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Research Article

The Mathematical Basis of the Inverse Scattering Problem for Cracks from Near-Field Data

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We consider the acoustic scattering problem from a crack which has Dirichlet boundary condition on one side and impedance boundary condition on the other side. The inverse scattering problem in this paper tries to determine the shape of the crack and the surface impedance coefficient from the near-field measurements of the scattered waves, while the source point is placed on a closed curve. We firstly establish a near-field operator and focus on the operator's mathematical analysis. Secondly, we obtain a uniqueness theorem for the shape and surface impedance. Finally, by using the operator's properties and modified linear sampling method, we reconstruct the shape and surface impedance.

1. Introduction

In this paper, we consider the scattering of an electromagnetic time-harmonic plane wave by an infinite cylinder having an open crack as cross section in R^2. We assume that the cylinder is coated on one side by a material with surface impedance λ. This corresponds to the situation when the boundary or more generally a portion of the boundary is coated with an unknown material in order to avoid detection. Assuming that the electric field is polarized in the TM mode (see [1–3]), this leads to a mixed boundary value problem for the Helmholtz equation defined in the exterior of an open arc in R^2.

Briefly speaking, let $\Gamma \subset R^2$ be an oriented piecewise smooth nonintersecting crack without cups; that is, $\Gamma = \rho(s) : s \in [s_0, s_1]$, where $\rho : [s_0, s_1] \to R^2$ is an injective piecewise C^1 function and the crack Γ is contained in a closed curve Λ. Then the mixed boundary value problem for the Helmholtz equation in R^2 can be formulated as follows:

$$\begin{aligned} \Delta u + k^2 u &= 0, \quad \text{in } R^2 \setminus \overline{\Gamma}, \\ u_+ &= 0, \quad \text{on } \Gamma,\ z \in \Lambda, \\ \frac{\partial u_-}{\partial \nu} + i\lambda u_- &= 0, \quad \text{on } \Gamma,\ z \in \Lambda, \end{aligned} \tag{1}$$

where $k > 0$ is the wave number and $\lambda > 0$ is the surface impedance. u is the total wave of the scattered wave u^s and the incident wave $u^i = \Phi(x, z)$; that is, $u = u^i + u^s$, and $\Phi(x, z)$ is the fundamental solution to the Helmholtz equation defined by

$$\Phi(x, z) = \frac{i}{4} H_0^{(1)}(k|x - z|), \tag{2}$$

with $H_0^{(1)}$ being a Hankel function of the first kind of order zero. The scattered field u^s is required to satisfy the Sommerfeld radiation condition

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u^s}{\partial \nu} - iku^s \right) = 0 \tag{3}$$

uniformly in $\hat{x} = x/|x|$ with $r = |x|$.

Remark 1. $u_\pm = \lim_{h \to 0^+} u(x \pm h\nu)$ for $x \in \Gamma$, and $\partial u_\pm / \partial \nu = \lim_{h \to 0^+} \nu \cdot \nabla u(x \pm h\nu)$ for $x \in \Gamma$ (for the details, see Section 2). In the following discussion, $(\cdot)_\pm$ means the limit approaching the boundary from outside and inside the domain.

The inverse scattering problem in this paper is trying to determine the shape of the arc (or crack) and the surface impedance coefficient from the near-field measurements of the scattered waves, while the source point is placed on a closed curve, the results are as follows.

Inverse Problem (Ip). In this paper, the inverse problem we are concerned about is to determine the crack Γ and the surface impedance λ from these measurements $u^s(x,z)$ for $x, z \in \Lambda$.

