

Research Article

Analytical Solution of Space-Time Fractional Fokker-Planck Equation by Homotopy Perturbation Sumudu Transform Method

Ravi Shanker Dubey,¹ Badr Saad T. Alkahtani,² and Abdon Atangana³

¹Department of Mathematics, Yagyavalkya Institute of Technology, Jaipur 302022, India

²Mathematics Department, College of Science, King Saud University, P.O. Box 1142, Riyadh 11989, Saudi Arabia

³Institute for Groundwater Studies, Faculty of Natural and Agricultural Sciences, University of the Free State, Bloemfontein 9300, South Africa

Correspondence should be addressed to Ravi Shanker Dubey; ravimath13@gmail.com

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An efficient approach based on homotopy perturbation method by using Sumudu transform is proposed to solve some linear and nonlinear space-time fractional Fokker-Planck equations (FPEs) in closed form. The space and time fractional derivatives are considered in Caputo sense. The homotopy perturbation Sumudu transform method (HPSTM) is a combined form of Sumudu transform, homotopy perturbation method, and He's polynomials. The nonlinear terms can be easily handled by the use of He's polynomials. Some examples show that the HPSTM is an effective tool for solving many space time fractional partial differential equations.

1. Introduction

Fokker-Planck equation (FPE) was introduced by Adriaan Fokker and Max Planck to describe the time evolution of the probability density function of position and velocity of a particle, which is one of the classical widely used equations of statistical physics [1]. FPE arises in a number of different fields in natural sciences; Brownian motion [2] and the diffusion model of chemical reactions [3] are now largely employed, in various generalized forms, in physics, chemistry, engineering, and biology [1]. The FPE arises in kinetic theory [4] where it describes the evolution of the one-particle distribution function of a dilute gas with long-range collisions, such as a Coulomb gas. Some applications of this type of equations can be worked out in the works of He and Wu [5], Jumarie [6], Kamitani and Matsuba [7], Xu et al. [8], and Zak [9].

The general FPE for the motion of a concentration field $v(x, t)$ of one space variable x at time t has the form [1]

$$\frac{\partial v(x, t)}{\partial t} = \left[-\frac{\partial A(x)}{\partial x} + \frac{\partial^2 B(x)}{\partial x^2} \right] v(x, t) \quad (1)$$

with initial condition

$$v(0, x) = f(x), \quad x \in \mathcal{R}, \quad (2)$$

where $A(x)$ and $B(x) > 0$ are called the drift and diffusion coefficients. This equation is also called the forward Kolmogorov equation. The drift and diffusion coefficients may also depend on time as

$$\frac{\partial v(x, t)}{\partial t} = \left[-\frac{\partial A(x, t)}{\partial x} + \frac{\partial^2 B(x, t)}{\partial x^2} \right] v(x, t). \quad (3)$$

There is a more general form of Fokker-Planck equation which is called the nonlinear Fokker-Planck equation. The nonlinear Fokker-Planck equation has important applications in various areas such as plasma physics, surface physics, population dynamics, biophysics, engineering, neurosciences, nonlinear hydrodynamics, polymer physics, laser physics, pattern formation, psychology, and so forth [10]. In the one variable case, the nonlinear FPE can be written as

$$\frac{\partial v(x, t)}{\partial t} = \left[-\frac{\partial A(x, t, v)}{\partial x} + \frac{\partial^2 B(x, t, v)}{\partial x^2} \right] v(x, t). \quad (4)$$

Due to vast range of applications of the FPE, a lot of work has been done to find numerical solution with this equation. In this context, the works of Buet et al. [11], Harrison [12], Palleschi et al. [13], Vanaja [14], and Zorzano et al. [15] are worth mentioning.

