consequence of Theorem A.3; since

$q(t) = \alpha_4 t^4 + \alpha_3 t^3 + \alpha_2 t^2 + \alpha_1 t + \alpha_0$ is a Hadamardized Hurwitz polynomial, then

$$\frac{\alpha_0\alpha_3}{\alpha_1} < \left(\sqrt{\frac{\alpha_1\alpha_4}{\alpha_3}} - \sqrt{\alpha_2}\right)^2, \tag{2}$$

and this inequality holds for small perturbations of the coefficients. Since it is not known necessary and sufficient conditions (in terms of its coefficients) for an n-degree Hurwitz polynomial to have a Hadamard factorization, we need a result that allows us to prove the case for polynomials of arbitrary degree. Such a result is the Open Mapping Theorem.

Theorem 1. *$\mathscr{H}$(Had) is an open set.*

Proof. Let $p(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$ be a Hadamardized Hurwitz polynomial. Then, there are two Hurwitz polynomials $f(t) = b_n t^n + b_{n-1} t^{n-1} + \cdots + b_0$ and $g(t) = c_n t^n + c_{n-1} t^{n-1} + \cdots + c_0$ such that $(f * g)(t) = p(t)$. Consider the operator $T_f : \mathscr{P}_n \to \mathscr{P}_n$, where $\mathscr{P}_n$ is the set of n-degree polynomials, defined by $T_f(h(t)) = (f * h)(t)$. T is a linear operator and if $h(t) = d_n t^n + d_{n-1} t^{n-1} + \cdots + d_1 t + d_0$, then

$$\begin{aligned} \left\| T_f(h(t)) \right\| &= \left\| (b_n d_n, \ldots, b_1 d_1, b_0 d_0) \right\| \\ &\leq \left\| (b_n, \ldots, b_1, b_0) \right\| \left\| (d_n, \ldots, d_1, d_0) \right\|. \end{aligned} \tag{3}$$

That is, T_f is a bounded linear operator. It is clear that T_f is a surjective mapping because if $q(t) = q_n t^n + q_{n-1} t^{n-1} + \cdots + q_1 t + q_0$ is an arbitrary polynomial, then

$$T_f\left(\frac{q_n}{b_n} t^n + \frac{q_{n-1}}{b_{n-1}} t^{n-1} + \cdots + \frac{q_1}{b_1} t + \frac{q_0}{b_0}\right) = q(t). \tag{4}$$

On the other hand, since $\mathscr{H}_n$ (n-degree Hurwitz polynomials set) is an open set, let U be an open set such that $g \in U \subset \mathscr{H}_n$. Then, $T_f[U] \subset \mathscr{H}_n$ (see [10]), and, by the Open Mapping Theorem [16], $T_f[U]$ is an open set. Since $p(t) \in T_f[U]$, we have that $p(t) \in T_f[U] \subset \mathscr{H}$(Had), and $\mathscr{H}$(Had) is an open set. □

2.1.1. Application. Consider the system $\dot{x} = A(\lambda)x$ where $A(\lambda) \in \mathscr{M}_{n\times n}(\mathscr{R})$ and $\lambda \in \mathscr{R}^n$ is a parameter. Let $p_\lambda(t) = a_n(\lambda)t^n + a_{n-1}(\lambda)t^{n-1} + \cdots + a_1(\lambda) + a_0(\lambda)$ be the characteristic polynomial of the matrix $A(\lambda)$.

Suppose that P_{λ_0} is a Hadamardized Hurwitz polynomial. By Theorem 1, there is an open ball $B_\epsilon(p_{\lambda_0})$ such that every element of $B_\epsilon(p_{\lambda_0})$ is a Hadamardized Hurwitz polynomial. Then, we can find two Hurwitz polynomials:

$$\begin{aligned} f_\lambda(t) &= b_n(\lambda) t^n + b_{n-1}(\lambda) t^{n-1} + \cdots + b_1(\lambda) t \\ &\quad + b_0(\lambda), \\ g_\lambda(t) &= c_n(\lambda) t^n + c_{n-1}(\lambda) t^{n-1} + \cdots + c_1(\lambda) t + c_0(\lambda) \end{aligned} \tag{5}$$

such that $f_\lambda * g_\lambda = P_\lambda$ for $\lambda \approx \lambda_0$. The idea is to obtain a Hadamard factorization of $P_\lambda(t)$ where the coefficients of $f_\lambda(t)$ and $g_\lambda(t)$ are simpler than those of $P_\lambda(t)$. To illustrate this idea, we give the following example.

