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Research Article

$\mathscr{H}_\infty$ Performance and Stability Analysis of Linear Systems with Interval Time-Varying Delays and Stochastic Parameter Uncertainties

M. J. Park,[1] O. M. Kwon,[1] Ju H. Park,[2] S. M. Lee,[3] and E. J. Cha[4]

[1] *School of Electrical Engineering, Chungbuk National University, 52 Naesudong-ro, Cheongju 361-763, Republic of Korea*
[2] *Nonlinear Dynamics Group, Department of Electrical Engineering, Yeungnam University, 280 Daehak-ro, Gyeongsan 712-749, Republic of Korea*
[3] *School of Electronic Engineering, Daegu University, Gyeongsan 712-714, Republic of Korea*
[4] *Department of Biomedical Engineering, School of Medicine, Chungbuk National University, 52 Naesudong-ro, Cheongju 361-763, Republic of Korea*

Correspondence should be addressed to O. M. Kwon; madwind@chungbuk.ac.kr

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This paper deals with the problems of $\mathscr{H}_\infty$ performance and stability analysis for linear systems with interval time-varying delays. It is assumed that the parameter uncertainties are of stochastic properties to represent random change of various environments. By constructing a newly augmented Lyapunov-Krasovskii functional, less conservative criteria of the concerned systems are introduced with the framework of linear matrix inequalities (LMIs). Four numerical examples are given to show the improvements over the existing ones and the effectiveness of the proposed methods.

1. Introduction

The mathematical models representing physical systems are generally not exact due to the various reasons such as noises and parameter changes in electrical elements. For this reason, in some cases, the stability of the mathematical model cannot guarantee the stability of the physical systems. In order to take into account such problem, the parameter uncertainties should be considered in the theoretical stability analysis for various systems. The aforementioned parameter uncertainties are the internal sources of the model, whereas the disturbances can be their external sources. Then, the objective of an $\mathscr{H}_\infty$ performance analysis is to find a saddle point of objective functional calculus depending on the disturbance. In other words, we find its minimum for the worst-case disturbances. Moreover, from the point of view of stability, it is also needed to pay attention to a delay in the time. It is well known that time delays frequently occur in various systems due to the finite speed limit of information processing and transmission in the implementation of the systems. For this reason, the undesirable dynamic behaviors such as poor performance and instability can be caused by the wake of the delay.

In this regard, $\mathscr{H}_\infty$ performance and/or stability of time-delay systems were dealt with in the literature [1–12]. Above all, in [5], the robust $\mathscr{H}_\infty$ performance conditions for uncertain networked control systems with time-delay were derived by the use of some slack matrix variable. Jeong et al. [6] introduced the improved conditions of $\mathscr{H}_\infty$ performance analysis and stability for systems with interval time-varying delays and uncertainties. In [10], the robust $\mathscr{H}_\infty$ performance analysis and stability problems of linear systems with interval time-varying delays were investigated by constructing some new augmented Lyapunov-Krasovskii functional. Also, in order to obtain tighter lower bounds of integral terms of quadratic form, Wirtinger-based inequality in [11] is the recent remarkable tool in reducing the conservatism of delay-dependent stability criteria for dynamic systems with time delays. Therefore, there are scopes for further improved results in stability analysis with time-delay.

Returning to the concept of parameter uncertainties, in this work, it is assumed that the parameter uncertainties occur by stochastic property to represent random change of various environments. This exemplifies why considering the stochastic property includes the fact that when the earthquake happens, although the seismic intensity is the same, at all times, its wavy pattern and effects are different. However, the systems with stochastic parameter uncertainties have not been fully investigated yet. Specially, in this work, two stochastic indexes, the mean and the variance, are utilized. Thus, the concerned problems highlighting the difference between the effects of the mean and the variance on the systems will be dealt in this work.

With this motivation mentioned above, in this paper, the $\mathscr{H}_\infty$ performance and stability problems to get improved sufficient conditions for uncertain systems with interval time-varying delays and stochastic parameter uncertainties are studied. Here, stability of system with interval time-varying delays has been a focused topic of theoretical and practical importance [13]. The interval time-varying delays mean that its lower bounds which guarantee the stability of system are not restricted to be zero and include networked control system as one of typical examples. To achieve this, by construction of a suitable augmented Lyapunov-Krasovskii functional and utilization of Wirtinger-based inequality [11], an $\mathscr{H}_\infty$ performance condition is derived in Theorem 8 with the framework of LMIs which can be formulated as convex optimization algorithms which are amenable to computer solution [14]. Also, inspired by the works of [4, 12], the reciprocally convex and zero equality approaches are utilized with some decision variables to reduce the conservatism of the concerned condition. Based on the result of Theorem 8, $\mathscr{H}_\infty$ performance condition with deterministic parameter uncertainties and an improved stability condition for the nominal form without parameter uncertainties and disturbances will be proposed, respectively, in Theorem 11 and Corollary 12. Finally, four numerical examples are included to show the effectiveness of the proposed methods.

Notation. The notations used throughout this paper are fairly standard. $\mathbb{R}^n$ is the n-dimensional Euclidean space, and $\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ real matrices. $\mathscr{L}_2[0,\infty)$ is the space of square integrable vector on $[0,\infty)$. For symmetric matrices X and Y, $X > Y$ means that the matrix $X - Y$ is positive definite, whereas $X \geq Y$ means that the matrix $X - Y$ is nonnegative definite. I_n, 0_n, and $0_{n\cdot m}$ denote $n \times n$ identity matrix, $n \times n$ and $n \times m$ zero matrices, respectively. $\|\cdot\|$ refers to the Euclidean vector norm or the induced matrix norm. $\mathtt{diag}\{\cdots\}$ denotes the block diagonal matrix. For square matrix X, $\mathtt{sym}\{X\}$ means the sum of X and its transposed matrix X^T, that is, $\mathtt{sym}\{X\} = X + X^T$. $\mathtt{col}\{x_1, x_2, \ldots, x_n\}$ means $[x_1^T, x_2^T, \ldots, x_n^T]^T$. $X_{[f(t)]} \in \mathbb{R}^{m\times n}$ means that the elements of matrix $X_{[f(t)]}$ include the scalar value of $f(t)$, that is, $X_{[f_0]} = X_{[f(t)=f_0]}$. $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation operator. $\Pr\{A\}$ means the occurrence probability of the event A.

2. Preliminaries and Problem Statement

Consider the uncertain systems with time-varying delays and disturbances given by

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A(t))\, x(t) \\ &\quad + \left(A_d + \Delta A_d(t)\right) x(t - h(t)) + B_1 w(t), \\ z(t) &= Cx(t) + C_d x(t - h(t)) + B_2 w(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $z(t) \in \mathbb{R}^{n_z}$ is the output vector, and $w(t) \in \mathbb{R}^{n_w}$ is the disturbances; A, $A_d \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times n_w}$, $B_2 \in \mathbb{R}^{n_z\times n_w}$, and C, $C_d \in \mathbb{R}^{n_z\times n}$ are the system matrices, and $\Delta A(t)$ and $\Delta A_d(t)$ are the parameter uncertainties of the form

$$\left[\Delta A(t), \Delta A_d(t)\right] = DF(t)\left[E_a, E_d\right], \tag{2}$$

where $D \in \mathbb{R}^{n\times n_f}$, $E_a \in \mathbb{R}^{n_u\times n}$, and $E_d \in \mathbb{R}^{n_u\times n}$ are real known constant matrices and $F(t) \in \mathbb{R}^{n_f\times n_u}$ is a real uncertain matrix function with Lebesgue measurable elements satisfying $F^T(t)F(t) \leq I_{n_u}$.

The delay $h(t)$ is a time-varying continuous function satisfying

$$0 \leq h_m \leq h(t) \leq h_M, \quad \dot{h}(t) \leq d_M, \tag{3}$$

where h_m, h_M, and d_M are known constant values.

For simplicity of system representation, the system can be formulated as follows:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t - h(t)) + B_1 w(t) + Dp(t), \\ p(t) &= F(t)\, q(t), \\ q(t) &= E_a x(t) + E_d x(t - h(t)), \\ z(t) &= Cx(t) + C_d x(t - h(t)) + B_2 w(t). \end{aligned} \tag{4}$$

Also, the following definition and lemmas will be used in main results.

Assumption 1. The parameter uncertainties are changed with the stochastic sequences $\rho(t)$, which are a family of time functions depending on the outcome of the set of experimental outcomes. Then, the uncertainty term, $q(t)$, is represented by

$$q(t) = \rho(t)\left(E_a x(t) + E_d x(t - h(t))\right), \tag{5}$$

where $\rho(t)$ satisfies $\mathbb{E}\{\rho(t)\} = \rho_0$ and $\mathbb{E}\{(\rho(t) - \rho_0)^2\} = \sigma^2$. Here, ρ_0 and σ^2 are mean and variance of $\rho(t)$, respectively.

Remark 2. After the introduction of the Bernoulli property to engineering, it has been applied in many situations such as random delays [15] and sensors fault [16]. In very recent times, various forms of randomly occurring concept, for example, randomly occurring uncertainties, randomly occurring nonlinearities, randomly occurring delays, and so on, are represented by the Bernoulli property [17, 18]. Besides, the Markov property, which is a favorite stochastic sequence, is used to describe the unexpected changes of parameters in hybrid systems [19–21]. It should be noted that the existing results utilizing Bernoulli and Markov property have not utilized the information about the variance. However, in this work, the system parameter uncertainties are described by the general stochastic property with its two indexes, mean and variance. By defining $\rho(t)$ in (5), the variance value of $\rho(t)$ will be considered in analyzing the robust $\mathscr{H}_\infty$ performance of system (4). The necessity of these considerations will be explained in Example 1. Therefore, to analyze this problem mentioned above, in this work, the stochastic parameter uncertainties are dealt by adopting the property of the stochastic sequence, which contains two indexes, mean and variance, instead of studying the problem of previous stochastic analysis method considering Wiener process, that is, the form $(E_a x(t) + E_d x(t-h(t)))d\omega(t)$, where $\omega(t)$ is Wiener process. Moreover, by utilizing the proposed model, the dynamic behavior of practical problem nearer to the random change of real environment will become accessible.

