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Research Article

Two Mathematical Comments on the Thevenin Theorem: An "Algebraic Ideal" and the "Affine Nonlinearity"

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We discuss the most important and simple concept of basic circuit theory—the concept of the unideal source—or the Thevenin circuit. It is explained firstly how the Thevenin circuit can be interpreted in the algebraic sense. Then, we critically consider the common opinion that it is a linear circuit, showing that linearity (or nonlinearity) depends on the use of the port. The difference between the cases of a source being an input or an internal element (as it is in Thevenin's circuit) is important here. The distinction in the definition of linear operator in algebra (here in system theory) and in geometry is also important for the subject, and we suggest the wide use of the concept of "affine nonlinearity." This kind of nonlinearity should be relevant for the development of complicated circuitry (perhaps in a biological modeling context) with nonprescribed definition of subsystems, when the interpretation of a port as input or output can become dependent on the local intensity of a process.

1. Introduction

For the first time, the equivalent circuit was suggested by Herman von Helmholtz in 1853. Later, it was rediscovered (and proved for complicated linear 1-ports) by Léon Charles Thevenin (in 1883) and then by Edward Lawry Norton and Hans Ferdinand Mayer (both in 1926) [1]. Since passing from the series circuit of Thevenin, including a voltage source, to the parallel Norton version, including a current source, is an immediate application of the equivalent generator theorem, we will always speak about "Thevenin theorem," or just "theorem," meaning the series circuit.

Though the rigor proof (in [2, 3] using the basic substitution theorem) of the theorem is nontrivial, the very result is not surprising. Indeed, when having some voltage at the port of a circuit, one can naturally try to use the circuit as a voltage source. Since, furthermore, it is difficult to create a good voltage source, the idea of an equivalent circuit with an internal resistor (or an internal impedance $Z(s)$, or $Z(j\omega)$, but we will use the simplest classical model with a usual resistor) providing the dependence of the output voltage on the load, that is, the nonideality of the source, naturally appears.

What is really surprising is not the technical but the logical side, namely, the fact that Helmholtz, who was able to formulate in 1847 the law of conservation of energy for both mechanical and electrical systems, suggested in 1853 the theorem only for electrical systems, while it is not difficult to replace the voltage source by a source of a mechanical force and find mechanical equivalent for the 1-port circuit. Thus, Helmholtz (who, of course, knew the circuit equations suggested by Kirchhoff in 1847 and could expect intensive development of circuit theory) saw in electrical engineering something relevant to the theorem, which is not found in mechanics. It is clear today that this "something" is associated with the concept of port (input, output), involved in the formulation of the theorem, because this concept is flexible not in mechanics, but in electrical engineering that easily creates very complicated structures that can be seen as composed of some subsystems relevant to the theorem. That is, the point is technology, but we will stress the theoretical side. See also [4].

The closely associated question to be asked is that of whether a subsystem is an active one, having a load, or by itself is a load of a stronger circuit. The first case is more common in the applications of the Thevenin theorem, but the "affine nonlinearity," with which we will be concerned, is better seen in the second case, and it is methodologically important that the question about linearity or nonlinearity of a subsystem

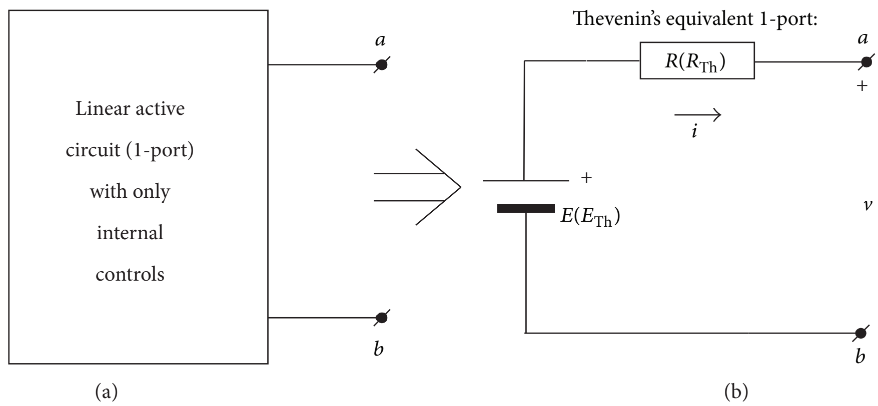

FIGURE 1: (b) The commonly used Thevenin equivalent of a linear active circuit (a). The rigorous proof of (a) $\rightarrow$ (b) is found in [2, 3].