In 1995, Kress considered the inverse scattering problem for cracks with sound-soft boundary condition in [4]. The case of a sound-hard crack was considered by Monch in 1997 in [5]. Both of the authors used Newton's method to reconstruct the shape of the crack from a knowledge of the far-field pattern. In 2003, Cakoni and Colton considered an inverse scattering problem by cracks, and they reconstructed the cracks by using the linear sampling method in [1]. In 2005, Colton and Haddar discussed similar inverse scattering problem by cracks, and they recovered the cracks by using the reciprocity gap functional method in [6]. Zeev and Cakoni considered the inverse scattering problem for a crack embedded in a known inhomogeneous background and recovered the crack (with a point source as incident field) in 2009 in [3]. More related research works can be found in [2, 3, 7] and the references therein.

This paper is arranged as follows. In the next section, we formulate the scattering problem mathematically and prove that the associated near-field operator is injective with dense range under appropriate assumptions. In Section 3, we show that the crack Γ and the surface impedance λ are uniquely determined from the near-field measurements of the scattered waves, while the source point is placed on a closed curve. We modify the linear sampling method and reconstruct the shape of the crack (or the surface impedance coefficient) in Section 4.

2. The Formulation of the Problem

We suppose that the crack Γ can be extended to an arbitrary piecewise smooth, simply connected, and closed curve $\partial\Omega$ enclosing a bounded domain Ω such that the normal vector ν on Γ coincides with outward normal vector on $\partial\Omega$ which we again denote by ν. Λ is a closed curve; we denote by D the domain surrounded by Λ. We suppose that Ω is completely contained in D, and we assume the normal vector ν on $\partial\nu$ and ∂D is mapped to the exteriors of the domain Ω and the domain D, respectively.

In order to formulate our scattering problem more precisely, we need to properly define the trace space on $\partial\Omega$ and ∂D. Let U be a bounded domain and let Σ be an open subset of the boundary ∂U. If $L^2(\partial U)$, $H^{1/2}(\partial U)$, and $H^{-1/2}(\partial U)$ denote the usual Sobolev spaces, we define the following spaces [8]:

$$
\begin{aligned}
L^2(\Sigma) &= \left\{ u|_\Sigma : u \in L^2(\partial U) \right\}, \\
H^{1/2}(\Sigma) &= \left\{ u|_\Sigma : u \in H^{1/2}(\partial U) \right\}, \\
\widetilde{H}^{1/2}(\Sigma) &= \left\{ u \in H^{1/2}(\partial U) : \operatorname{supp} u \subseteq \overline{\Sigma} \right\}, \\
H^{-1/2}(\Sigma) &= \left(\widetilde{H}^{1/2}(\Sigma) \right)', \\
&\qquad \text{the dual space of } \widetilde{H}^{1/2}(\Sigma), \\
\widetilde{H}^{-1/2}(\Sigma) &= \left(H^{1/2}(\Sigma) \right)', \\
&\qquad \text{the dual space of } H^{1/2}(\Sigma),
\end{aligned}
\tag{4}
$$

and we have the chain

$$
\begin{aligned}
\widetilde{H}^{1/2}(\Sigma) \subset H^{1/2}(\Sigma) \subset L^2(\Sigma) \subset \widetilde{H}^{-1/2}(\Sigma) \\
\subset H^{-1/2}(\Sigma).
\end{aligned}
\tag{5}
$$

Then problem (1) can be rewritten as

$$
\begin{aligned}
\Delta u^s + k^2 u^s &= 0, \quad \text{in } R^2 \setminus \overline{\Gamma}, \\
u^s_+ &= -\Phi(\cdot, z), \quad \text{on } \Gamma,\ z \in \Lambda, \\
\frac{\partial u^s_-}{\partial \nu} + i\lambda u^s_- &= -\frac{\partial \Phi(\cdot, z)}{\partial \nu} - i\lambda \Phi(\cdot, z), \quad \text{on } \Gamma,\ z \in \Lambda,
\end{aligned}
\tag{6}
$$

and u^s is required to satisfy Sommerfeld radiation condition (3).

Remark 2. By using similar method in [1], we can obtain the existence and uniqueness of solution to the direct problem (6). Here we use the point source as incident wave, while the plane wave was used as the incident in [1].