Example 2. Consider the system $\dot{x} = Ax$, where the entries $a_{ij}(\lambda)$ of A are given by

$$\begin{aligned} a_{11} &= -\lambda^2 - 8\lambda - 15, \\ a_{12} &= -\lambda^2 - 11\lambda - 29, \\ a_{13} &= -\lambda^2 - 12\lambda - 36, \\ a_{14} &= 2\lambda + 7, \\ a_{21} &= 0, \\ a_{22} &= 0, \\ a_{23} &= 1, \\ a_{24} &= 0, \\ a_{31} &= 0, \\ a_{32} &= 0, \\ a_{33} &= 0, \\ a_{34} &= 1, \\ a_{41} &= -\lambda^2 - 8\lambda - 15, \\ a_{42} &= -\lambda^2 - 11\lambda - 30, \\ a_{43} &= -\lambda^2 - 12\lambda - 36, \\ a_{44} &= \lambda + 7. \end{aligned} \tag{6}$$

The characteristic polynomial is $P_\lambda(t) = t^4 + (\lambda^2 + 6\lambda + 8)t^3 + (\lambda^2 + 12\lambda + 36)t^2 + (\lambda^2 + 11\lambda + 30)t + (\lambda^2 + 8\lambda + 15)$. For $\lambda = 0$, $P_0(t) = t^4 + 8t^3 + 36t^2 + 30t + 15$ is a Hadamardized Hurwitz polynomial by Theorem A.4.

By Theorem 1, we can find two Hurwitz polynomials such that the Hadamard product of them is equal to $P_\lambda(t)$ for small enough λ. Consider the Hurwitz polynomials:

$$\begin{aligned} F_0(t) &= t^4 + 4t^3 + 6t^2 + 6t + 5, \\ G_0(t) &= t^4 + 2t^3 + 6t^2 + 5t + 3. \end{aligned} \tag{7}$$

Note that $F_0 * G_0 = t^4 + 8t^3 + 36t^2 + 30t + 15$. The families of polynomials

$$\begin{aligned} F_\lambda(t) &= t^4 + (\lambda+4)t^3 + (\lambda+6)t^2 + (\lambda+6)t \\ &\quad + (\lambda+5), \\ G_\lambda(t) &= t^4 + (\lambda+2)t^3 + (\lambda+6)t^2 + (\lambda+5)t \\ &\quad + (\lambda+3) \end{aligned} \tag{8}$$

are Hurwitz stable for small enough λ, the Hadamard product $F_\lambda * G_\lambda$ satisfies $F_\lambda * G_\lambda = P_\lambda(t)$, and the coefficients of F_λ and G_λ are simpler than those of P_λ.

2.2. The Set of Hadamardized Hurwitz Polynomials Is a Nonconvex Set. We have that $\mathscr{H}(\text{Had}) \subseteq \mathscr{H}$ (see [13]), where $\mathscr{H}(\text{Had}) \neq \mathscr{H}$ and $\mathscr{H}$ is the set of Hurwitz polynomials, and it is known that the set of Hurwitz polynomials is a nonconvex set (see [17, 18]). However, this does not imply that the set of Hadamardized Hurwitz polynomials is also a nonconvex set. Thus, the nonconvexity of $\mathscr{H}(\text{Had})$ has to be proven.

Considered the ray of polynomials:

$$F_k(t) = t^4 + t^3 + 6.9t^2 + 2t + 3 + k\left(t^3 + t^2 + 2t + 1\right); \quad (9)$$

that is

$$\begin{aligned} F_k(t) &= t^4 + (1+k)t^3 + (6.9+k)t^2 + (2+2k)t \\ &\quad + (3+k). \end{aligned} \quad (10)$$

Then, $F_k(t)$ is Hurwitz if and only if

$$\begin{aligned} &1+k > 0, \\ &6.9+k > 0, \\ &2+2k > 0, \\ &3+k > 0, \\ &(1+k)(6.9+k)(2+2k) - (2+2k)^2 \\ &\quad - (1+k)^2(3+k) > 0. \end{aligned} \quad (11)$$

This inequality holds if and only if $k > -1$, and $F_k(t)$ is Hadamardized if and only if

$$\frac{(3+k)(1+k)}{2+2k} < \left(\sqrt{\frac{2+2k}{1+k}} - \sqrt{6.9+k}\right)^2. \quad (12)$$

Thus, $F_k(t)$ is Hurwitz and Hadamardized if and only if $k \in (-1, 1.2-\sqrt{3.2}) \cup (1.2+\sqrt{3.2}, \infty)$, and $F_k(t)$ is Hurwitz but not Hadamardized if and only if $k \in [1.2-\sqrt{3.2}, 1.2+\sqrt{3.2}]$.