For generalization of classical integer order of FPE (4), the following equation can be introduced. The Fokker-Planck equation with fractional space derivative is a particular case of anomalous diffusion and Lévy flights (see[16–19]). This equation is called nonlinear FPE with space-time fractional derivatives [20]:

$$\frac{\partial^\alpha v(x, t)}{\partial t^\alpha} = \left[-\frac{\partial A(x, t, v)}{\partial x} + \frac{\partial^2 B(x, t, v)}{\partial x^2} \right] v(x, t), \quad (5)$$

where $t > 0$, $x > 0$, and $0 < \alpha \leq 1$. It can be obtained from the general Fokker-Planck equation by replacing the space and time derivatives by fractional derivatives operator. The function $v(x, t)$ is assumed to be a causal function of time and space. Particularly, for $\alpha = 1$, the fractional FPE (5) reduces to the classical nonlinear FPE given by (4) in the case $x > 0$.

In recent years, researchers have studied the fractional partial differential equations and the fractional FPE to investigate various scientific models [21–24]. In the present paper we obtain closed form solutions of a linear-nonlinear time fractional FPE using homotopy perturbation Sumudu transform method (HPSTM); see [25].

2. Some Mathematical Preliminaries and Definitions

2.1. Fundamental Properties of Fractional Calculus. Firstly, we mention some of the fundamental properties of the fractional calculus, that is, fractional derivatives and integrals.

The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of a function $f(t) \in C_\mu$, $\mu \geq -1$, is defined by implementing the integration operator J^α in the following manner:

$$\begin{aligned} J^\alpha [f(t)] &= \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad \alpha > 0, \\ J^0 [f(t)] &= f(t), \\ J^\alpha J^\beta [f(t)] &= J^{\alpha+\beta} f(t), \\ J^\beta J^\alpha [f(t)] &= J^\alpha J^\beta f(t), \\ J^\alpha (t^\gamma) &= \frac{\Gamma(\gamma + 1)}{\Gamma(\alpha + \gamma + 1)} t^{\alpha+\gamma}. \end{aligned} \quad (6)$$

The fundamental properties of fractional integration and fractional differentiation have been introduced to the literature by Podlubny [26].

The fractional derivative of $f(t)$ in the Caputo sense is defined by the following relation [27]:

$$\begin{aligned} D_t^\alpha f(t) &= J^{m-\alpha} D^m f(t) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{-\alpha+m-1} f^{(m)}(\tau) d\tau, \quad (7) \\ m-1 &< \alpha \leq m, \end{aligned}$$

where $m \in N$ and $t > 0$.

The relation between Riemann-Liouville fractional integral operator and the Caputo fractional derivative operator is

$$\begin{aligned} J_t^\alpha D_t^\alpha f(t) &= f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0+) t^k}{k!}, \\ D^{-\alpha} [f(t)] &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad (8) \\ 0 &< \alpha \leq 1. \end{aligned}$$

Further properties and applications can be found in [26, 28–31].

2.2. The Sumudu Transform. In early 90s, Watugala [32] introduced a new integral transform named the Sumudu transform and applied it to the solution of ordinary differential equation in control engineering problems. The Sumudu transform is defined over the set of functions

$$\begin{aligned} A = \{f(t) \mid \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{|\tau_j|}, \\ \text{if } t \in (-1)^j \times [0, \infty)\}; \end{aligned} \quad (9)$$

the Sumudu transform is defined by

$$\tilde{G}(u) = S[f(t)] = \int_0^\infty f(ut) e^{-t} dt, \quad u \in (-\tau_1, \tau_2). \quad (10)$$

Sumudu transform has many useful and important properties like linear property, scale properties, shifting properties, duality with Laplace transforms, and so forth. Further detail and properties about this transform can be found in [25, 33–37].

By using the Sumudu transform of multiple differentiation, we obtain

$$\begin{aligned} S[D_t^\alpha f(t)] &= u^{m-\alpha} \left[\frac{\tilde{G}(u)}{u^m} - \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{u^{m-k}} \right] \\ &= \left[\frac{\tilde{G}(u)}{u^\alpha} - \sum_{k=0}^{m-1} \frac{f^{(k)}(0+)}{u^{\alpha-k}} \right], \quad (m-1 < \alpha \leq m), \end{aligned} \quad (11)$$

where $\tilde{G}(u) = S[f(t)]$.