We define the near-field operator $F : L^2(\Lambda) \to L^2(\Lambda)$ by

$$
(Fg)(x) = \int_\Lambda u^s(x, z)\, g(z)\, ds(z), \qquad g \in L^2(\Lambda),\ x \in \Lambda,
\tag{7}
$$

where the function $u^s(x,z)$ is the solution of problem (6). According to Green's representation formula, we have

$$
\begin{aligned}
u^s(x, z) &= \int_\Gamma \left[u^s_+(y, z) \frac{\partial \Phi(x, y)}{\partial \nu(y)} - \frac{\partial u^s_+(y, z)}{\partial \nu(y)} \Phi(x, y) \right] ds(y) \\
&\quad + \int_{\partial\Omega \setminus \Gamma} \left[u^s_+(y, z) \frac{\partial \Phi(x, y)}{\partial \nu(y)} - \frac{\partial u^s_+(y, z)}{\partial \nu(y)} \Phi(x, y) \right] ds(y), \quad x \in R^2 \setminus \Omega,
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
0 &= \int_\Gamma \left[u^s_-(y, z) \frac{\partial \Phi(x, y)}{\partial \nu(y)} - \frac{\partial u^s_-(y, z)}{\partial \nu(y)} \Phi(x, y) \right] ds(y) \\
&\quad + \int_{\partial\Omega \setminus \Gamma} \left[u^s_-(y, z) \frac{\partial \Phi(x, y)}{\partial \nu(y)} - \frac{\partial u^s_-(y, z)}{\partial \nu(y)} \Phi(x, y) \right] ds(y), \quad x \in R^2 \setminus \Omega.
\end{aligned}
\tag{9}
$$

On the boundary $\partial\Omega \setminus \overline{\Gamma}$, we have

$$u_+^s(x,z) = u_-^s(x,z), \quad \frac{\partial u_+^s(x,z)}{\partial \nu(x)} = \frac{\partial u_-^s(x,z)}{\partial \nu(x)}. \tag{10}$$

Then, by substituting (9) into (8), we have

$$\begin{aligned} u^s(x,z) = {} & \int_\Gamma \left[u_+^s(y,z) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u_+^s(y,z)}{\partial \nu(y)} \Phi(x,y) \right] ds(y) \\ & - \int_\Gamma \left[u_-^s(y,z) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u_-^s(y,z)}{\partial \nu(y)} \Phi(x,y) \right] ds(y), \\ & x \in R^2 \setminus \Omega. \end{aligned} \tag{11}$$

By changing the order of x and z, we have

$$\begin{aligned} u^s(z,x) = {} & \int_\Gamma \left[u_+^s(y,x) \frac{\partial \Phi(z,y)}{\partial \nu(y)} - \frac{\partial u_+^s(y,x)}{\partial \nu(y)} \Phi(z,y) \right] ds(y) \\ & - \int_\Gamma \left[u_-^s(y,x) \frac{\partial \Phi(z,y)}{\partial \nu(y)} - \frac{\partial u_-^s(y,x)}{\partial \nu(y)} \Phi(z,y) \right] ds(y), \\ & z \in R^2 \setminus \Omega. \end{aligned} \tag{12}$$

Then we have the following result.

Lemma 3. *For the problem (6), one has $u^s(x,z) = u^s(z,x)$ for $x, z \in \Lambda$.*

Proof. Applying Green's second theorem and (3), we obtain

$$\begin{aligned} 0 = {} & \int_\Gamma \left[u_+^s(y,z) \frac{\partial u_+^s(y,x)}{\partial \nu(y)} - u_+^s(y,x) \frac{\partial u_+^s(y,z)}{\partial \nu(y)} \right] ds(y) \\ & - \int_\Gamma \left[u_-^s(y,z) \frac{\partial u_-^s(y,x)}{\partial \nu(y)} - u_-^s(y,x) \frac{\partial u_-^s(y,z)}{\partial \nu(y)} \right] ds(y), \end{aligned} \tag{13}$$