(or of the whole system) can be influenced by the thus-seen degree of activity of the circuit. This circumstance, perhaps unexpected for many, is, in fact, not surprising, because we always consider linearity (or nonlinearity) of a certain system, and any correct definition of a system must include definition of its ports. (See also [4] and references there, for a development of this outlook, showing, in particular, that nonlinearity does not just mean a curviness of a characteristic.)

The classical theorem, related to the most basic circuit-theory concept of nonideal source, is taught at the very beginning of the standard electrical engineering education, but revising just the most basic concepts is most useful, and this relates also to the Thevenin theorem. In the relatively recent works [5, 6] a criterion is suggested, not appearing in the classical theory, defining the conditions when a 1-port, including a dependent source, is, as a whole, an ideal, or a nonideal source, and here we make some new steps.

Regarding the use here of the "unpopular" concepts of affine nonlinearity and algebraic ideal, it should be noted that in [4] a state of the practically very important fluorescent lamp circuit is classified using the concept of affine nonlinearity. Without this interpretation, the (actually very strong) nonlinearity of the lamp circuit is not obvious. The fact that the circuit specialists should pay more attention to the concept of affine nonlinearity is not only because of the Thevenin circuit.

The present completion of the conceptual frame can attract mathematical students to circuit theory and, on the contrary, the electrical engineering students to some additional mathematics, and the vision of a complicated system, suggested in Section 4.2, may be relevant for a biological modeling. Hopefully, our discussion will finally motivate some applications that would become some "tools in hand" for an ordinary electrical engineer.

2. The Circuit

Consider the well-known Thevenin circuit to which many linear 1-port circuits are reduced [1–3]. It is an equivalent circuit, which means that it influences the external circuit just as the original circuit does.

For simplicity, we will consider the Thevenin circuit as it is usually introduced, that is, for the simple resistive circuits. See Figure 1.

For a circuit including inductors and capacitors we can pass on to the domain of the Laplace-variable s in which all the linear circuits become "algebraic" just as the resistive circuits directly are in the time domain. Then the Thevenin equivalent includes some $V_{\text{Th}}(s)$ instead of $v_{\text{Th}}(t)$ and some $Z_{\text{Th}}(s)$ instead of R_{Th}. As is well known [2, 3], dynamic (in the time domain) linear circuits can be treated in the s-domain using all of the network theorems, among which the Thevenin theorem is extremely important, and the transfer to the s-domain does not change the circuit's topology.

The circuit of Figure 1(b) interests us from two points of view. First of all (Section 3), using the mathematical concept of *ideal*, we observe the universal simplicity of this circuit that can be extended in a linear form, by adding more linear elements and also sources, and then again contracted to this simple form. See Figure 2. The possibility of the contraction gives this simple topology the "absorption feature" typical for an algebraic "ideal."

Then, in Section 4, we consider the 1-port nature of the equivalent circuit, which allows one to use the circuit as a load for a stronger circuit, and argue that the fact (associated with the 1-port nature) that the circuit's source is an internal one means a nonlinearity.

Not yet coming to the point of the nonlinearity, we speak in Section 3—as it is traditionally done—only about linear circuit.

3. An Algebraic Outlook

For transferring to the mathematical point of view, associated with the concept of ideal [7–9], it is worthwhile first to try to understand why the standard mathematical education of electrical engineers includes only linear algebra and not the general algebra where ideals, rings, and groups are taught.