Theorem 3. *The set of Hadamardized Hurwitz polynomials is a nonconvex set.*

Proof. In the discussion above, take $k = -0.5$ and $k = 2.9$. Then we get the Hadamardized Hurwitz polynomials $r(t) = t^4 + t^3 + 6.9t^2 + 2t + 3 - 0.5(t^3 + t^2 + 2t + 1)$ and $h(t) = t^4 + t^3 + 6.9t^2 + 2t + 3 + 2.9(t^3 + t^2 + 2t + 1)$. Now consider the convex combination $\lambda r(t) + (1-\lambda)h(t)$. For $\lambda = 1/2$, we get the polynomial $t^4 + 2.2t^3 + 8.1t^2 + 4.4t + 4.2$, which is not Hadamardized. □

2.3. The Set of Hadamardized Hurwitz Polynomials Is an Unbounded Set

Theorem 4. *There are rays of polynomials contained in the set of Hadamardized Hurwitz polynomials. Hence, it is an unbounded set.*

Let

$$\begin{aligned} G_k(t) &= t^4 + t^3 + 7t^2 + 2t + 3 + k\left(t^3 + t^2 + 2t + 1\right) \\ &= t^4 + (1+k)t^3 + (7+k)t^2 + (2+2k)t \\ &\quad + (3+k). \end{aligned} \quad (13)$$

Then, $G_k(t)$ is Hurwitz if and only if

$$\begin{aligned} &1+k > 0, \\ &7+k > 0, \\ &2+2k > 0, \\ &3+k > 0, \\ &(1+k)(7+k)(2+2k) - (2+2k)^2 - (1+k)^2(3+k) \\ &\quad > 0, \end{aligned} \quad (14)$$

and $G_k(t)$ is Hadamardized if and only if

$$\frac{(3+k)(1+k)}{2+2k} < \left(\sqrt{\frac{2+2k}{1+k}} - \sqrt{7+k}\right)^2. \quad (15)$$

Consequently, $G_k(t)$ is a Hadamardized Hurwitz polynomial for $k > -1$ and $\mathscr{H}(\text{Had})$ is a unbounded set.

The existence of rays of Hadamardized Hurwitz polynomials allows us to show an application in the designing of stabilizing controls.

2.3.1. Application. The following example illustrates the designing of a state feedback control that stabilizes the system for each positive value of the parameter. Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & -2 & -7 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u; \quad (16)$$

and the state feedback control

$$u(x) = k(-1, -2, -1, -1)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad (17)$$

where $k \in \mathscr{R}$ is a parameter. Then, the controlled system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3-k & -2-2k & -7-k & -1-k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \quad (18)$$

The characteristic polynomial $G_k(t) = t^4 + t^3 + 7t^2 + 2t + 3 + k(t^3 + t^2 + 2t + 1)$ is a Hadamardized Hurwitz polynomial for all $k > -1$, and consequently the system is stable for all $k > -1$. Additionally, $G_k(t)$ is Hadamardized for all $k > -1$.

2.4. The Set of Hurwitz Hadamardized Polynomials with Positive Coefficients Is Arc-Connected. To prove the main result of this section, we need the following lemma.

Lemma 5. *$\mathscr{H}_n^+$ is an arc-connected set.*

Proof. Let $P_1(t)$ and $P_2(t)$ be two Hurwitz polynomials with positive coefficients. Suppose that $n = 2m$. Then, $P_1(t)$ and $P_2(t)$ can be written as

$$\begin{aligned} P_1(t) &= b_0\left(t^2 + b_1 t + b_2\right) \\ &\quad \cdot \left(t^2 + b_3 t + b_4\right)\cdots\left(t^2 + b_{2m-1}t + b_{2m}\right), \\ P_2(t) &= a_0\left(t^2 + a_1 t + a_2\right) \\ &\quad \cdot \left(t^2 + a_3 t + a_4\right)\cdots\left(t^2 + a_{2m-1}t + a_{2m}\right), \end{aligned} \tag{19}$$

where $b_i > 0$ and $a_i > 0$ for all $i = 0, 1, \ldots, 2m$. Define

$$\begin{aligned} H_\lambda(t) &= \left(\lambda a_0 + (1-\lambda) b_0\right)\left(t^2 + \left(\lambda a_1 + (1-\lambda) b_1\right) t\right. \\ &\quad \left. + \left(\lambda a_2 + (1-\lambda) b_2\right)\right)\left(t^2 + \left(\lambda a_3 + (1-\lambda) b_3\right) t\right. \\ &\quad \left. + \left(\lambda a_4 + (1-\lambda) b_4\right)\right)\cdots\left(t^2\right. \\ &\quad + \left(\lambda a_{2m-1} + (1-\lambda) b_{2m-1}\right) t \\ &\quad \left. + \left(\lambda a_{2m} + (1-\lambda) b_{2m}\right)\right) \end{aligned} \tag{20}$$

with $\lambda \in [0, 1]$. We have that $H_0(t) = P_1(t)$, $H_1(t) = P_2(t)$, and $H_\lambda(t)$ is a Hurwitz polynomial for all $\lambda \in [0, 1]$. Therefore, $\mathscr{H}_n^+$ is arc-connected set. □