$$\begin{aligned} 0 = {} & \int_\Gamma \left[\Phi(y,z) \frac{\partial \Phi(y,x)}{\partial \nu(y)} - \Phi(y,x) \frac{\partial \Phi(y,z)}{\partial \nu(y)} \right] ds(y) \\ & + \int_{\partial\Omega\setminus\Gamma} \left[\Phi(y,z) \frac{\partial \Phi(y,x)}{\partial \nu(y)} - \Phi(y,x) \frac{\partial \Phi(y,z)}{\partial \nu(y)} \right] ds(y). \end{aligned} \tag{14}$$

Substituting (13) into (11) and (14) into (12), respectively, we have

$$\begin{aligned} u^s(x,z) - u^s(z,x) = {} & \int_\Gamma \left[w_+^s(y,z) \frac{\partial w_+^s(y,x)}{\partial \nu(y)} - w_+^s(y,x) \frac{\partial w_+^s(y,z)}{\partial \nu(y)} \right] ds(y) \\ & - \int_\Gamma \left[w_-^s(y,z) \frac{\partial w_-^s(y,x)}{\partial \nu(y)} - w_-^s(y,x) \frac{\partial w_-^s(y,z)}{\partial \nu(y)} \right] ds(y), \end{aligned} \tag{15}$$

where $w^s = u^s + \Phi$. By using the boundary conditions in (6), we have

$$w_+^s\big|_\Gamma = 0, \qquad \left(\frac{\partial w_-^s}{\partial \nu} + i\lambda w_-^s \right)\bigg|_\Gamma = 0. \tag{16}$$

This implies that $u^s(x,z) - u^s(z,x) = 0$. So, we complete the proof of this lemma. □

Theorem 4. *The near-field operator F defined by (7) is injective and has dense range.*

Proof. From $u^s(x,z) = u^s(z,x)$, the L^2 adjoint of F is given by

$$\begin{aligned} (F^* h)(x) &= \int_\Lambda \overline{u^s(z,x)} h(z)\, ds(z) \\ &= \overline{\int_\Lambda u^s(x,z) \overline{h(z)} ds(z)}, \\ & \qquad x \in \Lambda,\ h \in L^2(\Lambda). \end{aligned} \tag{17}$$

Then we have $(F^* h)(x) = \overline{(Fg)(x)}$, where $g(z) = \overline{h(z)}$. Thus, operator F is injective if and only if F^* is injective. Since $N(F^*)^\perp = \overline{F(L^2(\Lambda))}$ in a Hilbert space, our proof will be finished by only showing that operator F is injective.

Let $Fg = 0$; we need to show that $g = 0$. Define

$$v(x) = \int_\Lambda u^s(x,z)\, g(z)\, ds(z), \quad x \in R^2 \setminus \overline{\Gamma}. \tag{18}$$

It is easy to verify that v satisfies the exterior problem

$$\begin{aligned} \Delta v + k^2 v = 0, &\quad \text{in } R^2 \setminus \overline{D}, \\ v = 0, &\quad \text{on } \Lambda, \\ \lim_{r\to\infty} \sqrt{r} \left(\frac{\partial v}{\partial r} - ikv \right) = 0. & \end{aligned} \tag{19}$$

This exterior Dirichlet problem has only zero solution (see [9]). Then the unique continuation principle now yields $v = 0$ in $R^2 \setminus \overline{\Gamma}$. Therefore,

$$v_+\big|_\Gamma = 0, \quad \left(\frac{\partial v_-}{\partial \nu} + i\lambda v_- \right)\bigg|_\Gamma = 0. \tag{20}$$

Now define

$$w(x)=\int_{\Lambda}\Phi(x,z)\,g(z)\,ds(z),\qquad x\in R^2\setminus\overline{\Gamma}. \tag{21}$$