Electrical engineering students and specialists are used to working with expressions of the type

$$a_1x_1 + a_2x_2 + \cdots , \tag{1}$$

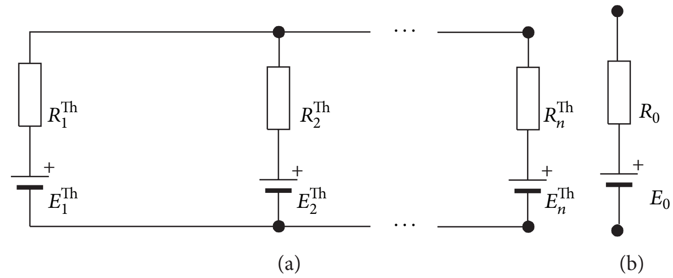

Figure 2: (a) Any such "ladder-type" circuit, and even a much more complicated 1-port, can be equivalently (for the external circuitry) reduced to (b). E_0 and R_0 are easily calculated [1, 2]. So, (b) is the "ideal"; considering additions of linear elements to such a circuit as (b) as an interaction (i.e., a binary operation) with (b), we obtain the form of (b) again.

where $\{a_k\}$ are numbers from a *field* [7] A and $\{x_k\}$ are the vectors (functions) of interest. One says that the set of the vectors is given *over field A*. This scheme of the linear algebra includes only multiplication of the type ax. Such mathematical objects as ideals, rings, and groups, where the xx-type multiplication arises, are not included in the education of the electrical specialists.

Group theory demonstrates some very important applications of the "xx-operation" to physics and chemistry [10–16], reflecting some physical symmetry of a real system. No geometric symmetry is the present point, but in mathematics itself, groups reflect preservation of some properties of the elements of a set, which is close to the circuit situation under the discussion, because every linear "autonomous" 1-port has the feature of the simple Thevenin circuit, and preservation of this feature over the set of linear 1-ports (or under the linear construction operations) is our focus.

By the formal definition, ideal I is a subset of a set $M \supset I$, with elements g_k, so that for all $m \in M$ and for all $g \in I$, $mg \in I$. It is the specific "absorption" property of I, and one says that "I is an ideal of M." Since it can be, in particular, that $m \in I$, every ideal is also a group, but the specific for the ideal feature of the "absorption" is exhibited only for $m \notin I$.

If we consider the operation of the circuit simplification, that is, of replacing a circuit by its Thevenin equivalent, as a here-relevant operation (however far it is from arithmetic multiplication), then Thevenin 1-port is the "ideal" of all of the linear 1-ports. See Figure 2 again.

Note also that the "absorption" nature of the ideal has some recursive feature, as it is with the specific fractal feature of a (any) 1-port, discussed in [17]. (Every branch of a 1-port is also a 1-port, and it is possible to repeat the given structure of the whole 1-port in every of its branches and to continue this procedure recursively.)

Applications of modern algebra to system theory, made by professional mathematicians, for example, [18], are too difficult for engineers, even to circuit theorists, and some compromise of the "logistic" of these applications has to be found. Classical works, for instance, [19, 20], do not include our points as is necessary. That is, the material is too difficult on the one hand and too simple (weak) on the other.

An immediate reason for this omission in the education of the EE specialists is seen if we pay attention to the problem with the physical dimensions. Consider, for instance, the basic condition defining a group G: if we multiply two elements of the group, g_1 and g_2, then the result $g_3 = g_1 g_2$ also belongs to G. This is not a simple requirement for an engineer, since if the elements of G are measured, say, in volts (V), that is,

$$[g_1] = [g_2] = V, \tag{2}$$

then

$$[g_3] = [g_1 g_2] = [g_1][g_2] = V^2; \tag{3}$$

that is, g_3 has a different dimension (units) and should not belong to G. One can overcome this problem, using, together with the multiplication, a special factor k, having the dimension $1/V$, or including $\sqrt{k}$ as a factor in each element, thus "normalizing" the elements in the dimensional sense. Then,

$$g_3 = k g_1 g_2 \in G, \tag{4}$$

without any dimensional problem, but one faces then the problem of the physical sense of k; this parameter is to be a universal one, so that (4) would be correct for all of the elements of G.

However, the algebraic "multiplication," appearing in the definition of the group, need not be the simple arithmetic multiplication related to numbers; the statement is just that, *for two elements of a set, one finds, by some rule, the third element that belongs to a prescribed set*. It is even said in [21] that in some cases one can consider *addition* being such a "multiplication." Then, for instance, "walking" along a straight line is such an addition of the distance, which can be seen as a group. This example can be extended, furthermore, to any translation symmetry.