2.3. *The Mittag-Leffler Function.* The Mittag-Leffler function which is a generalization of exponential function (see [38]) is as follows:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \quad (\alpha \in \mathbb{C}, \Re(\alpha) > 0). \quad (12)$$

2.4. *Adomian Decomposition Method.* The Adomian decomposition method (ADM) is a creative and effective method for exactly solving functional equations of various kinds. The method was developed by Adomian [39]. It is important to note that a large amount of research work has been devoted to the application of the ADM in a wide class of linear, nonlinear, ordinary, or partial differential equations. The decomposition method provides the solution as an infinite series in which each term can be determined easily.

3. Solution of Fractional Differential Equations by Homotopy Perturbation Sumudu Transform Method (HPSTM)

The homotopy perturbation method (HPM), introduced by He (see [40–43]), is a series expansion method used in the solution of nonlinear partial differential equations. The HPM uses a so-called convergence-control parameter to guarantee the convergence of approximation series over a given interval of physical parameters.

We illustrate the basic idea of this method by considering a general fractional nonlinear nonhomogeneous partial differential equation

$$D_t^\alpha U(x, t) = LU(x, t) + NU(x, t) + f(x, t), \quad \alpha > 0, \quad (13)$$

with initial conditions

$$\begin{aligned} D_0^k U(x, 0) &= g_k, \quad (k = 0, 1, \dots, n-1), \\ D_0^n U(x, 0) &= 0, \quad n = [\alpha], \end{aligned} \quad (14)$$

where $D_t^\alpha U(x, t)$ is the Caputo fractional derivative of the function $U(x, t)$, L is the linear differential operator, N represents the general nonlinear differential operator, and $f(x, t)$ is a known function.

Applying the Sumudu transform on both sides of (13), we get

$$S[D_t^\alpha U(x, t)] = S[LU(x, t)] + S[NU(x, t)] + S[f(x, t)]. \quad (15)$$

Using the property of the Sumudu transform, we have

$$S[U(x, t)] = u^\alpha S[LU(x, t) + NU(x, t)] + g(x, t). \quad (16)$$

Operating with the inverse Sumudu transform on both sides of (16), we get

$$U(x, t) = G(x, t) + S^{-1}[u^\alpha S[LU(x, t) + NU(x, t)]], \quad (17)$$

where $G(x, t)$ represents the term arising from the function and the prescribed initial conditions.

Now we use the homotopy perturbation method

$$U(x, t) = \sum_{n=0}^{\infty} p^n U_n(x, t) \quad (18)$$

and the nonlinear term can be decomposed as

$$NU(x, t) = \sum_{n=0}^{\infty} p^n H_n(U) \quad (19)$$

on using He's polynomials $H_n(U)$ (see [10, 44]) which are given by

$$H_n(U_0, U_1, \dots, U_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[N \left(\sum_{i=0}^{\infty} p^i U_i \right) \right]_{p=0}. \quad (20)$$

Substituting (18) and (19) in (17), we get

$$\begin{aligned} &\sum_{n=0}^{\infty} p^n U_n(x, t) \\ &= G(x, t) \\ &+ p \left(S^{-1} \left[u^\alpha S \left[L \sum_{n=0}^{\infty} p^n U_n(x, t) + \sum_{n=0}^{\infty} p^n H_n(U) \right] \right] \right) \end{aligned} \quad (21)$$

which shows the coupling of the Sumudu transform and the HPM by using He's polynomials.

On collecting the coefficients of powers of p , we obtain

$$\begin{aligned} p^0 : U_0(x, t) &= G(x, t) \\ p^1 : U_1(x, t) &= S^{-1} [u^\alpha S \{LU_0(x, t) + H_0(U)\}] \\ p^2 : U_2(x, t) &= S^{-1} [u^\alpha S \{LU_1(x, t) + H_1(U)\}] \end{aligned} \quad (22)$$

and similarly

$$p^n : U_n(x, t) = S^{-1} [u^\alpha S \{LU_{n-1}(x, t) + H_{n-1}(U)\}]. \quad (23)$$

Finally, we approximate the analytical solution $U(x, t)$, by truncated series

$$U(x, t) = \lim_{N \rightarrow \infty} \sum_{n=0}^N U_n(x, t). \quad (24)$$

However, as we have seen in many cases, the exact solution in a closed form may be obtained. In addition, the decomposition series solutions generally converge very rapidly. Abbaoui and Cherruault [45] had defined the classical approach of convergence of this type of series.