By the boundary conditions (6) and (20) together with the jump relationship of the single-layer potential on the boundary Λ, we conclude that w satisfies the following problem:

$$\begin{aligned}
\Delta w+k^2w&=0,\quad \text{in } R^2\setminus\overline{D},\\
\Delta w+k^2w&=0,\quad \text{in } D\setminus\overline{\Gamma},\\
w_+-w_-&=0,\quad \text{on } \Lambda,\\
\frac{\partial w_+}{\partial\nu}-\frac{\partial w_-}{\partial\nu}&=-g,\quad \text{on } \Lambda,\\
w_+&=0,\quad \text{on } \Gamma,\\
\frac{\partial w_-}{\partial\nu}+i\lambda w_-&=0,\quad \text{on } \Gamma,\\
\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial w}{\partial r}-ikw\right)&=0.
\end{aligned} \tag{22}$$

On the boundary $\partial\Omega$, due to the jump relationship of the single-layer potential, we have $w_+=w_-$ and $\partial w_+/\partial\nu=\partial w_-/\partial\nu$. So, we get $w_-|_\Gamma=0$ and $(\partial w_-/\partial\nu)|_\Gamma=0$ by the boundary condition of (22) on Γ. Then Holmgren's principle and uniqueness continuation principle imply that $w=0$ on D. From this, we obtain that $w_+|_\Lambda=w_-|_\Lambda=0$.

Therefore, in the domain $R^2\setminus\overline{D}$, we have the following problem for w:

$$\begin{aligned}
\Delta w+k^2w&=0,\quad \text{in } R^2\setminus\overline{D},\\
w_+&=0,\quad \text{on } \Lambda,\\
\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial w}{\partial r}-ikw\right)&=0.
\end{aligned} \tag{23}$$

The problem has only zero solution which implies that $(\partial w_+/\partial\nu)|_\Lambda=0$; thus $-g=(\partial w_+/\partial\nu)|_\Lambda-(\partial w_-/\partial\nu)|_\Lambda=0$. Hence, F is injective. □

3. Uniqueness for the Inverse Problem

Based on the idea of [10, 11], we firstly conclude that Γ is uniquely determined from $u^s|_\Lambda$ without knowing λ a priori. Secondly, we show that the surface impedance λ can be uniquely determined by $u^s|_\Lambda$ (see [12]).

Theorem 5. *Assume that Γ_1 and Γ_2 are two cracks with corresponding impedance λ_1 and impedance λ_2 such that for a fixed wave number the corresponding scattering fields $u_1^s(\cdot,z)$ and $u_2^s(\cdot,z)$ coincide on Λ for all point sources $z\in\Lambda$. Then $\Gamma_1=\Gamma_2$ and $\lambda_1=\lambda_2$.*

Proof. We consider problem (6) with $z\in\Lambda$ replaced by $z\in R^2\setminus\overline{\Gamma}_i$. By using the same method in Lemma 3 in Section 2, we have

$$u_i^s(x,z)=u_i^s(z,x)\qquad \forall x,z\in R^2\setminus\overline{\Gamma}_i, \tag{24}$$

where $i=1,2$.

In the domain $G=R^2\setminus(\overline{\Gamma}_1\cup\overline{\Gamma}_2)$, let $w^s=u_1^s-u_2^s$. Then w^s satisfies problem (19) replacing the corresponding domain D with G; that is,

$$\begin{aligned}
\Delta w^s+k^2w^s&=0,\quad \text{in } G,\\
w^s&=0,\quad \text{on } \Lambda,\\
\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial w^s}{\partial r}-ikw^s\right)&=0,
\end{aligned} \tag{25}$$

where we used the condition $u_1^s(x,z)=u_2^s(x,z)$ on Λ for all point sources $z\in\Lambda$.

The only zero solution of this problem and the unique continuation principle imply that $w^s=0$ in G; that is, $u_1^s(x,z)=u_2^s(x,z)$ for all $x\in G$ and $z\in\Lambda$.