Theorem 6. *The set of Hadamardized Hurwitz polynomials with positive coefficients is arc-connected.*

Proof. Let $P_1(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0$ and $P_2(t) = b_n t^n + b_{n-1} t^{n-1} + \cdots + b_0$ Hadamardized Hurwitz polynomials with positive coefficients; then, there are

$$\begin{aligned} f_1(t) &= \alpha_n t^n + \alpha_{n-1} t^{n-1} + \cdots + \alpha_0, \\ f_2(t) &= \beta_n t^n + \beta_{n-1} t^{n-1} + \cdots + \beta_0, \\ g_1(t) &= \gamma_n t^n + \gamma_{n-1} t^{n-1} + \cdots + \gamma_0, \\ g_2(t) &= \delta_n t^n + \delta_{n-1} t^{n-1} + \cdots + \delta_0 \end{aligned} \tag{21}$$

such that f_1, f_2, g_1, and g_2 are Hurwitz polynomials with positive coefficients and $(f_1 * g_1)(t) = P_1(t)$ and $(f_2 * g_2)(t) = P_2(t)$; that is,

$$\begin{aligned} &\alpha_n\gamma_n t^n + \alpha_{n-1}\gamma_{n-1} t^{n-1} + \cdots + \alpha_0\gamma_0 \\ &\quad = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0, \\ &\beta_n\delta_n t^n + \beta_{n-1}\delta_{n-1} t^{n-1} + \cdots + \beta_0\delta_0 \\ &\quad = b_n t^n + b_{n-1} t^{n-1} + \cdots + b_0. \end{aligned} \tag{22}$$

Since $\mathscr{H}_n^+$ is arc-connected set, then there exist

$$\begin{aligned} G_{1,\lambda}(t) &= c_n(\lambda) t^n + c_{n-1}(\lambda) t^{n-1} + \cdots + c_0(\lambda), \\ G_{2,\lambda}(t) &= d_n(\lambda) t^n + d_{n-1}(\lambda) t^{n-1} + \cdots + d_0(\lambda) \end{aligned} \tag{23}$$

such that

$$\begin{aligned} G_{1,0}(t) &= c_n(0) t^n + c_{n-1}(0) t^{n-1} + \cdots + c_0(0) \\ &= f_1(t), \\ G_{2,0}(t) &= d_n(0) t^n + d_{n-1}(0) t^{n-1} + \cdots + d_0(0) \\ &= g_1(t), \\ G_{1,1}(t) &= c_n(1) t^n + c_{n-1}(1) t^{n-1} + \cdots + c_0(1) \\ &= f_2(t), \\ G_{2,1}(t) &= d_n(1) t^n + d_{n-1}(1) t^{n-1} + \cdots + d_0(1) \\ &= g_2(t) \end{aligned} \tag{24}$$

and each $G_{i,\lambda}(t) \in \mathscr{H}_n^+$ for all $\lambda \in [0, 1]$. Now, consider

$$\begin{aligned} \left(G_{1,\lambda} * G_{2,\lambda}\right)(t) &= c_n(\lambda) d_n(\lambda) t^n \\ &\quad + c_{n-1}(\lambda) d_{n-1}(\lambda) t^{n-1} + \cdots \\ &\quad + c_0(\lambda) d_0(\lambda) \end{aligned} \tag{25}$$

and then $(G_{1,\lambda} * G_{2,\lambda})(t)$ is a Hadamardized Hurwitz polynomial for all $\lambda \in [0, 1]$ (see [10]) and

$$\begin{aligned} \left(G_{1,0} * G_{2,0}\right)(t) &= \left(f_1 * g_1\right)(t) = P_1(t), \\ \left(G_{1,1} * G_{2,1}\right)(t) &= \left(f_2 * g_2\right)(t) = P_2(t). \end{aligned} \tag{26}$$

Consequently, the set of Hadamardized Hurwitz polynomials whit positive coefficients is arc-connected. □

3. The Kharitonov Theorem and the Set of Hadamardized Hurwitz Polynomials

The Kharitonov theorem (see [19]) has motivated a lot of research on the subject of stable polynomials. Many papers have been published since Kharitonov's theorem was reported (see [20–24]). We now address the following question. Suppose we have an interval family of Hurwitz polynomials. If the four Kharitonov polynomials are Hadamardized polynomials, is each element of the family a Hadamardized polynomial? We will give the answer in the case of 4-degree polynomials. First, we need the following result. Although the proof is simple, we have decided to include it.