The aim of this paper is to investigate the $\mathscr{H}_\infty$ performance and stability analysis of system (4) with interval time-varying delays and stochastic parameter uncertainties. Before deriving our main results, the following definition and lemmas are introduced.

Definition 3. $\mathscr{H}_\infty$-optimization seeks a state-feedback controller that minimizes the $\mathscr{H}_\infty$-norm of the system's closed-loop transfer function between the controlled output $z(t)$ and the external disturbance $w(t)$, which belongs to $\mathscr{L}_2[0,\infty)$; that is, $\|G_{zw}\|_\infty = \sup_{\|w(t)\|_2 \neq 0}(\|z(t)\|_2/\|w(t)\|_2)$. Then, an equivalent definition of the $\mathscr{H}_\infty$-norm is

$$\|G_{zw}\|_\infty^2 = \sup_{w\neq 0}\frac{\int_0^\infty z^T(t)\,z(t)\,dt}{\int_0^\infty w^T(t)\,w(t)\,dt}, \qquad (6)$$

where it is assumed that $x(0)=0$. Therefore, $\|G_{zw}\|_\infty$ is the maximum possible gain in signal energy. This fact can be used to express constraints on the $\mathscr{H}_\infty$-norm in terms of LMIs. From the above it follows that $\|G_{zw}\|_\infty < \gamma$ is equivalent to

$$J = \int_0^\infty \left(z^T(t)\,z(t) - \gamma^2 w^T(t)\,w(t)\right)dt < 0. \qquad (7)$$

Lemma 4 (see [11]). *The following inequality holds for a given matrix $M > 0$ and all continuously differentiable functions x in $[a,b] \to \mathbb{R}^n$:*

$$\int_a^b x^T(s)\,Mx(s)\,ds \geq \frac{1}{b-a}\xi_1^T M\xi_1 + \frac{3}{b-a}\xi_2^T M\xi_2, \qquad (8)$$

where $\xi_1 = \int_a^b x(s)ds$ and $\xi_2 = \int_a^b x(s)ds - (2/(b-a))\int_a^b\int_a^s x(u)\,du\,ds$.

Lemma 5 (see [12]). *For any vectors x_1, x_2, constant matrices M, S, and real scalars $0<\alpha<1$ satisfying that $\left[\begin{smallmatrix} M & S \\ S^T & M\end{smallmatrix}\right] \geq 0$, the following inequality holds:*

$$\frac{1}{\alpha}x_1^T M x_1 + \frac{1}{1-\alpha}x_2^T M x_2 \geq \begin{bmatrix} x_1 \\ x_2\end{bmatrix}^T \begin{bmatrix} M & S \\ S^T & M\end{bmatrix}\begin{bmatrix} x_1 \\ x_2\end{bmatrix}. \qquad (9)$$

Lemma 6 (see [22]). *Let $\zeta \in \mathbb{R}^n$, $\Phi = \Phi^T \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{m\times n}$ such that $\mathrm{rank}(B) < n$. Then, the following statements are equivalent:*

(i) $\zeta^T\Phi\zeta < 0$, $B\zeta = 0$, $\zeta \neq 0$,

(ii) $B^{\perp T}\Phi B^{\perp} < 0$, *where $B^{\perp}$ is a right orthogonal complement of B.*

Lemma 7 (see [23]). *For any matrices $\Omega > 0$, Ξ, matrix Λ, the following statements are equivalent:*

(i) $\Xi - \Lambda^T\Omega\Lambda < 0$,

(ii) $\exists F : \left[\begin{smallmatrix} \Xi+\Lambda^T F + F^T\Lambda & F^T \\ F & -\Omega\end{smallmatrix}\right] < 0$.

3. Main Results

In this section, some new $\mathscr{H}_\infty$ performance and stability criteria for the system (4) will be derived. For convenience, the notations of several matrices are defined as follows:

$$\begin{aligned}
\zeta(t) = \mathtt{col}\Big\{ & x(t), x(t-h(t)), x(t-h_m), x(t-h_M), \\
& \dot{x}(t), \dot{x}(t-h_m), \\
& \dot{x}(t-h_M), \frac{1}{h_m}\int_{t-h_m}^{t} x(s)\,ds, \\
& \frac{1}{h(t)-h_m}\int_{t-h(t)}^{t-h_m} x(s)\,ds, \\
& \frac{1}{h_M-h(t)}\int_{t-h_M}^{t-h(t)} x(s)\,ds, \\
& \frac{1}{h_m}\int_{t-h_m}^{t}\int_{t-h_m}^{s} x(u)\,du\,ds, \\
& \frac{1}{h(t)-h_m}\int_{t-h(t)}^{t-h_m}\int_{t-h(t)}^{s} x(u)\,du\,ds, \\
& \frac{1}{h_M-h(t)}\int_{t-h_M}^{t-h(t)}\int_{t-h_M}^{s} x(u)\,du\,ds, p(t)\Big\},
\end{aligned}$$

$$\begin{aligned}
\Pi_{1[h(t)]} = {} & [e_1, e_3, e_4, e_8, e_9, e_{10}] \\
& \cdot \left[\begin{array}{c|c|c} I_{3n} & 0_{3n\cdot n} & 0_{4n\cdot n} \\ 0_{3n} & h_m I_n & (h(t)-h_m)\,I_n \\ & 0_{2n\cdot n} & (h_M-h(t))\,I_n \end{array}\right],
\end{aligned}$$

$$\mathscr{V}_{1[h(t)]} = \mathtt{sym}\left\{\Pi_{1[h(t)]}\mathscr{R}[e_5, e_6, e_7, e_1-e_3, e_3-e_4]^T\right\},$$

$$\mathscr{V}_2 = \begin{bmatrix} [e_1, e_5]^T \\ \hline [e_3, e_6]^T \\ \hline [e_4, e_7]^T \end{bmatrix}^T \cdot \mathtt{diag}\left\{\mathscr{N}_1, \mathscr{N}_2-\mathscr{N}_1, -\mathscr{N}_2\right\}\begin{bmatrix} [e_1, e_5]^T \\ \hline [e_3, e_6]^T \\ \hline [e_4, e_7]^T \end{bmatrix},$$

$$\mathscr{V}_{3[h(t)]} = \left[\frac{[e_3, e_0]^T}{[e_2, e_3 - e_2]^T}\right]^T \cdot \mathtt{diag}\{\mathscr{G}, -(1-d_M)\mathscr{G}\}\left[\frac{[e_3, e_0]^T}{[e_2, e_3 - e_2]^T}\right]^T + (h(t) - h_m)\,\mathtt{sym}\left\{[e_9, e_3 - e_9]\left[\frac{G_{12}}{G_{22}}\right]e_6^T\right\},$$

$$\mathscr{V}_{4,1} = [e_5, e_1]\,h_m^2\mathscr{Q}_1[e_5, e_1]^T - \left[\frac{[e_1 - e_3, h_m e_8]^T}{\left[\frac{(e_1 + e_3 - 2e_8)^T}{(h_m e_8 - 2e_{11})^T}\right]}\right]^T \cdot \mathtt{diag}\{\mathscr{Q}_1, 3\mathscr{Q}_1\}\left[\frac{[e_1 - e_3, h_m e_8]^T}{\left[\frac{(e_1 + e_3 - 2e_8)^T}{(h_m e_8 - 2e_{11})^T}\right]}\right],$$

$$\Theta_z = [e_3, e_2, e_4](h_M - h_m) \cdot \mathtt{diag}\{Z_1, Z_2 - Z_1, -Z_2\}[e_3, e_2, e_4]^T,$$

$$\mathscr{V}_{4,2} = [e_6, e_3](h_M - h_m)^2\mathscr{Q}_2[e_6, e_3]^T + (h_M - h_m)\Theta_z,$$

$$\mathscr{V}_4 = \mathscr{V}_{4,1} + \mathscr{V}_{4,2},$$

$$\mathbf{Q}_{2,1} = \mathscr{Q}_2 + \left[\begin{array}{c|c} 0_n & Z_1 \\ \hline Z_1 & 0_n \end{array}\right],$$

$$\mathbf{Q}_{2,2} = \mathscr{Q}_2 + \left[\begin{array}{c|c} 0_n & Z_2 \\ \hline Z_2 & 0_n \end{array}\right],$$

$$\Omega = \left[\begin{array}{c|c} \mathtt{diag}\{\mathbf{Q}_{2,1}, 3\mathbf{Q}_{2,1}\} & \mathscr{S} \\ \hline \mathscr{S}^T & \mathtt{diag}\{\mathbf{Q}_{2,2}, 3\mathbf{Q}_{2,2}\} \end{array}\right],$$

$$\Lambda_{[h(t)]} = \left[\begin{array}{c} [e_3 - e_2, (h(t) - h_m)e_9]^T \\ \hline [e_3 + e_2 - 2e_9, (h(t) - h_m)e_9 - 2e_{12}]^T \\ \hline [e_2 - e_4, (h_M - h(t))e_{10}]^T \\ \hline [e_2 + e_4 - 2e_{10}, (h_M - h(t)) * e_{10} - 2e_{13}]^T \end{array}\right],$$