4. Solution of Fokker-Planck Fractional Differential Equation

Solution of the following nonlinear space-time fractional Fokker-Planck equations (FPE) is

$$\frac{\partial^\alpha v(x, t)}{\partial t^\alpha} = \left[-\frac{\partial A(x, t, v)}{\partial x} + \frac{\partial^2 B(x, t, v)}{\partial x^2} \right] v(x, t) \quad (25)$$

with the initial condition

$$v(x, 0) = f(x). \tag{26}$$

Applying the Sumudu transform on both sides of (25), subject to initial condition (26), we have

$$S[v(x, t)] = f(x) + u^\alpha S \times \left[-\frac{\partial A(x, t, v)}{\partial x} v(x, t) + \frac{\partial^2 B(x, t, v)}{\partial x^2} v(x, t) \right], \tag{27}$$

taking inverse Sumudu transform

$$v(x, t) = f(x) + S^{-1} \left[u^\alpha S \left\{ -\frac{\partial}{\partial x} \sum_{n=0}^{\infty} A_n + \frac{\partial^2}{\partial x^2} \sum_{n=0}^{\infty} B_n \right\} \right], \tag{28}$$

where

$$\sum_{n=0}^{\infty} A_n = A(x, t, v) \cdot v(x, t), \tag{29}$$

$$\sum_{n=0}^{\infty} B_n = B(x, t, v) \cdot v(x, t).$$

Using HPM method, we get

$$\sum_{n=0}^{\infty} p^n v_n(x, t) = f(x) + p \times \left(S^{-1} \left[u^\alpha S \left\{ -\sum_{n=0}^{\infty} p^n H_n(x, t, v) + \sum_{n=0}^{\infty} p^n H_n^*(x, t, v) \right\} \right] \right), \tag{30}$$

where $H_n(x, t, v)$ and $H_n^*(x, t, v)$ are He's polynomials that represent the nonlinear terms. The He's polynomials are given by

$$\sum_{n=0}^{\infty} H_n(x, t, v) = \frac{\partial}{\partial x} \left(\sum_{n=0}^{\infty} A_n \right), \tag{31}$$

$$\sum_{n=0}^{\infty} H_n^*(x, t, v) = \frac{\partial^2}{\partial x^2} \left(\sum_{n=0}^{\infty} B_n \right).$$

Using the above equation we can collect the coefficient of power of p

$$p^0 : v_0(x, t) = f(x),$$

$$p^1 : v_1(x, t) = S^{-1} [u^\alpha S \{[-H_0 + H_0^*]\}], \tag{32}$$

$$p^2 : v_2(x, t) = S^{-1} [u^\alpha S \{[-H_1 + H_1^*]\}]$$

and similarly

$$p^n : v_n(x, t) = S^{-1} [u^\alpha S \{[-H_{n-1} + H_{n-1}^*]\}]. \tag{33}$$

Finally, we approximate the analytical solution $v_n(x, t)$ by truncated series [25]

$$v(x, t) = \lim_{N \rightarrow \infty} \sum_{n=0}^N v_n(x, t); \tag{34}$$

the series solutions of the above equation converge very rapidly [25, 45].

5. Numerical Examples

In this section we will illustrate the HPSTM techniques by several examples. These examples are somewhat artificial in the sense that the exact answer, for the special case $\alpha = 1$, is known in advance and the initial and boundary conditions are directly taken from this answer. Nonetheless, such an approach is needed to evaluate the accuracy of the analytical techniques and to examine the effect of varying the order of the space- and time-fractional derivatives on the behavior of the solution. All the results are calculated by using the symbolic calculus software MATLAB.

Example 1. Consider the nonlinear time fractional FPE

$$D_t^\alpha v(x, t) = \left[-\frac{\partial}{\partial x} \left(3v - \frac{x}{2} \right) + \frac{\partial^2}{\partial x^2} (xv) \right] v(x, t), \tag{35}$$

where $t > 0, x > 0, 0 < \alpha \leq 1$, and D_t^α is the Caputo fractional derivative defined by (7) and initial condition is

$$v(x, 0) = x. \tag{36}$$

Then solution is given by $v(x, t) = xE_\alpha(t^\alpha)$.