By reciprocity (24) we have that $u_1^s(z,x)=u_2^s(z,x)$ for all $z\in\Lambda$ and $x\in G$. Then again arguing as above we have that $u_1^s(z,x)=u_2^s(z,x)$ for all $z,x\in G$.

Now we assume that $\Gamma_1\neq\Gamma_2$. Without loss of generality, there exists $x^*\in\Gamma_1$ but $x^*\notin\Gamma_2$. Choose $h>0$ such that the sequence $x_n=x^*+(h/n)\nu(x^*)$, $n=1,2,\ldots$, is contained in G, where the unit normal vector to the boundary $\partial\Omega_1$ is directed into the exterior of Ω_1. Considering $u_{n,j}^s$ $(j=1,2)$ as the solution of problem (6) with $z\in\Lambda$ replaced by $x_n\in G$ corresponding to $\lambda=\lambda_j$ and $\Gamma=\Gamma_j$, then $u_{n,1}^s(x)=u_{n,2}^s(x)$ for $x\in G$. For simplicity, we use $u_n^s(x)$ to denote the solution in G; that is,

$$u_n^s(x)=u_{n,1}^s(x)=u_{n,2}^s(x),\qquad x\in G. \tag{26}$$

Consider the crack Γ_2; then $u_n^s(x)=u_{n,2}^s(x)$, and, on the two sides of Γ_2, we know that $u_{n+}^s(\cdot)=-\Phi(\cdot,x_n)$ and $\partial u_{n-}^s(\cdot)/\partial\nu+i\lambda_2u_{n-}^s(\cdot)=-\partial\Phi(\cdot,x_n)/\partial\nu+i\lambda_2\Phi(\cdot,x_n)$ on Γ_2 are uniformly bounded.

The well-posedness of the solution to the corresponding problem implies that the limit $\|u_n^s(x^*)\|_{H^1_{\mathrm{loc}}(R^2\setminus\overline{\Gamma}_2)}$ is bounded as $n\to\infty$, so by the trace theorem $\|u_n^s(x^*)\|_{H^{1/2}(B_r(x^*)\cap\Gamma_1)}$ is uniformly bounded with respect to n, where $B_r(x^*)$ is a small neighborhood centered at x^* not intersecting Γ_2.

On the other hand, consider the solution of (6) with respect to the crack Γ_1; in this case $u_n^s(x)=u_{n,1}^s(x)$. From the boundary condition $u_{n+}^s(\cdot)=-\Phi(\cdot,x_n)$ on the crack Γ_1, we have that $\|u_n^s\|_{H^{1/2}(B_r(x^*)\cap\Gamma_1)}\to\infty$ as $n\to\infty$ since $\|\Phi(\cdot,x_n)\|_{H^{1/2}(B_r(x^*)\cap\Gamma_1)}\to\infty$ as $n\to\infty$. This is a contradiction. Therefore $\Gamma_1=\Gamma_2$.

Now let $\Gamma=\Gamma_1=\Gamma_2$ and assume that $\lambda_1(x)\neq\lambda_2(x)$ for $x\in\Gamma$. Then, from relation (24) and the unique continuation principle, we know that $w^s=0$ in $R^2\setminus\overline{\Gamma}$. Then $w_\pm^s=0$ and $\partial w_\pm^s/\partial\nu=0$ on Γ; that is, $u_{1\pm}^s=u_{2\pm}^s$ and $\partial u_{1\pm}^s/\partial\nu=\partial u_{2\pm}^s/\partial\nu$ on Γ. Let $w(\cdot,z)=u_1^s+\Phi(\cdot,z)$; from the boundary condition (6), we have

$$\frac{\partial w_-}{\partial\nu}+i\lambda_jw_-=0,\qquad j=1,2,\ \text{on } \Gamma, \tag{27}$$

and thus

$$(\lambda_1 - \lambda_2)\, w_-(\cdot, z) = 0 \quad \text{on } \Gamma. \tag{28}$$