$$\Delta_{[\rho_0,\sigma]} = \epsilon\left\{(\rho_0^2 + \sigma^2)\left([E_a, E_d][e_1, e_2]^T\right)^T \cdot [E_a, E_d][e_1, e_2]^T - e_{14}I_{n_f}e_{14}^T\right\},$$

$$\Psi = \left[\begin{array}{c|c} [e_1, e_2][C, C_d]^T[C, C_d][e_1, e_2]^T & [e_1, e_2][C, C_d]^T B_2 \\ \hline B_2^T[C, C_d][e_1, e_2]^T & B_2^T B_2 \end{array}\right],$$

$$\Upsilon = [A, A_d, -I_n, D][e_1, e_2, e_5, e_{14}]^T,$$

$$\mathscr{V}_{[h(t)]} = \mathscr{V}_{1[h(t)]} + \mathscr{V}_2 + \mathscr{V}_{3[h(t)]} + \mathscr{V}_4,$$

$$\Xi_{[h(t),\rho_0,\sigma]} = [\Upsilon, B_1]^{\perp T} \cdot \left(\mathtt{diag}\left\{\mathscr{V}_{[h(t)]} + \Delta_{[\rho_0,\sigma]}, 0_{n_w}\right\} + \Psi\right)[\Upsilon, B_1]^{\perp} + \mathtt{sym}\left\{\mathcal{F}^T\mathtt{diag}\left\{\Lambda_{[h(t)]}, I_{n_w}\right\}[\Upsilon, B_1]^{\perp}\right\}, \tag{10}$$

where $e_i \in \mathbb{R}^{(13n+n_f)\times n}$ $(i = 1, 2, \ldots, 14)$ are defined as block entry matrices, for example, $e_3^T\zeta(t) = x(t - h_m)$.

Then, the following theorem is given by the main result.

Theorem 8. *For given scalars $0 \le h_m < h_M$ and d_M, the system (4) is stochastically stable with $\mathscr{H}_\infty$ performance γ, stochastic parameters ρ_0 and σ, for $0 \le h_m \le h(t) \le h_M$ and $\dot{h}(t) \le d_M$, if there exist positive definite matrices $\mathscr{R} \in \mathbb{R}^{5n\times 5n}$, $\mathscr{N}_i \in \mathbb{R}^{2n\times 2n}$ $(i = 1, 2)$, $\mathscr{G} = [G_{ij}] \in \mathbb{R}^{2n\times 2n}$, and $\mathscr{Q}_i \in \mathbb{R}^{2n\times 2n}$ $(i = 1, 2)$ and a positive scalar ϵ, any symmetric matrices $Z_i \in \mathbb{R}^{n\times n}$ $(i = 1, 2)$, and any matrices $\mathscr{S} \in \mathbb{R}^{4n\times 4n}$ and $\mathcal{F} \in \mathbb{R}^{(8n+n_w)\times(12n+n_f+n_w)}$ satisfying the following LMIs:*

$$\left[\begin{array}{c|c} \Xi_{i[\rho_0,\sigma]} & \mathcal{F}^T \\ \hline \mathcal{F} & -\mathtt{diag}\left\{\Omega, \gamma^2 I_{n_w}\right\} \end{array}\right] < 0, \quad (i = 1, 2), \tag{11}$$

$$\mathtt{diag}\left\{\Omega, \gamma^2 I_{n_w}\right\} > 0, \tag{12}$$

$$\mathbf{Q}_{2,i} \ge 0, \quad (i = 1, 2), \tag{13}$$

where $\Xi_{i[\rho_0,\sigma]}$ are the two vertices of $\Xi_{[h(t),\rho_0,\sigma]}$ with the bounds of $h(t)$. That is, $h(t) = h_M$ when $i = 1$ and $h(t) = h_m$ when $i = 2$.

Proof. Let us consider the following Lyapunov-Krasovskii functional candidate as follows:

$$V(t) = V_1 + V_2 + V_3 + V_4, \tag{14}$$

where

$$\begin{aligned} V_1 &= \nu_1^T(t)\mathscr{R}\nu_1(t), \\ V_2 &= \int_{t-h_m}^{t} \nu_2^T(s)\mathscr{N}_1\nu_2(s)\,ds + \int_{t-h_M}^{t-h_m} \nu_2^T(s)\mathscr{N}_2\nu_2 x(s)\,ds, \\ V_3 &= \int_{t-h(t)}^{t-h_m} \nu_3^T(t,s)\mathscr{G}\nu_3(t,s)\,ds, \\ V_4 &= h_m\int_{t-h_m}^{t}\int_s^t \nu_4^T(u)\mathscr{Q}_1\nu_4(u)\,du\,ds + (h_M - h_m)\int_{t-h_M}^{t-h_m}\int_s^{t-h_m} \nu_4^T(u)\mathscr{Q}_2\nu_4(u)\,du\,ds \end{aligned} \tag{15}$$

with

$$\begin{aligned} \nu_1(t) &= \mathtt{col}\left\{x(t), x(t-h(t)), x(t-h_M), \int_{t-h_m}^{t} x(s)\,ds, \int_{t-h_M}^{t-h_m} x(s)\,ds\right\}, \\ \nu_2(s) &= \mathtt{col}\{x(s), \dot{x}(s)\}, \\ \nu_3(t,s) &= \mathtt{col}\left\{x(s), \int_s^{t-h_m}\dot{x}(u)\,du\right\}, \\ \nu_4(u) &= \mathtt{col}\{\dot{x}(u), x(u)\}. \end{aligned} \tag{16}$$

By infinitesimal operator $\mathbb{L}$ in [24], the $\mathbb{L}V_i$ $(i = 1, 2, 3)$ can be calculated as follows:

$$\mathbb{L}V_1 = 2\underbrace{\begin{bmatrix} x(t) \\ x(t-h_m) \\ x(t-h_M) \\ \frac{h_m}{h_m}\int_{t-h_m}^{t} x(s)\,ds \\ \left(\begin{array}{c}\frac{h(t)-h_m}{h(t)-h_m}\int_{t-h(t)}^{t-h_m} x(s)\,ds \\ +\frac{h_M-h(t)}{h_M-h(t)}\int_{t-h_M}^{t-h(t)} x(s)\,ds\end{array}\right)\end{bmatrix}^T}_{=\zeta^T(t)\Pi_{1[h(t)]}} \cdot \mathscr{R}\begin{bmatrix} \dot{x}(t) \\ \dot{x}(t-h_m) \\ \dot{x}(t-h_M) \\ x(t)-x(t-h_n) \\ x(t-h_m)-x(t-h_M)\end{bmatrix}$$

$$= \zeta^T(t)\mathscr{V}_{1[h(t)]}\zeta(t),$$

$$\begin{aligned}\mathbb{L}V_2 &= \nu_2^T(t)\mathscr{N}_1\nu_2(t) - \nu_2^T(t-h_m)\mathscr{N}_1\nu_2(t-h_m) + \nu_2^T(t-h_m)\mathscr{N}_2\nu_2 x(t-h_m) - \nu_2^T(t-h_M)\mathscr{N}_2\nu_2 x(t-h_M) \\ &= \zeta^T(t)\mathscr{V}_2\zeta(t),\end{aligned}$$

$$\begin{aligned}\mathbb{L}V_3 &= \nu_3^T(t, t-h_m)\mathscr{G}\nu_3(t, t-h_m) - \left(1-\dot{h}(t)\right)\nu_3^T(t, t-h(t))\mathscr{G}\nu_3(t, t-h(t)) + 2\int_{t-h(t)}^{t-h_m}\nu_3^T(t,s)\mathscr{G}\left(\frac{\partial}{\partial t}\nu_3(t,s)\right)ds \\ &\le \begin{bmatrix} x(t-h_m) \\ \int_s^{t-h_m}\dot{x}(s)\,ds|_{s=t-h_m}\end{bmatrix}^T \cdot \mathscr{G}\begin{bmatrix} x(t-h_m) \\ \int_s^{t-h_m}\dot{x}(s)\,ds|_{s=t-h_m}\end{bmatrix} \\ &\quad - (1-d_M)\begin{bmatrix} x(t-h(t)) \\ \int_s^{t-h_m}\dot{x}(s)\,ds|_{s=t-h(t)}\end{bmatrix}^T \cdot \mathscr{G}\begin{bmatrix} x(t-h(t)) \\ \int_s^{t-h_m}\dot{x}(s)\,ds|_{s=t-h(t)}\end{bmatrix} \\ &\quad + 2\int_{t-h(t)}^{t-h_m}\nu_3^T(t,s)\begin{bmatrix} G_{11} & G_{12} \\ G_{12}^T & G_{22}\end{bmatrix}\begin{bmatrix} 0_{n\cdot 1} \\ \dot{x}(t-h_m)\end{bmatrix}ds \\ &= \begin{bmatrix} x(t-h_m) \\ 0_{n\cdot 1}\end{bmatrix}^T \mathscr{G}\begin{bmatrix} x(t-h_m) \\ 0_{n\cdot 1}\end{bmatrix} - (1-d_M)\begin{bmatrix} x(t-h(t)) \\ x(t-h_m)-x(t-h(t))\end{bmatrix}^T \cdot \mathscr{G}\begin{bmatrix} x(t-h(t)) \\ x(t-h_m)-x(t-h(t))\end{bmatrix} \\ &\quad + 2\begin{bmatrix} \int_{t-h(t)}^{t-h_m} x(s)\,ds \\ \int_{t-h(t)}^{t-h_m}\int_s^{t-h_m}\dot{x}(u)\,du\,ds\end{bmatrix}^T \begin{bmatrix} G_{12} \\ G_{22}\end{bmatrix}\dot{x}(t-h_m) \\ &= \begin{bmatrix} x(t-h_m) \\ 0_{n\cdot 1}\end{bmatrix}^T \mathscr{G}\begin{bmatrix} x(t-h_m) \\ 0_{n\cdot 1}\end{bmatrix} - (1-d_M)\begin{bmatrix} x(t-h(t)) \\ x(t-h_m)-x(t-h(t))\end{bmatrix}^T \cdot \mathscr{G}\begin{bmatrix} x(t-h(t)) \\ x(t-h_m)-x(t-h(t))\end{bmatrix} + 2(h(t)-h_m) \\ &\quad \cdot \begin{bmatrix} \frac{1}{h(t)-h_m}\int_{t-h(t)}^{t-h_m} x(s)\,ds \\ x(t-h_m) - \frac{1}{h(t)-h_m}\int_{t-h(t)}^{t-h_m} x(s)\,ds\end{bmatrix}^T \cdot \begin{bmatrix} G_{12} \\ G_{22}\end{bmatrix}\dot{x}(t-h_m) = \zeta^T(t)\mathscr{V}_{3[h(t)]}\zeta(t).\end{aligned} \tag{17}$$