Theorem 7. *The Hurwitz polynomial $f(t) = \alpha_4 t^4 + \alpha_3 t^3 + \alpha_2 t^2 + \alpha_1 t + \alpha_0$ is Hadamardized if and only if $g(t) = \sqrt{\alpha_4} t^4 + \sqrt{\alpha_3} t^3 + \sqrt{\alpha_2} t^2 + \sqrt{\alpha_1} t + \sqrt{\alpha_0}$ is a Hurwitz polynomial.*

Proof. The necessary condition is immediate. To prove sufficient condition, we suppose that $f(t)$ is Hadamardized, and then (see [14, 15])

$$\frac{\alpha_0\alpha_3}{\alpha_1} < \left(\sqrt{\frac{\alpha_1\alpha_4}{\alpha_3}} - \sqrt{\alpha_2}\right)^2. \tag{27}$$

Note that $\alpha_2\alpha_3 - \alpha_1\alpha_4 > 0$ (since f is Hurwitz), and then

$$\sqrt{\alpha_2} - \sqrt{\frac{\alpha_1\alpha_4}{\alpha_3}} > 0. \tag{28}$$

Consequently, if $f(t)$ is Hadamardized, then

$$\sqrt{\frac{\alpha_0\alpha_3}{\alpha_1}} < \sqrt{\alpha_2} - \sqrt{\frac{\alpha_1\alpha_4}{\alpha_3}}. \tag{29}$$

Whence, $0 < \sqrt{\alpha_1\alpha_2\alpha_3} - \alpha_1\sqrt{\alpha_4} - \sqrt{\alpha_0\alpha_3}$, which implies that $g(t)$ is a Hurwitz polynomial. □

The last theorem let us establish the following result.

Theorem 8. *Consider the interval family $F(t) = [a_4, b_4]t^4 + [a_3, b_3]t^3 + [a_2, b_2]t^2 + [a_1, b_1]t + [a_0, b_0]$ which consists of Hurwitz polynomials. If the four Kharitonov polynomials are Hadamardized polynomials, then each element of the family is a Hadamardized polynomial.*

Proof. The four Kharitonov polynomials are

$$\begin{aligned} k_1(t) &= a_4t^4 + b_3t^3 + b_2t^2 + a_1t + a_0,\\ k_2(t) &= a_4t^4 + a_3t^3 + b_2t^2 + b_1t + a_0,\\ k_3(t) &= b_4t^4 + b_3t^3 + a_2t^2 + a_1t + b_0,\\ k_4(t) &= b_4t^4 + a_3t^3 + a_2t^2 + b_1t + b_0. \end{aligned} \tag{30}$$

Each one is a Hurwitz polynomial. If the four Kharitonov polynomials are Hadamardized, then, applying the previous theorem, the polynomials

$$\begin{aligned} g_1(t) &= \sqrt{a_4}t^4 + \sqrt{b_3}t^3 + \sqrt{b_2}t^2 + \sqrt{a_1}t + \sqrt{a_0},\\ g_2(t) &= \sqrt{a_4}t^4 + \sqrt{a_3}t^3 + \sqrt{b_2}t^2 + \sqrt{b_1}t + \sqrt{a_0},\\ g_1(t) &= \sqrt{b_4}t^4 + \sqrt{b_3}t^3 + \sqrt{a_2}t^2 + \sqrt{a_1}t + \sqrt{b_0},\\ g_1(t) &= \sqrt{b_4}t^4 + \sqrt{a_3}t^3 + \sqrt{a_2}t^2 + \sqrt{b_1}t + \sqrt{b_0} \end{aligned} \tag{31}$$

are Hurwitz. Now, for Kharitonov's theorem, the polynomial

$$\begin{aligned} G(t) = {} & \left[\sqrt{a_4}, \sqrt{b_4}\right]t^4 + \left[\sqrt{a_3}, \sqrt{b_3}\right]t^3\\ & + \left[\sqrt{a_2}, \sqrt{b_2}\right]t^2 + \left[\sqrt{a_1}, \sqrt{b_1}\right]t\\ & + \left[\sqrt{a_0}, \sqrt{b_0}\right] \end{aligned} \tag{32}$$

is also Hurwitz. Let $f(t) = c_4t^4 + c_3t^3 + c_2t^2 + c_1t + c_0$ be an arbitrary element of $F(t)$, where $c_i \in [a_i, b_i]$ for $i = 1, 2, 3, 4$, and consider the polynomial in $G(t)$

$$g(t) = \sqrt{c_4}t^4 + \sqrt{c_3}t^3 + \sqrt{c_2}t^2 + \sqrt{c_1}t + \sqrt{c_0} \tag{33}$$

and then $g(t)$ is Hurwitz. Hence,

$$f(t) = g(t) * g(t) \tag{34}$$

and the theorem is proved. □

Remark 9. The discussion above allows us to propose the following open problem.