In the theory of symmetry, groups have wide applications without any dimensional problem. This problem is avoided, at least seemingly, if we use, for instance, the operators acting on phases (angles of a symmetric-obstacle position) as a group of rotations:

$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}, \tag{5}$$

even though there still can be some amplitudes (numerically taken as 1) having a special dimension.

Thus, when we have the elements of a group as an *operation that only interests us*, as in the case of the rotation, then (4) is legitimized for $k = 1$. However, the main constructive point for the rotations and shifts and also for the circuit problem in mind is that the result of the operation, that is, g_3, is as simple as the involved operations g_1 and g_2.

For the circuit application, the concept that is unusual for mathematics is that of "construction," that is, of realization of the simplified circuit. However, any application of mathematics to reality requires some constraints and idealizations, and nobody asks to the point; for instance, how physically difficult is it to rotate a symmetric massive, or a microscopic, body and what instruments are necessary for doing this? Similarly to the known in statistical physics image of "Maxwell's Demon," who just does his job, one could suggest speaking about "Thevenin (Helmholtz) Technician" simplifying the circuits as needed, who would be a quite realistic figure, though the details of his job would be similarly irrelevant to the mathematicians.

4. Linear and Nonlinear

Consider now another point, certainly practically important, also missed regarding Thevenin equivalent in the classical theory, at least as this theory is traditionally taught and presented in the popular textbooks.

The well-known definition of the linear operator, acting on some time-functions $\{f_k(t)\}$, relevant to *system theory and algebra*, is

$$\hat{L}\left[\sum_{1}^{N} a_k f_k\right] = \sum_{1}^{N} a_k \hat{L}\left[f_k\right]. \tag{6}$$

In particular,

$$\hat{L}\left[af\right] = a\hat{L}\left[f\right], \tag{6a}$$

and setting $a = 0$ we have

$$\hat{L}\left[0\right] = 0. \tag{7}$$

This algebraic definition of linearity, requiring $0 \rightarrow 0$, differs from the definition of linearity in *geometry* where a shift can be added, and it is important that the system-theory definition leaves the *affine dependence*, *that is*, the *geometric linearity* (an obvious generalization of $y = ax + b$):

$$\hat{L}\left[\sum_{1}^{N} a_k f_k(t)\right] = \sum_{1}^{N} a_k \hat{L}\left[f_k(t)\right] + \varphi(t), \tag{8}$$

where $\varphi(t)$ is a known function, *as a possibility of system nonlinearity*. This simple nonlinearity becomes just trivial if we assume the Thevenin circuit to be a load for a stronger circuit (which is a quite natural use for a 1-port) and compare the circuit of Figure 1(b) with the circuit of Figure 3 in which the Thevenin source is replaced by a voltage hard-limiter.

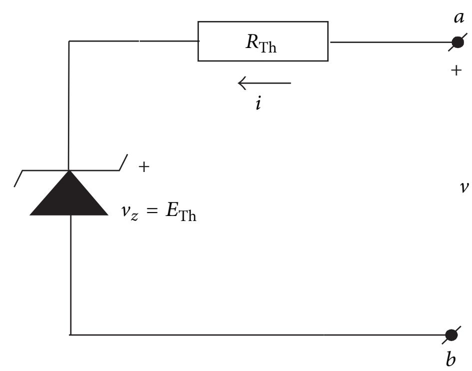

Figure 3: An equivalent circuit, with Zener-diode, if the external circuit with a stronger battery (applying its voltage as shown, the plus up) uses our Thevenin circuit as a load. (Notice the direction of the current.)

It is a very simple comparison/situation, but it is unusual for one to realize that when a source ceases to be applied to a circuit via a special input and becomes an internal circuit element playing a role of a constant term of the type of the function $\varphi(t)$ in (8), then this source becomes nothing else but a strongly nonlinear element. (Notice that $\varphi(t)$ in (8) is *not* any input.)

The problem of the definition of circuit inputs becomes thus the crucial one in the classification of a circuit as linear or nonlinear. Consider also Figure 4.