Prior to obtaining the bound of $\mathbb{L}V_4$, the V_4 is divided into the following two parts:

$$\begin{aligned} V_{4,1} &= h_m\int_{t-h_m}^{t}\int_s^t \nu_4^T(u)\mathscr{Q}_1\nu_4(u)\,du\,ds. \\ V_{4,2} &= (h_M-h_m)\int_{t-h_M}^{t-h_m}\int_s^{t-h_m}\nu_4^T(u)\mathscr{Q}_2\nu_4(u)\,du\,ds.\end{aligned} \tag{18}$$

Inspired by the work of [4], the following zero equalities with any symmetric matrices Z_1 and Z_2 are considered as a tool of reducing the conservatism of criterion:

$$\begin{aligned} 0 &= x^T(t-h_m)Z_1x(t-h_m) - x^T(t-h(t))Z_1x(t-h(t)) - 2\int_{t-h(t)}^{t-h_m} x^T(s)Z_1\dot{x}(s)\,ds \\ &\quad + x^T(t-h(t))Z_2x(t-h(t)) - x^T(t-h_M)Z_2x(t-h_M) - 2\int_{t-h_M}^{t-h(t)} x^T(s)Z_2\dot{x}(s)\,ds \\ &= \zeta^T(t)\Theta_z\zeta(t) - \xi_1(t),\end{aligned} \tag{19}$$

where

$$\xi_1(t) = 2\int_{t-h(t)}^{t-h_m} x^T(s) Z_1 \dot{x}(s)\,ds + 2\int_{t-h_M}^{t-h(t)} x^T(s) Z_2 \dot{x}(s)\,ds. \tag{20}$$

By utilizing Lemma 4, calculating the $\mathbb{L}V_{4,1}$ and $\mathbb{L}V_{4,2}$, and adding (19) into the $\mathbb{L}V_{4,2}$, the following relations can be obtained as follows:

$$\begin{aligned}
\mathbb{L}V_{4,1} &= h_m^2 \nu_4^T(t)\mathcal{Q}_1\nu_4(t) - h_m\int_{t-h_m}^{t}\nu_4^T(s)\mathcal{Q}_1\nu_4(s)\,ds \\
&\le h_m^2\nu_4^T(t)\mathcal{Q}_1\nu_4(t) - \left(\int_{t-h_m}^{t}\nu_4(s)\,ds\right)^T\mathcal{Q}_1\left(\int_{t-h_m}^{t}\nu_4(s)\,ds\right) \\
&\quad - 3\left(\int_{t-h_m}^{t}\nu_4(s)\,ds - \frac{2}{h_m}\int_{t-h_m}^{t}\int_{t-h_m}^{s}\nu_4(u)\,du\,ds\right)^T\mathcal{Q}_1 \\
&\quad \cdot\left(\int_{t-h_m}^{t}\nu_4(s)\,ds - \frac{2}{h_m}\int_{t-h_m}^{t}\int_{t-h_m}^{s}\nu_4(u)\,du\,ds\right) \\
&= h_m^2\nu_4^T(t)\mathcal{Q}_1\nu_4(t) - \begin{bmatrix} x(t)-x(t-h_m) \\ \int_{t-h_m}^{t}x(s)\,ds\end{bmatrix}^T\mathcal{Q}_1\begin{bmatrix} x(t)-x(t-h_m) \\ \int_{t-h_m}^{t}x(s)\,ds\end{bmatrix} \\
&\quad - 3\begin{bmatrix} x(t)+x(t-h_m)-\frac{2}{h_m}\int_{t-h_m}^{t}x(s)\,ds \\ \int_{t-h_m}^{t}x(s)\,ds - \frac{2}{h_m}\int_{t-h_m}^{t}\int_{t-h_m}^{s}x(u)\,du\,ds\end{bmatrix}^T\mathcal{Q}_1 \\
&\quad \cdot\begin{bmatrix} x(t)+x(t-h_m)-\frac{2}{h_m}\int_{t-h_m}^{t}x(s)\,ds \\ \int_{t-h_m}^{t}x(s)\,ds - \frac{2}{h_m}\int_{t-h_m}^{t}\int_{t-h_m}^{s}x(u)\,du\,ds\end{bmatrix} \\
&= \zeta^T(t)\mathscr{V}_{4,1}\zeta(t),
\end{aligned}$$

$$\begin{aligned}
&\mathbb{L}V_{4,2} + (h_M-h_m)\underbrace{\left\{\zeta^T(t)\Theta_z\zeta(t) - \xi_1(t)\right\}}_{=0} \\
&\quad = (h_M-h_m)^2\nu_4^T(t-h_m)\mathcal{Q}_2\nu_4(t-h_m) + (h_M-h_m)\zeta^T(t)\Theta_z\zeta(t) - (h_M-h_m) \\
&\qquad \cdot\left\{\int_{t-h(t)}^{t-h_m}\nu_4^T(s)\mathcal{Q}_2\nu_4(s)\,ds - 2\int_{t-h(t)}^{t-h_m}x^T(s)Z_1\dot{x}(s)\,ds\right\} \\
&\qquad - (h_M-h_m)\left\{\int_{t-h_M}^{t-h(t)}\nu_4^T(s)\mathcal{Q}_2\nu_4(s)\,ds - 2\int_{t-h_M}^{t-h(t)}x^T(s)Z_2\dot{x}(s)\,ds\right\} \\
&\quad = \zeta^T(t)\mathscr{V}_{4,2}\zeta(t) - \xi_2(t),
\end{aligned} \tag{21}$$

where

$$\xi_2(t) = (h_M-h_m)\int_{t-h(t)}^{t-h_m}\nu_4^T(s)\mathbf{Q}_{2,1}\nu_4(s)\,ds + (h_M-h_m)\int_{t-h_M}^{t-h(t)}\nu_4^T(s)\mathbf{Q}_{2,2}\nu_4(s)\,ds. \tag{22}$$

By Lemmas 4 and 5, the integral terms, $\xi_2(t)$, of the $\mathbb{L}V_{4,2}$ are bounded as follows:

$$\begin{aligned}
\xi_2(t) &\ge \frac{h_M-h_m}{h(t)-h_m}\varpi_{1,1}^T(t)\mathbf{Q}_{2,1}\varpi_{1,1}(t) + 3\frac{h_M-h_m}{h(t)-h_m}\varpi_{1,2}^T(t)\mathbf{Q}_{2,1}\varpi_{1,2}(t) \\
&\quad + \frac{h_M-h_m}{h_M-h(t)}\varpi_{2,1}^T(t)\mathbf{Q}_{2,2}\varpi_{2,1}(t) + 3\frac{h_M-h_m}{h_M-h(t)}\varpi_{2,2}^T(t)\mathbf{Q}_{2,2}\varpi_{2,2}(t) \\
&= \frac{1}{\alpha(t)}\begin{bmatrix}\varpi_{1,1}(t)\\ \varpi_{1,2}(t)\end{bmatrix}^T \mathtt{diag}\{\mathbf{Q}_{2,1}, 3\mathbf{Q}_{2,1}\} \cdot\begin{bmatrix}\varpi_{1,1}(t)\\ \varpi_{1,2}(t)\end{bmatrix} + \frac{1}{1-\alpha(t)}\begin{bmatrix}\varpi_{2,1}(t)\\ \varpi_{2,2}(t)\end{bmatrix}^T \\
&\quad \cdot\mathtt{diag}\{\mathbf{Q}_{2,2}, 3\mathbf{Q}_{2,2}\}\begin{bmatrix}\varpi_{2,1}(t)\\ \varpi_{2,2}(t)\end{bmatrix} \\
&\ge \begin{bmatrix}\varpi_{1,1}(t)\\ \varpi_{1,2}(t)\end{bmatrix}^T \mathtt{diag}\{\mathbf{Q}_{2,1}, 3\mathbf{Q}_{2,1}\} \cdot\begin{bmatrix}\varpi_{1,1}(t)\\ \varpi_{1,2}(t)\end{bmatrix} + \begin{bmatrix}\varpi_{1,1}(t)\\ \varpi_{1,2}(t)\end{bmatrix}^T\mathcal{S}\begin{bmatrix}\varpi_{2,1}(t)\\ \varpi_{2,2}(t)\end{bmatrix} \\
&\quad + \begin{bmatrix}\varpi_{2,1}(t)\\ \varpi_{2,2}(t)\end{bmatrix}^T\mathcal{S}^T\begin{bmatrix}\varpi_{1,1}(t)\\ \varpi_{1,2}(t)\end{bmatrix} + \begin{bmatrix}\varpi_{2,1}(t)\\ \varpi_{2,2}(t)\end{bmatrix}^T\mathtt{diag}\{\mathbf{Q}_{2,2}, 3\mathbf{Q}_{2,2}\} \cdot\begin{bmatrix}\varpi_{2,1}(t)\\ \varpi_{2,2}(t)\end{bmatrix} \\
&= \zeta^T(t)\Lambda_{[h(t)]}^T\Omega\Lambda_{[h(t)]}\zeta(t),
\end{aligned} \tag{23}$$