Open Problem 1. Consider the Hurwitz polynomial $f(t) = \alpha_nt^n + \cdots + \alpha_1t + \alpha_0$ with $n \geq 5$; if $f(t)$ is a Hadamardized polynomial, then $g(t) = \sqrt{\alpha_n}t^n + \cdots + \sqrt{\alpha_1}t + \sqrt{\alpha_0}$ is a Hurwitz polynomial.

Remark 10. Note that, by repeated application of the square root, the polynomial $\sqrt[2m]{\alpha_n}t^n + \cdots + \sqrt[2m]{\alpha_1}t + \sqrt[2m]{\alpha_0}$ converges to the polynomial $t^n + t^{n-1} + \cdots + t + 1$ when $m \to \infty$, which has equally spaced roots on the unit circle and consequently $t^n + t^{n-1} + \cdots + 1$ is not a Hurwitz polynomial. It let us establish the following problem.

Open Problem 2. Let $\alpha_nt^n + \cdots + \alpha_1t + \alpha_0$ and $\sqrt{\alpha_n}t^n + \cdots + \sqrt{\alpha_1}t + \sqrt{\alpha_0}$ be Hurwitz polynomials; what is the maximum m such that $\sqrt[2m]{\alpha_n}t^n + \cdots + \sqrt[2m]{\alpha_1}t + \sqrt[2m]{\alpha_0}$ is a Hurwitz polynomial?

Remark 11. Since for n degree greater than or equal to 5 we do not have a similar result to Theorem 8, then we can propose the following two open problems.

Open Problem 3. Consider the family of Hurwitz polynomials with degree $n \geq 5$: $[a_n, b_n]t^n + \cdots + [a_1, b_1]t + [a_0, b_0]$. If the four Kharitonov polynomials are Hadamardized polynomials, every element is a Hadamardized polynomial.

Open Problem 4. If all of the vertices of an interval family of polynomials with degree $n \geq 5$ are Hadamardized Hurwitz polynomials, does the family consist of Hadamardized Hurwitz polynomials?

4. The Stability Test, the Set of Hadamardized Hurwitz Polynomials, and Differential Topology

The stability test is a criterion for verifying whether a given polynomial is Hurwitz. We explain it. Consider the n-degree polynomial with positive coefficients:

$$P(t) = A_nt^n + A_{n-1}t^{n-1} + \cdots + A_1t + A_0. \tag{35}$$

Defining the $n-1$-degree polynomial

$$\begin{aligned} R(t) = {} & A_{n-1}^2t^{n-1} + \left(A_{n-1}A_{n-2} - A_nA_{n-3}\right)t^{n-2}\\ & + A_{n-1}A_{n-3}t^{n-3}\\ & + \left(A_{n-1}A_{n-4} - A_nA_{n-5}\right)t^{n-4} + \cdots, \end{aligned} \tag{36}$$

then we have the following result.

Theorem 12. *$P(t)$ is a Hurwitz polynomial if and only if $R(t)$ is a Hurwitz polynomial.*

Proof. See [1]. □

The importance of the stability test was appreciated when Kharitonov used it to obtain his celebrated theorem. Moreover, it is worth mentioning that other important properties of the set of Hurwitz polynomials were shown with the stability test (see [6, 13]). Now we use the stability test for studying the following subset of the Hadamardized Hurwitz polynomial set:

$$\begin{aligned}\mathscr{H}(\mathrm{Had})_n^* = \{ & f(t) = a_n t^n + \cdots + a_0 \in \mathscr{H}_n^+ : h(t) \\ & = \sqrt{a_n} t^n + \cdots + \sqrt{a_0} \in \mathscr{H}_n^+ \}.\end{aligned} \tag{37}$$

As was observed in Open Problem 1 of Section 3, at the moment, we do not know the answer to the following question: $\mathscr{H}(\mathrm{Had})_n^* \subsetneqq \mathscr{H}(\mathrm{Had})_n$ or $\mathscr{H}(\mathrm{Had})_n^* = \mathscr{H}(\mathrm{Had})_n$?

In this section, we get a different test, which is applicable to studying the set $\mathscr{H}(\mathrm{Had})_n^*$. This special test let us obtain the following property: $\mathscr{H}(\mathrm{Had})_n^*$ is a vector bundle on the set $\mathscr{H}(\mathrm{Had})_{n-1}^*$. There are previous works (see [6, 7]), where some ideas of differential topology have been used to study the space of Hurwitz polynomials. A presentation of the concepts used in this section can be consulted in [6, 7, 25, 26].