Solution 1. On using the method defined in Section 4, we get the coefficient of powers of p

$$p^0 : v_0(x, t) = x,$$

$$p^1 : v_1(x, t) = \frac{xt^\alpha}{\Gamma(\alpha + 1)},$$

$$p^2 : v_2(x, t) = \frac{xt^{2\alpha}}{\Gamma(2\alpha + 1)}, \tag{37}$$

$$p^3 : v_3(x, t) = \frac{xt^{3\alpha}}{\Gamma(3\alpha + 1)};$$

hence, $v(x, t)$ is

$$v(x, t) = x \left(1 + \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + \dots \right)$$

$$v(x, t) = xE_\alpha(t^\alpha). \tag{38}$$

Remark 2. Setting $\alpha = 1$, Example 1 reduces to nonlinear FPE

$$\frac{\partial}{\partial t} v(x, t) = \left[-\frac{\partial}{\partial x} \left(3v - \frac{x}{2} \right) + \frac{\partial^2}{\partial x^2} (xv) \right] v(x, t) \tag{39}$$

with initial condition

$$v(x, 0) = x, \tag{40}$$

and a solution as

$$v(x, t) = xe^t. \tag{41}$$

Example 3. Consider the nonlinear time fractional FPE

$$D_t^\alpha v(x, t) = \left[-\frac{\partial}{\partial x} \left(\frac{4v}{x} - \frac{x}{3} \right) + \frac{\partial^2}{\partial x^2} (v) \right] v(x, t), \quad x > 0, t > 0, \tag{42}$$

where $t > 0, x > 0, 0 < \alpha \leq 1$, and D_t^α is Caputo fractional derivative defined by (7) and initial condition is

$$v(x, 0) = x^2. \tag{43}$$

Then solution is given by $v(x, t) = x^2 E_\alpha(t^\alpha)$.

Solution 2. By using the method defined in Section 4, we get the coefficient of power of p

$$\begin{aligned} p^0 : v_0(x, t) &= x^2, \\ p^1 : v_1(x, t) &= \frac{x^2 t^\alpha}{\Gamma(\alpha + 1)}, \\ p^2 : v_2(x, t) &= \frac{x^2 t^{2\alpha}}{\Gamma(2\alpha + 1)}, \\ p^3 : v_3(x, t) &= \frac{x^2 t^{3\alpha}}{\Gamma(3\alpha + 1)} \dots; \end{aligned} \tag{44}$$

hence, $v(x, t)$ is

$$\begin{aligned} v(x, t) &= x^2 \left(1 + \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + \dots \right) \\ &= x^2 E_\alpha(t^\alpha). \end{aligned} \tag{45}$$

Remark 4. Setting $\alpha = 1$, Example 3 reduces to nonlinear FPE

$$\begin{aligned} \frac{\partial}{\partial t} v(x, t) &= \left[-\frac{\partial}{\partial x} \left(\frac{4v}{x} - \frac{x}{3} \right) + \frac{\partial^2}{\partial x^2} (v) \right] v(x, t), \quad x > 0, t > 0 \end{aligned} \tag{46}$$

with initial condition

$$v(x, 0) = x^2 \tag{47}$$

and a solution as

$$v(x, t) = x^2 e^t. \tag{48}$$

Example 5. Consider the nonlinear time fractional FPE

$$D_t^\alpha v(x, t) = \left[-\frac{\partial}{\partial x} (x) + \frac{\partial^2}{\partial x^2} \left(\frac{x^2}{2} \right) \right] v(x, t), \quad x > 0, t > 0, \tag{49}$$

where $t > 0, x > 0, 0 < \alpha \leq 1$, and D_t^α is Caputo fractional derivative defined by (7) and initial condition is

$$v(x, 0) = x. \tag{50}$$

Solution 3. By using the method defined in Section 4, we get the coefficient of p

$$\begin{aligned} p^0 : v_0(x, t) &= x, \\ p^1 : v_1(x, t) &= \frac{xt^\alpha}{\Gamma(\alpha + 1)}, \\ p^2 : v_2(x, t) &= \frac{xt^{2\alpha}}{\Gamma(2\alpha + 1)}, \\ p^3 : v_3(x, t) &= \frac{xt^{3\alpha}}{\Gamma(3\alpha + 1)}; \end{aligned} \tag{51}$$

hence, $v(x, t)$ is

$$\begin{aligned} v(x, t) &= x \left(1 + \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + \dots \right) \\ &= x E_\alpha(t^\alpha). \end{aligned} \tag{52}$$