where $1/\alpha(t) = (h_M - h_m) / (h(t) - h_m)$,

$$\varpi_{1,1}(t) = \int_{t-h(t)}^{t-h_m} \nu_4(s)\, ds = \begin{bmatrix} x(t-h_m) - x(t-h(t)) \\ \int_{t-h(t)}^{t-h_m} x(s)\, ds \end{bmatrix} = [e_3 - e_2, (h(t)-h_m) e_9]^T \zeta(t),$$

$$\varpi_{1,2}(t) = \int_{t-h(t)}^{t-h_m} \nu_4(s)\, ds - \frac{2}{h(t)-h_m} \cdot \int_{t-h(t)}^{t-h_m} \int_{t-h(t)}^{s} \nu_4(u)\, du\, ds = \begin{bmatrix} x(t-h_m) + x(t-h(t)) - \frac{2}{h(t)-h_m} \int_{t-h(t)}^{t-h_m} x(s)\, ds \\ \int_{t-h(t)}^{t-h_m} x(s)\, ds - \frac{2}{h(t)-h_m} \int_{t-h(t)}^{t-h_m} \int_{t-h(t)}^{s} x(u)\, du\, ds \end{bmatrix} = [e_3 + e_2 - 2e_9, (h(t)-h_m) e_9 - 2e_{12}]\, \zeta(t),$$

$$\varpi_{2,1}(t) = \int_{t-h_M}^{t-h(t)} \nu_4(s)\, ds = \begin{bmatrix} x(t-h(t)) - x(t-h_M) \\ \int_{t-h_M}^{t-h(t)} x(s)\, ds \end{bmatrix} = [e_2 - e_4, (h_M - h(t)) e_{10}]\, \zeta(t),$$

$$\varpi_{2,2}(t) = \int_{t-h_M}^{t-h(t)} \nu_4(s)\, ds - \frac{2}{h_M - h(t)} \cdot \int_{t-h_M}^{t-h(t)} \int_{t-h_M}^{s} \nu_4(u)\, du\, ds = \begin{bmatrix} x(t-h(t)) + x(t-h_M) - \frac{2}{h_M - h(t)} \int_{t-h_M}^{t-h(t)} x(s)\, ds \\ \int_{t-h(t)}^{t-h_m} x(s)\, ds - \frac{2}{h_M - h(t)} \int_{t-h_M}^{t-h(t)} \int_{t-h_M}^{s} x(u)\, du\, ds \end{bmatrix} = [e_2 + e_4 - 2e_{10}, (h_M - h(t)) e_{10} - 2e_{13}]\, \zeta(t), \tag{24}$$

and $\Lambda_{[h(t)]}\zeta(t) = \mathtt{col}\{\varpi_{1,1}(t), \varpi_{1,2}(t), \varpi_{2,1}(t), \varpi_{2,2}(t)\}$.

Hence,

$$\mathbb{L}V_4 \le \zeta^T(t) \mathscr{V}_4 \zeta(t) - \zeta^T(t) \Lambda_{[h(t)]}^T \Omega \Lambda_{[h(t)]} \zeta(t). \tag{25}$$

Here, when $\mathbf{Q}_{2,i} \ge 0$ $(i = 1, 2)$ hold, the bound of $\mathbb{L}V_4$ is valid.

In succession, with the relational expression between $p(t)$ and $q(t)$, $p^T(t)p(t) \le q^T(t)q(t)$, from the system (4), there exists any scalar $\epsilon > 0$ satisfying the following inequality

$$\begin{aligned} 0 &\le \mathbb{E}\left\{\epsilon \left[q^T(t) q(t) - p^T(t) p(t)\right]\right\} \\ &= \mathbb{E}\left\{\epsilon \left[\rho(t) \underbrace{\left(E_a x(t) + E_d x(t-h(t))\right)^T}_{=\varphi(t)} \cdot \rho(t) \varphi(t)\right] - \epsilon p^T(t) p(t)\right\} \\ &= \mathbb{E}\left\{\epsilon \left[\left(\rho_0 + (\rho(t) - \rho_0)\right) \varphi^T(t) \cdot \left(\rho_0 + (\rho(t) - \rho_0)\right) \varphi(t)\right] - \epsilon p^T(t) p(t)\right\} \\ &= \zeta^T(t) \Delta_{[\rho_0,\sigma]} \zeta(t), \end{aligned} \tag{26}$$

where $\phi(t) = \rho_0^2 + 2\rho_0(\rho(t) - \rho_0) + (\rho(t) - \rho_0)^2$ and its mathematical expectation is as follows:

$$\mathbb{E}\{\phi(t)\} = \mathbb{E}\left\{\rho_0^2\right\} + \mathbb{E}\left\{(\rho(t) - \rho_0)^2\right\} = \rho_0^2 + \sigma^2. \tag{27}$$

From (17) to (26), the $\mathbb{L}V(t)$ has a new upper bound as follows:

$$\mathbb{L}V(t) \le \zeta^T(t) \Big(\mathscr{V}_{1[h(t)]} + \mathscr{V}_2 + \mathscr{V}_{3[h(t)]} + \mathscr{V}_4 - \Lambda_{[h(t)]}^T \Omega \Lambda_{[h(t)]} + \Delta_{[\rho_0,\sigma]}\Big) \zeta(t). \tag{28}$$

From Definition 3, with the zero initial condition, we rewrite J as follows:

$$J = \mathbb{E}\left\{\int_{t=0}^{\infty} \left(z^T(t) z(t) - \gamma^2 w^T(t) w(t) + \mathbb{L}V(t) - \mathbb{L}V(t)\right) dt\right\}. \tag{29}$$

Here, we get

$$\gamma^2 w^T(t) w(t) = \begin{bmatrix} \zeta(t) \\ w(t) \end{bmatrix}^T \mathtt{diag}\left\{0_{13n+n_f}, \gamma^2 I_{n_w}\right\} \cdot \begin{bmatrix} \zeta(t) \\ w(t) \end{bmatrix},$$

$$\begin{aligned} z^T(t) z(t) &= \left(Cx(t) + C_d x(t-h(t)) + B_2 w(t)\right)^T \cdot \left(Cx(t) + C_d x(t-h(t)) + B_2 w(t)\right) \\ &= \left([[C, C_d], B_2] \begin{bmatrix} [e_1, e_2]^T \zeta(t) \\ w(t) \end{bmatrix}\right)^T \cdot \left([[C, C_d], B_2] \begin{bmatrix} [e_1, e_2]^T \zeta(t) \\ w(t) \end{bmatrix}\right) \\ &= \begin{bmatrix} [e_1, e_2]^T \zeta(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} [C, C_d]^T \\ B_2^T \end{bmatrix} \cdot [[C, C_d], B_2] \begin{bmatrix} [e_1, e_2]^T \zeta(t) \\ w(t) \end{bmatrix} \\ &= \begin{bmatrix} \zeta(t) \\ w(t) \end{bmatrix}^T \cdot \left[\begin{array}{c|c} \$_1 & [e_1, e_2] [C, C_d]^T B_2 \\ \hline B_2^T [C, C_d] [e_1, e_2]^T & B_2^T B_2 \end{array}\right] \cdot \begin{bmatrix} \zeta(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} \zeta(t) \\ w(t) \end{bmatrix}^T \Psi \begin{bmatrix} \zeta(t) \\ w(t) \end{bmatrix}, \end{aligned} \tag{30}$$

where $\$_1 = [e_1, e_2] [C, C_d]^T [C, C_d] [e_1, e_2]^T$.

From (29) with (30), considering $V(t)|_{t=0} = 0$ and $V(t)|_{t=\infty} \to 0$, the J is bounded as follows:

$$J \le \mathbb{E}\left\{\int_{t=0}^{\infty}\left(z^T(t)z(t) - \gamma^2 w^T(t)w(t) + \mathbb{L}V(t)\right)dt\right\}$$
$$= \int_{t=0}^{\infty}\left\{\begin{bmatrix}\zeta(t)\\ w(t)\end{bmatrix}^T\left(\Psi - \mathtt{diag}\left\{0_{13n+n_f}, \gamma^2 I_{n_w}\right\}\right)\cdot\begin{bmatrix}\zeta(t)\\ w(t)\end{bmatrix} + \begin{bmatrix}\zeta(t)\\ w(t)\end{bmatrix}^T\cdot\mathtt{diag}\left\{\mathscr{V}_{[h(t)]} - \Lambda^T_{[h(t)]}\Omega\Lambda_{[h(t)]} + \Delta_{[\rho_0,\sigma]}, 0_{n_w}\right\}\cdot\begin{bmatrix}\zeta(t)\\ w(t)\end{bmatrix}\right\}dt = J^*. \quad (31)$$

Thus, the following inequality

$$\begin{bmatrix}\zeta(t)\\ w(t)\end{bmatrix}^T\cdot\left(\Psi + \mathtt{diag}\left\{\mathscr{V}_{[h(t)]} - \Lambda^T_{[h(t)]}\Omega\Lambda_{[h(t)]} + \Delta_{[\rho_0,\sigma]}, -\gamma^2 I_{n_w}\right\}\right)\cdot\begin{bmatrix}\zeta(t)\\ w(t)\end{bmatrix} < 0 \quad (32)$$

is equivalent to the $J \le J^* < 0$. Therefore, if the condition (32) then system (4) is asymptotically stable with $\mathscr{H}_\infty$ performance γ.