4.1. A Test for $\mathscr{H}(\mathrm{Had})_n^*$. Consider the n-degree polynomial with positive coefficients as (35); define the $(n-1)$-degree polynomial

$$\begin{aligned}Q(t) = {} & A_{n-1}^2 t^{n-1} \\ & + \left(\sqrt{A_{n-1}}\sqrt{A_{n-2}} - \sqrt{A_n}\sqrt{A_{n-3}}\right)^2 t^{n-2} \\ & + A_{n-1}A_{n-3} t^{n-3} \\ & + \left(\sqrt{A_{n-1}}\sqrt{A_{n-4}} - \sqrt{A_n}\sqrt{A_{n-5}}\right)^2 t^{n-4} + \cdots\end{aligned} \tag{38}$$

and then we have the following result.

Theorem 13. *$P(t) \in \mathscr{H}(\mathrm{Had})_n^*$ if and only if $Q(t) \in \mathscr{H}(\mathrm{Had})_{n-1}^*$.*

Proof. If $P(t) \in \mathscr{H}(\mathrm{Had})_n^*$, then $\sqrt{A_n}t^n + \sqrt{A_{n-1}}t^{n-1} + \cdots + \sqrt{A_1}t + \sqrt{A_0} \in \mathscr{H}_n^+$, and by stability test

$$\begin{aligned} & A_{n-1}t^{n-1} + \left(\sqrt{A_{n-1}}\sqrt{A_{n-2}} - \sqrt{A_n}\sqrt{A_{n-3}}\right) t^{n-2} \\ & \quad + \sqrt{A_{n-1}}\sqrt{A_{n-3}} t^{n-3} \\ & \quad + \left(\sqrt{A_{n-1}}\sqrt{A_{n-4}} - \sqrt{A_n}\sqrt{A_{n-5}}\right) t^{n-4} + \cdots \\ & \in \mathscr{H}_{n-1}^+.\end{aligned} \tag{39}$$

Then, $Q(t)$ is a Hurwitz polynomial (see [10]). Consequently, $Q(t) \in \mathscr{H}(\mathrm{Had})_{n-1}^*$. This proves the sufficient condition. Now, if $Q(t) \in \mathscr{H}(\mathrm{Had})_{n-1}^*$, then we obtain (39) and by the stability test

$$\sqrt{A_n}t^n + \sqrt{A_{n-1}}t^{n-1} + \cdots + \sqrt{A_1}t + \sqrt{A_0} \in \mathscr{H}_n^+, \tag{40}$$

where $P(t) \in \mathscr{H}_n^+$ (see [10]) and then $P(t) \in \mathscr{H}(\mathrm{Had})_n^*$. This completes the proof. □

4.2. An Approach of Differential Topology in the Study of the Set of Hadamardized Hurwitz Polynomials. Suppose that the $(n-1)$-degree polynomial

$$Q(t) = b_{n-1}t^{n-1} + b_{n-2}t^{n-2} + \cdots + b_1 t + b_0 \tag{41}$$

is an element of $\mathscr{H}(\mathrm{Had})_{n-1}^*$; then,

$$q(t) = \sqrt{b_{n-1}}t^{n-1} + \sqrt{b_{n-2}}t^{n-2} + \cdots + \sqrt{b_1}t + \sqrt{b_0} \tag{42}$$

is an element of $\mathscr{H}_{n-1}^+$. We can calculate the fiber (see [6, 13]) of $q(t)$:

$$\begin{aligned}P_s(t) = {} & st^n + b_{n-1}^{1/4}t^{n-1} + \left(\frac{\sqrt{b_{n-2}}}{b_{n-1}^{1/4}} + \frac{\sqrt{b_{n-3}}}{b_{n-1}^{1/2}}s\right)t^{n-2} \\ & + \frac{\sqrt{b_{n-3}}}{b_{n-1}^{1/4}}t^{n-3} + \left(\frac{\sqrt{b_{n-4}}}{b_{n-1}^{1/4}} + \frac{\sqrt{b_{n-5}}}{b_{n-1}^{1/2}}s\right)t^{n-4} \\ & + \cdots\end{aligned} \tag{43}$$

with $s > 0$; then, we have

$$\begin{aligned}&(P_s * P_s)(t) \\ & = s^2 t^n + \sqrt{b_{n-1}}t^{n-1} \\ & \quad + \left(\frac{b_{n-2}}{\sqrt{b_{n-1}}} + \frac{2\sqrt{b_{n-2}}\sqrt{b_{n-3}}}{b_{n-1}^{3/4}}s + \frac{b_{n-3}}{b_{n-1}}s^2\right)t^{n-2} \\ & \quad + \frac{b_{n-3}}{\sqrt{b_{n-1}}}t^{n-3} + \cdots.\end{aligned} \tag{44}$$

Then, $(P_s * P_s)(t)$ will be the fiber of $Q(t)$ and we have the following result.