Figure 5(b) compares the affine nonlinearity of the circuit with the linearity of the possible simple load (a passive resistor R_L) for the circuit; we need both lines to find the work-point. However if the Thevenin circuit is a load for such stronger circuits, we work with two affine lines.

In order to see the importance of the affine nonlinearity, let us consider two examples.

4.1. The Characteristic of a Solar Cell. It is worth noting that the curviness of the typical solar cell characteristic [22, 23] shown in Figure 6 is *not* the main reason for the nonlinearity of the characteristic. If this characteristic were the straight line of the type

$$i(v) = i(0) - kv, \quad k > 0, \; i(0) \neq 0, \tag{9}$$

similar to the line with the negative slope in Figure 5(b), we would still have the active element, and in fact not the curviness, but *the closeness of the characteristic to affine nonlinearity is the main part of the nonlinearity of the cell.*

4.2. A Model of a Living Medium with Growing Sources. The schematic model of Figure 7 also effectively demonstrates the role of the affine nonlinearity, but this time in view of the supposition that complicated circuitry may arise in the modeling of biological systems. We consider a medium, in which a voltage source can grow up, thus decreasing its internal resistance, that is, becoming a stronger (more ideal) source. Thus, the sources can compete with each other.

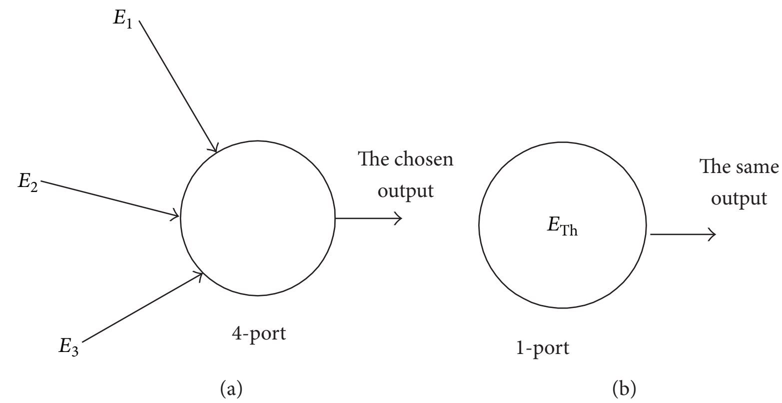

Figure 4: (a) A *linear* circuit with many inputs (may be some sensed physical influences, e.g., the sun's radiation). (b) The Thevenin equivalent of (a), which can be seen as an *affine-nonlinear* 1-port.

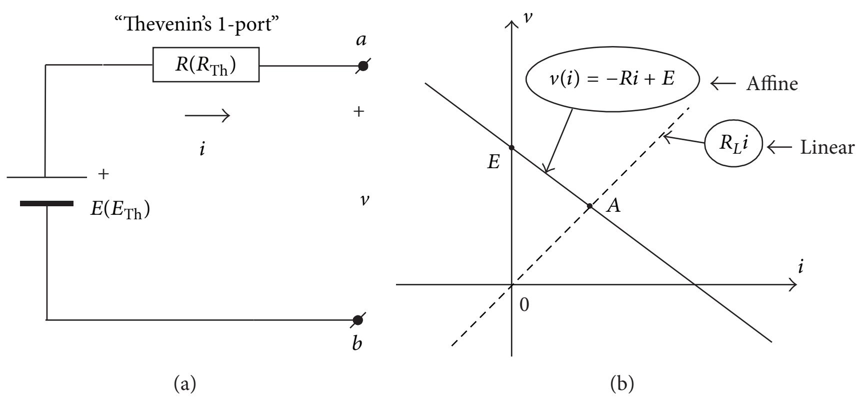

Figure 5: The affine action. (a) The circuit that can either have a load or be by itself a load for a stronger circuit (e.g., for a similar circuit with a stronger source or/and a smaller internal resistance). (b) The relevant graphs.