Applying Lemmas 6 and 7 to (32) with

$$0 = \mathbb{E}\left\{\Upsilon\zeta(t) + B_1 w(t)\right\} = [\Upsilon, B_1]\begin{bmatrix}\zeta(t)\\ w(t)\end{bmatrix} \quad (33)$$

leads to

$$[\Upsilon, B_1]^{\perp T}\cdot\left(\Psi + \mathtt{diag}\left\{\mathscr{V}_{[h(t)]} - \Lambda^T_{[h(t)]}\Omega\Lambda_{[h(t)]} + \Delta_{[\rho_0,\sigma]}, -\gamma^2 I_{n_w}\right\}\right)\cdot[\Upsilon, B_1]^{\perp} = [\Upsilon, B_1]^{\perp T}\cdot\left(\Psi + \mathtt{diag}\left\{\mathscr{V}_{[h(t)]} + \Delta_{[\rho_0,\sigma]}, 0_{n_w}\right\}\right)[\Upsilon, B_1]^{\perp} - [\Upsilon, B_1]^{\perp T}\mathtt{diag}\left\{\Lambda^T_{[h(t)]}\Omega\Lambda_{[h(t)]}, \gamma^2 I_{n_w}\right\}\cdot[\Upsilon, B_1]^{\perp} < 0$$
$$\Longleftrightarrow \left[\begin{array}{c|c}\$_2 & \mathcal{F}^T\\ \hline \mathcal{F} & -\mathtt{diag}\left\{\Omega, \gamma^2 I_{n_w}\right\}\end{array}\right] < 0 \quad (34)$$

for any matrix $\mathcal{F} \in \mathbb{R}^{(8n+n_w)\times(12n+n_f+n_w)}$, where $\$_2 = [\Upsilon, B_1]^{\perp T}(\Psi + \mathtt{diag}\{\mathscr{V}_{[h(t)]} + \Delta_{[\rho_0,\sigma]}, 0_{n_w}\})[\Upsilon, B_1]^{\perp} + \mathtt{sym}\{\mathcal{F}^T\mathtt{diag}\{\Lambda_{[h(t)]}, I_{n_w}\}[\Upsilon, B_1]^{\perp}\}$.

The above condition is affinely dependent on $h(t)$. Therefore, if LMIs (11) hold, then system (4) is stochastically stable with $\mathscr{H}_\infty$ performance γ and stochastic indexes ρ_0 and σ^2 for $0 \le h_m \le h(t) \le h_M$ and $\dot{h}(t) \le d_M$. It should be noted that the inequality (12) is satisfied if the inequalities (11) hold. This completes our proof. □

Remark 9. To achieve the less conservatism of stability condition, Wirtinger-based inequality with the basic Lyapunov-Krasovskii functional was introduced in [11]. However, a newly Lyapunov-Krasovskii functional was not proposed. In view of this, the main contribution in this work is the use of V_3 included in a new Lyapunov-Krasovskii functional (14). As a result, some cross terms such as $2(h(t) - h_m)\begin{bmatrix}(1/(h(t)-h_m))\int_{t-h(t)}^{t-h_m}x(s)ds\\ x(t-h_m)-((1/(h(t)-h_m))\int_{t-h(t)}^{t-h_m}x(s)ds)\end{bmatrix}^T\begin{bmatrix}G_{12}\\ G_{22}\end{bmatrix}\dot{x}(t - h_m)$ and $-(1 - d_M)\begin{bmatrix}x(t-h(t))\\ x(t-h_m)-x(t-h(t))\end{bmatrix}^T\mathscr{G}\begin{bmatrix}x(t-h(t))\\ x(t-h_m)-x(t-h(t))\end{bmatrix}$ are utilized in estimating the $\mathbb{L}V$.

Remark 10. In deriving lower bounds of $h_m\int_{t-h_m}^{t}\nu^T(s)\mathscr{Q}_1\nu(s)ds$ and $\xi_2(t)$ obtained by calculating the time-derivative values of V_4, Lemma 4 which is the remarkable result in reducing the conservatism of delay-dependent stability criteria is utilized. However, unlike the results in [11], the utilized vectors of the two quadratic integral terms $h_m\int_{t-h_m}^{t}\nu^T(s)\mathscr{Q}_1\nu(s)ds$ and $\xi_2(t)$ are $[\dot{x}^T(s), x^T(s)]^T$. As a result, some new integral terms such as $(1/h_m)\int_{t-h_m}^{t}\int_{t-h_m}^{s}x(u)\,du\,ds$, $(1/(h(t) - h_m))\int_{t-h(t)}^{t-h_m}\int_{t-h(t)}^{s}x(u)\,du\,ds$, and $(1/(h_M - h(t)))\int_{t-h_M}^{t-h(t)}\int_{t-h_M}^{s}x(u)\,du\,ds$ are utilized as elements of the augmented vector $\zeta(t)$, which is different from the works [11].

In case of the deterministic uncertainties, the following theorem can be obtained.

Theorem 11. *For given scalars $0 \le h_m < h_M$, d_M, the system (4) is asymptotically stable with $\mathscr{H}_\infty$ performance γ, for $0 \le h_m \le h(t) \le h_M$ and $\dot{h}(t) \le d_M$, if there exist positive definite matrices $\mathscr{R} \in \mathbb{R}^{5n\times5n}$, $\mathscr{N}_i \in \mathbb{R}^{2n\times2n}$ $(i = 1, 2)$, $\mathscr{G} = [G_{ij}] \in \mathbb{R}^{2n\times2n}$, and $\mathscr{Q}_i \in \mathbb{R}^{2n\times2n}$ $(i = 1, 2)$, a positive scalar ϵ, any symmetric matrices $Z_i \in \mathbb{R}^{n\times n}$ $(i = 1, 2)$, any matrices $\mathscr{S} \in \mathbb{R}^{4n\times4n}$ and $\mathcal{F} \in \mathbb{R}^{(8n+n_w)\times(12n+n_f+n_w)}$ satisfying the LMIs (13) and*

$$\left[\begin{array}{c|c}\Xi_{i[\rho_0=1,\sigma=0]} & \mathcal{F}^T\\ \hline \mathcal{F} & -\mathtt{diag}\left\{\Omega, \gamma^2 I_{n_w}\right\}\end{array}\right] < 0, \quad (i = 1, 2), \quad (35)$$

$$\mathtt{diag}\left\{\Omega, \gamma^2 I_{n_w}\right\} > 0, \quad (36)$$

where all notations were defined in (10).

Proof. When the mean, ρ_0, and the variance, σ^2, of $\rho(t)$ are, respectively, 1 and 0, it means the uncertainties are deterministic. Therefore, by setting $\rho_0 = 1$ and $\sigma = 0$ in the

TABLE 1: MADBs with fixed unknown d_M, fixed $h_m = 0$ and $\gamma = 1$ (Example 1).

ρ_0	σ^2								
	0.2	0.6	0.8	1.2	1.6	2.0	2.4	2.8	3.2
0.0	1.64	1.30	1.08	0.91	0.77	0.66	0.53	0.31	0
0.2	1.60	1.28	1.06	0.89	0.76	0.65	0.51	0.23	0
0.4	1.48	1.21	1.00	0.85	0.73	0.61	0.47	0	0
0.8	1.33	1.10	0.92	0.79	0.67	0.54	0.35	0	0
1.0	1.16	0.97	0.82	0.70	0.58	0.43	0	0	0
1.2	0.99	0.84	0.71	0.60	0.45	0	0	0	0
1.4	0.82	0.70	0.58	0.43	0	0	0	0	0
1.6	0.67	0.54	0.35	0	0	0	0	0	0
1.8	0.47	0	0	0	0	0	0	0	0
2.0	0	0	0	0	0	0	0	0	0

0: infeasible.

result of Theorem 8, LMIs (35) can be easily obtained. So, it is omitted. □

As a special case of Theorem 11, when the system (1) is the nominal form without parameter uncertainties and disturbances given by

$$\dot{x}(t) = Ax(t) + A_d x(t - h(t)), \quad (37)$$

then, based on same Lyapunov-Krasovskii functional candidate in (14), the following corollary can be obtained.