Theorem 14. *$\mathscr{H}(\mathrm{Had})_n^*$ is a vector bundle with base $\mathscr{H}(\mathrm{Had})_{n-1}^*$.*

Proof. Given $P(t) = A_n t^n + A_{n-1}t^{n-1} + \cdots + A_1 t + A_0 \in \mathscr{H}(\mathrm{Had})_n^*$, $Q(t)$ is the $(n-1)$-degree polynomial defined according to (38). Then, we can define $\varphi : \mathscr{H}(\mathrm{Had})_n^* \to \mathscr{H}(\mathrm{Had})_{n-1}^* \times \mathbb{R}$ as follows: $\varphi(P(t)) = (Q(t), l_n(b_n))$.

Now we define $\varphi^{-1} : \mathscr{H}(\mathrm{Had})_{n-1}^* \times \mathbb{R} \to \mathscr{H}(\mathrm{Had})_n^*$; given the polynomial $Q(t)$ defined in (41) and $l \in \mathbb{R}$, $\varphi^{-1}(Q(t), l)$ is the following polynomial $\varphi^{-1}(Q(t), l) = (P_{e^{l/2}} * P_{e^{l/2}})(t)$, obtained by replacing s by $e^{l/2}$ in (44).

Then, we obtain the following commutative diagram:

$$\begin{tikzcd} \mathscr{H}(\mathrm{Had})_n^* \arrow[rr, "\varphi"] \arrow[dr, "H"'] & & \mathscr{H}(\mathrm{Had})_{n-1}^* \times \mathbb{R} \arrow[dl] \\ & \mathscr{H}(\mathrm{Had})_{n-1}^*, & \end{tikzcd} \quad (45)$$

where H is determined by the new test established in Theorem 8; that is, $H(P(t)) = Q(t)$. Then, $\mathscr{H}(\mathrm{Had})_n^*$ is a vector bundle with base $\mathscr{H}(\mathrm{Had})_{n-1}^*$. □

Remark 15. By the last theorem, we can say that $\mathscr{H}(\mathrm{Had})_n^* \approx \mathscr{H}(\mathrm{Had})_{n-1}^* \times \mathbb{R}$ and it implies that

$$\mathscr{H}(\mathrm{Had})_n^* \approx \mathscr{H}(\mathrm{Had})_4 \times \mathbb{R}^{n-4}. \quad (46)$$

5. Conclusion

The study of the set of Hadamardized Hurwitz polynomials began with a work by Garloff and Shrinivasan in 1996. Later, Loredo-Villalobos and Aguirre-Hernández obtained other results in 2011. We now prove that the set of Hadamardized Hurwitz polynomials is an open, nonconvex, unbounded, and arc-connected set. Additionally, we have given a condition to check when a family of interval polynomials of fourth degree is a Hadamardized polynomial and we have proposed an approach of differential topology for studying the set of Hadamardized Hurwitz polynomials. We have presented some results that are valid for n-degree polynomials and other results that are valid for 4-degree polynomials and then we have established open problems for polynomials with degree greater than or equal to 5. We hope that some constructive techniques are discovered for solving the open problems.

Appendix

Open Mapping Theorem

Definition A.1. Let X and Y be metric spaces. Then, it is said that $T : D(T) \to Y$ (with $D(T) \subset X$) is an open mapping if for every open set in $D(T)$ the image is an open set in Y.

Definition A.2 (bounded linear operator). Let X and Y be normed spaces and $T : D(T) \to Y$ a linear operator, where $D(T) \subset X$. The operator T is said to be bounded if there is a real number c such that for all $x \in D(T)$ we have

$$\|Tx\| \leq c \|x\|. \quad (A.1)$$

Theorem A.3 (Open Mapping Theorem [16]). *A bounded linear operator T from a Banach space X onto a Banach space Y is an open mapping.*

Theorem A.4 (see [14, 15]). *We have that $f(t) = \alpha_4 t^4 + \alpha_3 t^3 + \alpha_2 t^2 + \alpha_1 t + \alpha_0$ is Hadamardized if and only if*

$$\frac{\alpha_0 \alpha_3}{\alpha_1} < \left(\sqrt{\frac{\alpha_1 \alpha_4}{\alpha_3}} - \sqrt{\alpha_2} \right)^2. \quad (A.2)$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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