Remark 6. Setting $\alpha = 1$ in problem, Example 5 reduces to nonlinear FPE

$$\begin{aligned} \frac{\partial}{\partial t} v(x, t) &= \left[-\frac{\partial}{\partial x} (x) + \frac{\partial^2}{\partial x^2} \left(\frac{x^2}{2} \right) \right] v(x, t), \\ & \quad x > 0, t > 0, \end{aligned} \tag{53}$$

with initial condition

$$v(x, 0) = x \tag{54}$$

and a solution as

$$v(x, t) = xe^t. \tag{55}$$

The approximate solutions for (42) and (49) shown in Tables 1 and 2, respectively, were obtained for different values of x and t by using the homotopy perturbation Sumudu transform method. The exact solutions of (42) and (49) are $v(x, t) = x^2 e^t$ and $v(x, t) = x e^t$, respectively. The accuracy of our approximate solutions can be improved by computing more terms of the approximate solutions. Both the methods viz. ADM and HPSTM are used adroitly to obtain the exact values and get comparisons. From the numerical results in Tables 1 and 2, it is easy to conclude that the solution continuously depends on the space-fractional derivative and the approximate solutions of (42) and (49) obtained by using

TABLE 1: Comparison study for solution of (42) for $\alpha = 1$, when $t = 0.2, 0.4, 0.6$ and $x = 0.25, 0.50, 0.75, 1.0$ by various methods.

t	x	HPSTM	ADM (see [20])	EXACT (see [20])
0.2	0.25	0.076333	0.076333	0.076338
0.2	0.50	0.305333	0.305333	0.305351
0.2	0.75	0.687000	0.687000	0.687039
0.2	1.00	1.221333	1.221333	1.221403
0.4	0.25	0.093167	0.093167	0.093239
0.4	0.50	0.372667	0.372667	0.372956
0.4	0.75	0.838500	0.838500	0.839151
0.4	1.00	1.490667	1.490667	1.491825
0.6	0.25	0.113500	0.113500	0.113882
0.6	0.50	0.454000	0.454000	0.455530
0.6	0.75	1.021500	1.021500	1.024942
0.6	1.00	1.816000	1.816000	1.822119

TABLE 2: Comparison study for solution of (49) for $\alpha = 1$, when $t = 0.2, 0.4, 0.6$ and $x = 0.25, 0.50, 0.75, 1.0$ by various methods.

t	x	HPSTM	ADM (see [20])	EXACT (see [20])
0.2	0.25	0.305333	0.305333	0.305351
0.2	0.50	0.610667	0.610667	0.610701
0.2	0.75	0.916000	0.916000	0.916052
0.2	1.00	1.221333	1.221333	1.221403
0.4	0.25	0.372667	0.372667	0.372956
0.4	0.50	0.745333	0.745333	0.745912
0.4	0.75	1.118000	1.118000	1.118869
0.4	1.00	1.490667	1.490667	1.491825
0.6	0.25	0.454000	0.454000	0.455530
0.6	0.50	0.908000	0.908000	0.911059
0.6	0.75	1.362000	1.362000	1.366589
0.6	1.00	1.816000	1.816000	1.822119

the Adomian decomposition method are the same as those obtained by the homotopy perturbation Sumudu transform method. It is to be noted that only the fourth-order term of the series of the abovesaid methods was used in evaluating the approximate solutions for Tables 1 and 2.

6. Conclusion

Within the scope of the HPSTM, we derive the solution of the nonlinear time fractional Fokker-Planck equations. We made use of the Caputo derivative because it allows usual initial conditions. The numerical result shows that the method used is very simple and straightforward to implement.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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