Corollary 12. *For given scalars $0 \le h_m < h_M$, d_M, the system (37) is asymptotically stable, for $0 \le h_m \le h(t) \le h_M$ and $\dot{h}(t) \le d_M$, if there exist positive definite matrices $\mathcal{R} \in \mathbb{R}^{5n\times5n}$, $\mathcal{N}_i \in \mathbb{R}^{2n\times2n}$ $(i = 1, 2)$, $\mathcal{G} = [G_{ij}] \in \mathbb{R}^{2n\times2n}$, and $\mathcal{Q}_i \in \mathbb{R}^{2n\times2n}$ $(i = 1, 2)$, any symmetric matrices $Z_i \in \mathbb{R}^{n\times n}$ $(i = 1, 2)$, and any matrices $\mathcal{S} \in \mathbb{R}^{4n\times4n}$ and $\widehat{\mathcal{F}} \in \mathbb{R}^{8n\times12n}$ satisfying the LMIs (13) and*

$$\left[\begin{array}{c|c} \widehat{\Xi}_i & \widehat{\mathcal{F}}^T \\ \hline \widehat{\mathcal{F}} & -\Omega \end{array}\right] < 0, \quad (i = 1, 2), \qquad \Omega > 0, \quad (38)$$

where $\widehat{\Xi}_i$ is the two vertices of $\widehat{\Xi}_{[h(t)]} = \widehat{\Upsilon}^{\perp^T}\mathcal{V}_{[h(t)]}\widehat{\Upsilon}^{\perp} + \mathtt{sym}\{\widehat{\mathcal{F}}^T\Lambda_{[h(t)]}\Upsilon^{\perp}\}$ with $\widehat{\Upsilon} = [A, A_d, -I_n][e_1, e_2, e_5]^T$ and other notations were defined in (10).

Proof. Upper bound of time-derivative of (14) can be calculated as follows:

$$\dot{V}(t) \le \widehat{\zeta}^T(t)\mathcal{V}_{[h(t)]}\widehat{\zeta}(t), \quad (39)$$

where $\mathcal{V}_{[h(t)]}$ was defined in (10) and

$$\begin{aligned}\widehat{\zeta}(t) = \mathtt{col}\Big\{& x(t), x(t-h(t)), x(t-h_m), x(t-h_M), \\ & \dot{x}(t), \dot{x}(t-h_m), \dot{x}(t-h_M), \\ & \frac{1}{h_m}\int_{t-h_m}^{t} x(s)\,ds, \\ & \frac{1}{h(t)-h_m}\int_{t-h(t)}^{t-h_m} x(s)\,ds, \\ & \frac{1}{h_M-h(t)}\int_{t-h_M}^{t-h(t)} x(s)\,ds, \\ & \frac{1}{h_m}\int_{t-h_m}^{t}\int_{t-h_m}^{s} x(u)\,du\,ds, \\ & \frac{1}{h(t)-h_m}\int_{t-h(t)}^{t-h_m}\int_{t-h(t)}^{s} x(u)\,du\,ds, \\ & \frac{1}{h_M-h(t)}\int_{t-h_M}^{t-h(t)}\int_{t-h_M}^{s} x(u)\,du\,ds\Big\}\end{aligned} \quad (40)$$

with replacing the block entry matrices to $e_i \in \mathbb{R}^{13n\times n}$ $(i = 1, \dots, 13)$, which is very similar to the proofs of Theorems 8 and 11, so it is omitted. □

Remark 13. When the information of $\dot{h}(t)$ is unknown, the corresponding results of Theorems 8, 11 and Corollary 12 can be obtained by choosing $\mathcal{G} = 0$, respectively.

4. Illustrative Examples

Example 1. Consider the system (1) with

$$\begin{aligned} A &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.01 \\ 0.05 \end{bmatrix}, \\ C &= [0.1, 0.2], \\ C_D &= 0_{1\cdot2}, \\ B_2 &= 0, \\ D &= I_2, \\ E_a &= \mathtt{diag}\{1.6, 0.05\}, \\ E_d &= \mathtt{diag}\{0.1, 0.3\}. \end{aligned} \quad (41)$$

For the above system, the maximum allowable delay bounds (MADBs) with various ρ_0 and σ^2, fixed $h_m = 0$ and $\gamma = 1$, and unknown d_M are listed in Table 1. When the stochastic indexes (the mean ρ_0 and the variance σ^2)

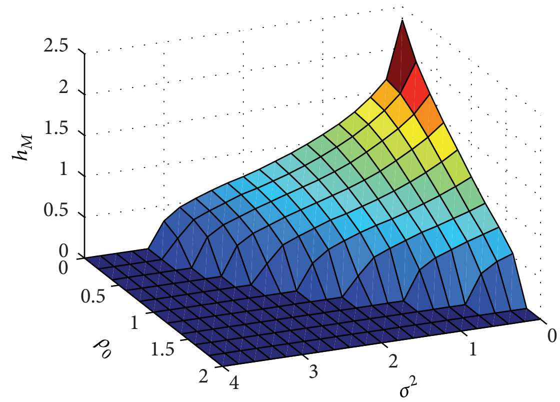

Figure 1: Effect of ρ_0 and σ^2 (Example 1).

Table 2: MADBs with fixed unknown d_M and $\gamma = 1$ (Example 2).

h_m	0.0	0.2	0.4	0.6	0.8	1.0
Yue et al. [5]	0.6695	0.7343	0.8118	0.8962	0.9852	1.0784
Jeong et al. [6]	1.0400	1.0404	1.0411	1.0426	1.0508	1.0794
Kwon et al. [10]	1.0736	1.0995	1.0955	1.0986	1.1254	1.1740
Theorem 11	1.0833	1.1019	1.1042	1.1198	1.1625	1.2296

Table 3: Minimized $\mathscr{H}_\infty$ performance γ with various ordered pair (h_m, h_M) and unknown d_M (Example 3).

(h_m, h_M)	(0, 0.8695)	(0.5695, 0.8695)	(0, 1)	(0.9, 1)
Yue et al. [5]	6.82	1.26	—	3.98
Jeong et al. [6]	0.87	0.81	4.05	2.59
Kwon et al. [10]	0.7810	0.7348	2.6153	1.8788
Theorem 11	0.7019	0.6718	1.9819	1.3521

increase, the MADBs become smaller, which means that Theorem 8 becomes more conservative. In order to verify this, the MADBs with ranges $\rho_0 = \{0, 0.2, \ldots, 2\}$ and $\sigma^2 = \{0, 0.2, \ldots, 4\}$ are shown in Figure 1. This figure demonstrates that a larger ρ_0 or σ^2 will lead to a smaller h_M. Then, from Table 1 and Figure 1, it can be seen that the mean ρ_0 and the variance σ^2 can be addressed in the parameter uncertainties since the MADBs for guaranteeing the $\mathscr{H}_\infty$ performance are influenced by the stochastic indexes. Moreover, Figures 2 and 3 are drawn to show the state trajectories with ρ_0 and σ^2. At this time, the initial condition $x(0) = [0, 0]^T$ and the disturbance $w(t)$ is 1 if $3 \le t \le 5$ and 0, otherwise, and the time-delays are $(0.67/2)\sin((2/0.67)\,t) + (0.67/2)$ and $(0.43/2)\sin((2/0.43)\,t) + (0.43/2)$, respectively, in Figures 2 and 3. Also, in order to verify the stochastic indexes, $F(t)$ is set as I_2.

These figures give the relations between state trajectories and σ^2 for the fixed $\rho_0 = 1.0$ and the relations between the state trajectories and ρ_0 for the fixed $\sigma^2 = 0.2$. Also, these figures show that a lager ρ_0 or σ^2 will lead to the poor performance of system.

Here, one of significant points is that the effect of the mean and the variance on system performance is different. From Table 1, it can be seen that the growth of stochastic indexes leads to conservatism, whereas, from Figures 2 and 3, one can confirm the following two facts: (i) the mean ρ_0 deteriorates the dynamic behavior of systems (see Figure 2) and (ii) the variance σ^2 influences the system performance (see Figure 3).

Example 2. Consider the system (1) with (41). For the above system, the results of MADBs with various h_m, fixed unknown d_M, and $\gamma = 1$ are listed in Table 2. By applying Theorem 11, it can be guaranteed that the MADBs under the same conditions are larger than the ones in the existing works which supports the fact that the proposed Lyapunov-Krasovskii functional and some utilized techniques effectively reduce the conservatism in $\mathscr{H}_\infty$ performance.

Example 3. Consider the system (1) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \qquad A_d = \begin{bmatrix} 0 & 0 \\ -0.375 & -1.15 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix},$$

$$C = [0, 1],$$

$$C_D = [-0.375, -1.15],$$

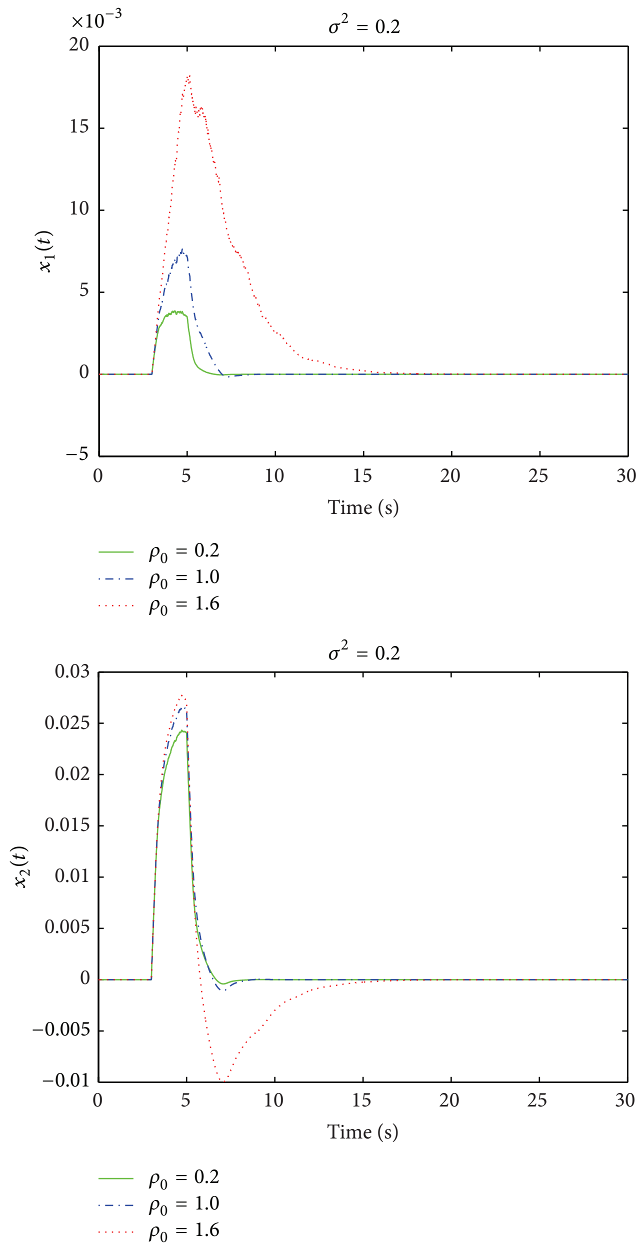

FIGURE 2: State trajectories with various ρ_0 and fixed $\sigma^2 = 0.2$ (Example 1).