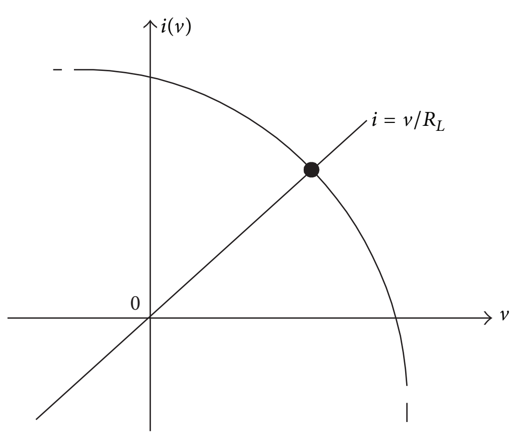

Figure 6: A schematic characteristic of a solar cell. The fact that $i(0)$ is nonzero is most important for the very fact of the nonlinearity and for the power application.

Such observations are relevant to complicated circuitry where definition of subsystems is not simple.

Another interesting example of affine nonlinearity, described in [4], gives a theory of fluorescent lamp circuits.

5. One More Important Application of the Thevenin Theorem

Our final comment, again in simple terms, relates to an unusual application of the concept of nonideal source. We usually see the nonideality of a source as its disadvantage, but in this example it is an advantage, because, in a dangerous, faulty situation, we have to "fight" against the source. This topic relates to electrical safety. Consider a faulty generator, having its metallic body electrified, and a student, who (while making an experiment with this generator in the power laboratory) touches this body. See Figure 8. A good grounding can save the student's life. The resistance of the grounding must be so small that the voltage division between the internal generator's and the grounding resistances (impedances) ensures the voltage on the body of the generator to be sufficiently low. It is not easy to technically realize a small value of the grounding resistor (this may require a special grounding for the laboratory), but this argument is important, because it clearly shows the necessity in the good grounding for a power laboratory in which the students check the working power equipment.

Since a more powerful generator (i.e., a more ideal voltages source) has smaller Z_{eq}, electrical safety either requires

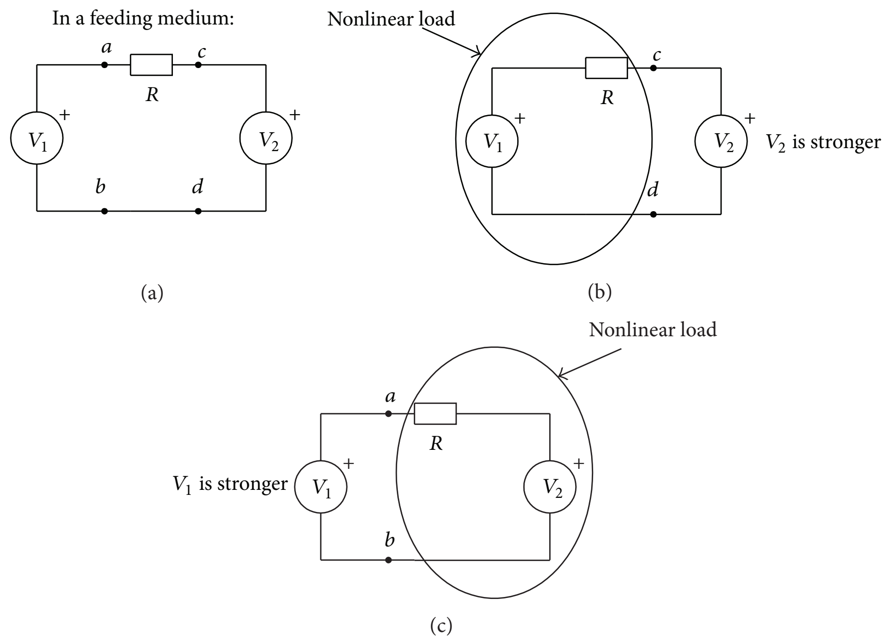

FIGURE 7: (a) We have two options for the sources "growing" in a feeding medium; namely, one of the sources may become dominant; (b) and (c): one of the sources has grown stronger, and we can see in it the ideal source with zero internal resistance and to consider this source to be the input for the rest of the circuit. The other source is weak, that is, its internal resistance (or impedance—see discussion of Figure 1) is significant; it is the (affine) nonlinear side of the circuit, which will have a response of type (8). Notice that in (b) and (c) we have only *one* input.