$$B_2 = 0,$$
$$D = E_a = E_d = 0_2. \tag{42}$$

For the above system, the minimized $\mathscr{H}_\infty$ performance γ with various ordered pair (h_m, h_M) and unknown d_M are listed in Table 3. In this table, the recent results [5, 6, 10] are compared with ones in this works. From Table 3, it is clear that our results for this example give smaller γ than the ones in [5, 6, 10].

Example 4. Consider the system (1) with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \qquad A_d = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}. \tag{43}$$

In Table 4, the results for different condition of various h_m and d_M for guaranteeing stability are compared with the results of the existing works. From Table 4, it can be shown that our result for this example gives larger MABD than the ones in [7–10].

5. Conclusions

The $\mathscr{H}_\infty$ performance and stability analysis for linear systems with interval time-varying delays and disturbances were studied in this paper. In Theorem 8, the $\mathscr{H}_\infty$ performance criterion for interval time-delayed systems with stochastic parameter uncertainties was proposed with the stochastic

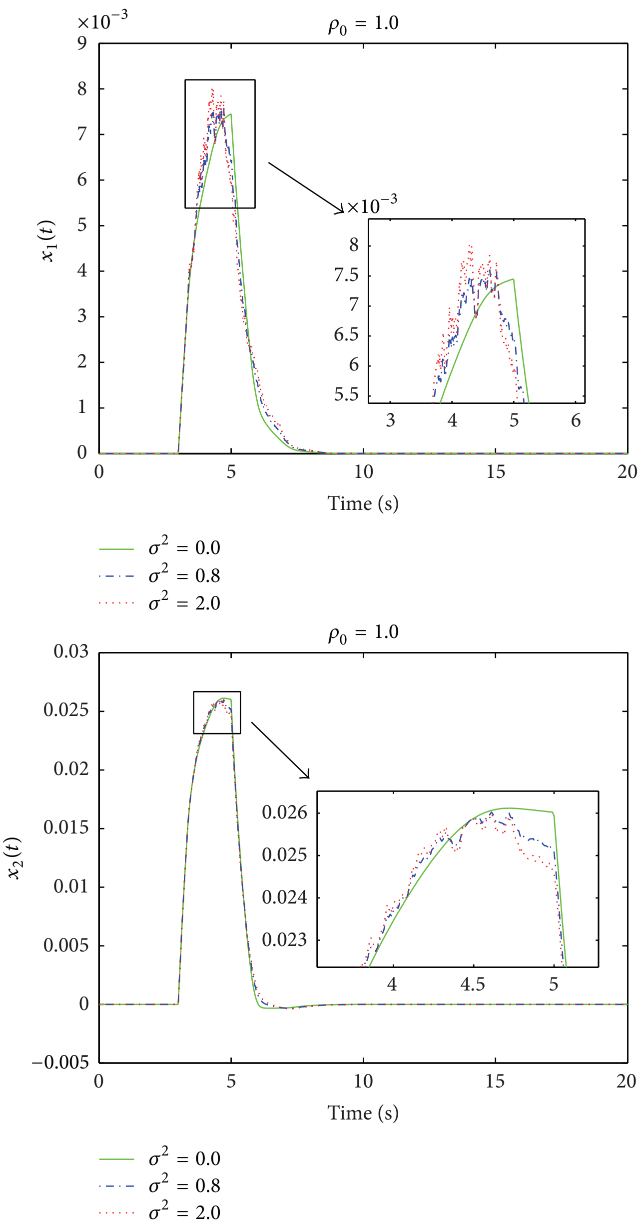

Figure 3: State trajectories with various σ^2 and fixed $\rho_0 = 1.0$ (Example 1).

Table 4: MADBs with various h_m and fixed $d_M = 0.3$ (Example 4).

	0.3	0.5	0.8	1
Lee and Park [9]	1.07	1.21	1.45	1.61
Sun et al. [7]	2.2634	2.2858	2.3078	2.3167
Liu et al. [8]	2.2887	2.3094	2.3370	2.3516
Kwon et al. [10]	2.4503	2.4756	2.5069	2.5279
Corollary 12	2.5198	2.5289	2.5492	2.5704

indexes, the mean ρ_0 and the variance σ^2. In Theorem 11, based on the result of Theorem 8, the interval time-delayed systems with deterministic parameter uncertainties were dealt. Afterward, in Corollary 12, the improved stability criterion for the nominal form of linear systems without parameter uncertainties and disturbances was derived. Four illustrative examples have been given to show the effectiveness and usefulness of the presented criteria. By utilizing the proposed criteria, future works will focus on solving various problems in [25–30].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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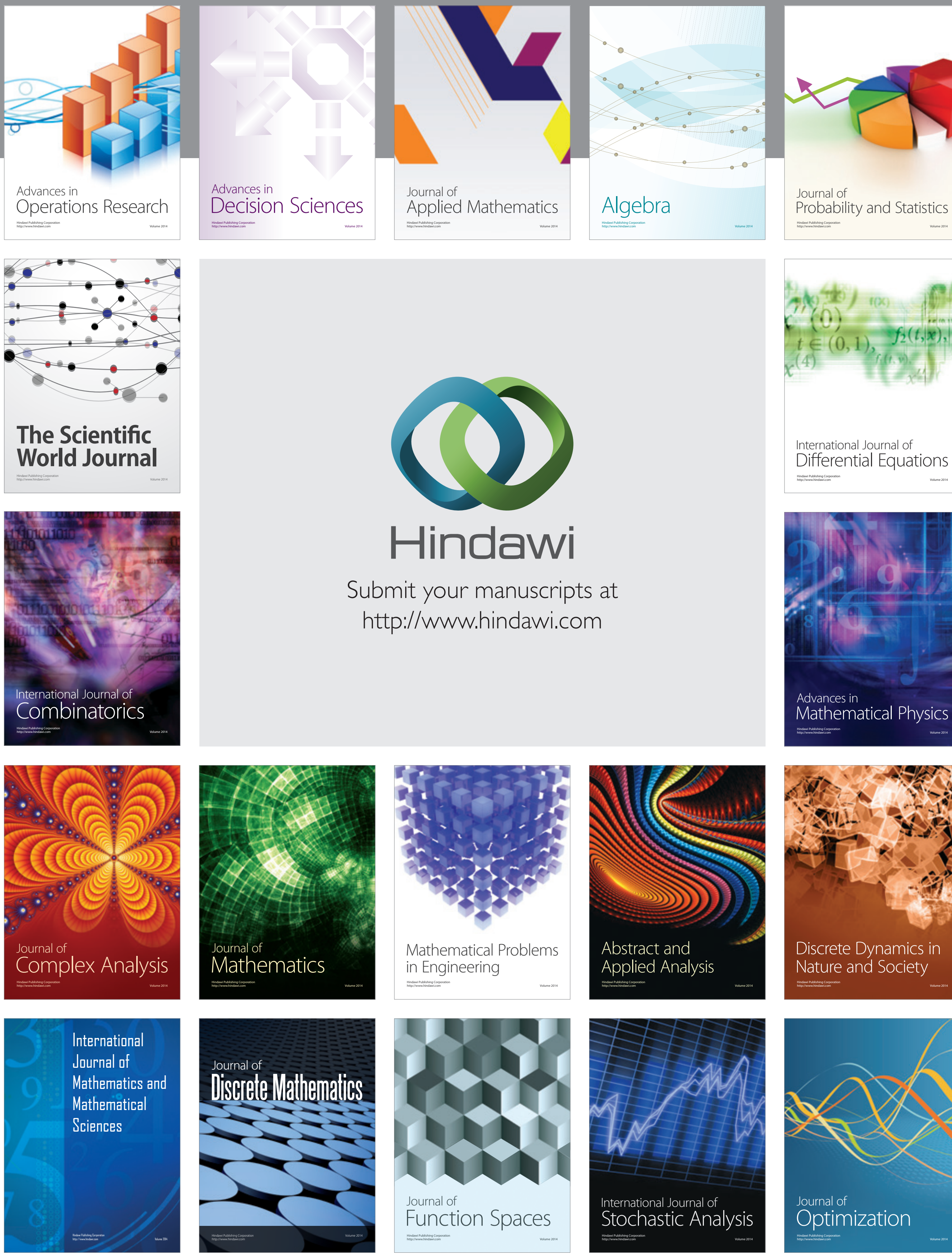
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