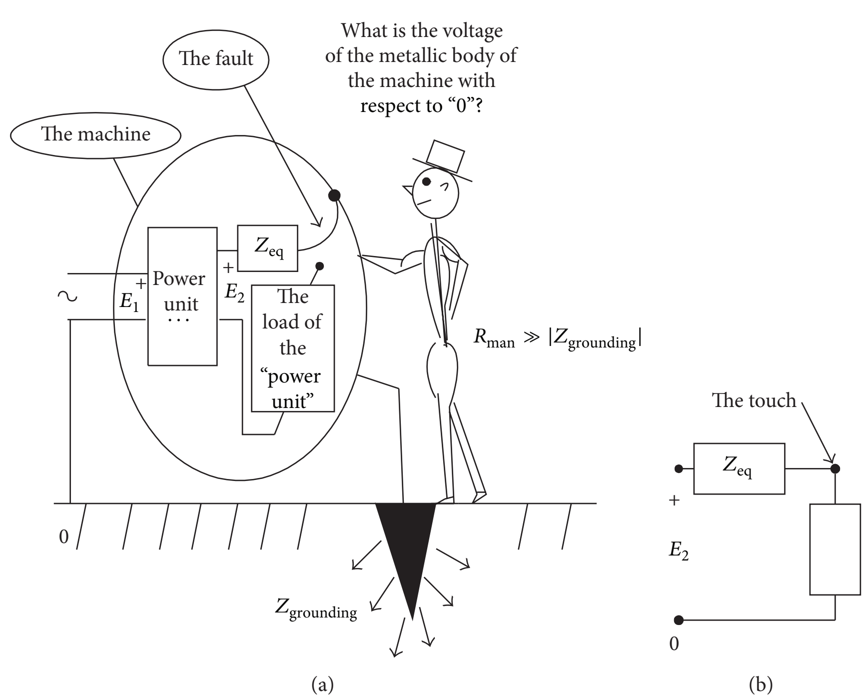

FIGURE 8: The situation of the electrical fault. Instead of being connected to the load of the power unit, Z_{eq} appeared to be connected to the metallic body of the generator (machine). Understanding the generator as a nonideal voltage source having internal impedance Z_{eq} makes it obvious that the ratio $Z_{\text{grounding}}/Z_{\text{eq}}$ must be sufficiently small.

making $Z_{\text{grounding}}$ sufficiently small, or the power of the equipment to be tested by the students, be sufficiently low.

Denoting as V_0 the highest permitted for touching voltage and as V_g the voltage of the generator, we have for the ratio $Z_{\text{grounding}}/Z_{\text{eq}}$ the condition

$$\left| V_g \frac{Z_{\text{grounding}}}{Z_{\text{eq}}} \right| < V_0. \quad (10)$$

For the typical values of $V_0 = 30$ Vrms and $V_g = 220$ Vrms, this gives $|Z_{\text{grounding}}/Z_{\text{eq}}| < 0.136$.

It remains to connect Z_{eq} with the nominal power of the generator. This is done in [24].

Obviously, such argument is relevant to organizing any new students' laboratory, and, on behalf of the pedagogical side, we have here an introduction to the generally important topic of electrical safety [3, 25–27], using the simple equivalent-circuit theorem.

6. Conclusions and Final Remark

The concept of Thevenin theorem and the associated concept of nonideal source are among the most basic concepts of electrical circuit theory. We have shown here and also in [5, 6] that these concepts still are unexplored for the analytical study.

The procedure of deriving Thevenin equivalent is interpreted as an operation related to the algebraic concept of ideal. This connection can stimulate a mathematician to become interested in the circuit theory and an electrical engineering specialist (student) to take interest in the general algebra.

We also strongly argued for the importance of observing affine nonlinearity in circuit theory (see also [4]), which is closely associated with the role of the choice of inputs, which was presumably predicted by Helmholtz 162 years ago. However simple and natural the theorem in focus is, it is, perhaps, the most important contribution to the circuit theory made after Gustav Robert Kirchhoff (a colleague of Helmholtz in Berlin University and the other outstanding teacher of Max Plank) introduced his circuits laws.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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