Hence $w_-(\cdot, z) = 0$ on Γ since $\lambda_1(x) \neq \lambda_2(x)$. Notice that $w_+(\cdot, z) = u^s_{1+} + \Phi(\cdot, z) = 0$ on Γ; then we have

$$\begin{aligned} &\Delta w + k^2 w = 0, \quad \text{in } R^2 \setminus (\overline{\Gamma} \cup \{z\}), \\ &w_\pm = 0, \quad \text{on } \Gamma, \\ &\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial w}{\partial r} - ikw\right) = 0. \end{aligned} \tag{29}$$

This problem has only zero solution; that is, $w = 0$ in $R^2 \setminus (\overline{\Gamma} \cup z)$.

Now choose $h > 0$ sufficiently small such that $w(x_n, z) = 0$; that is, $u^s_1(x_n) = -\Phi(x_n, z)$, where $x_n = z + (h/n)\nu(z)$, $n = 1, 2, \ldots$, and ν is the unit outward normal to Λ. Let $n \to \infty$; then the limit of $\|u^s_1(x_n)\|_{H^1_{\mathrm{loc}}(R^2\setminus\overline{\Gamma})}$ is bounded because that problem (6) is well posed, but $\|\Phi(x_n, z)\|_{H^1_{\mathrm{loc}}(R^2\setminus\overline{\Gamma})}$ is unbounded which leads to a contraction. So, we complete the proof of the theorem. □

4. The Linear Sampling Method

The inverse scattering problem in this paper is trying to determine the shape of the crack and the surface impedance coefficient from the near-field measurements of the scattered waves, while the source point is placed on a closed curve. In this part, we provide the mathematical basis to reconstruct the crack Γ from the knowledge of $u^s(x, z)$ for $x, z \in \Lambda$ by using the linear sampling method; that is, we want to determine Γ from a knowledge of $u^s(x, z)$ for $x, z \in \Lambda$, where Λ is a circle centered at the origin; that is, $\Lambda = \{x \in R^2, \ |x| = r_\Lambda > 0\}$. Based on the ideas of [2, 3], we introduce the near-field equation

$$\int_\Lambda u^s(x, z)\, g(z)\, ds(z) = \Phi^L(x), \qquad g \in L^2(\Lambda),\ x \in \Lambda; \tag{30}$$

that is,

$$(Fg(z))(x) = \Phi^L(x), \quad g \in L^2(\Lambda),\ x \in \Lambda, \tag{31}$$

where

$$\Phi^L(x) = \int_L \Phi(x, y)\, \alpha^L(y)\, ds(y) + \int_L \frac{\partial \Phi(x, y)}{\partial \nu(y)} \beta^L(y)\, ds(y) \tag{32}$$

and L is smooth nonintersecting arc and $\alpha^L(y) \in \widetilde{H}^{-1/2}(L)$ and $\beta^L(y) \in \widetilde{H}^{1/2}(L)$.

We want to characterize the crack Γ by using the behavior of an approximate solution g of the near-field equation (30).

Now consider the following problem:

$$\begin{aligned} &\Delta u + k^2 u = 0, \quad \text{in } R^2 \setminus \overline{\Gamma}, \\ &u_+ = p, \quad \text{on } \Gamma, \\ &\frac{\partial u_-}{\partial \nu} + i\lambda u_- = q, \quad \text{on } \Gamma, \\ &\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial w}{\partial r} - ikw\right) = 0, \end{aligned} \tag{33}$$

for $p \in H^{1/2}(\Gamma)$ and $q \in H^{-1/2}(\Gamma)$. From [1], we know that this problem has a unique solution $u \in H^1_{\mathrm{loc}}(R^2 \setminus \overline{\Gamma})$ such that $\|u\|_{H^1_{\mathrm{loc}}(R^2\setminus\overline{\Gamma})} \leq c(\|p\|_{H^{1/2}(\Gamma)} + \|q\|_{H^{-1/2}(\Gamma)})$, where $c > 0$ is a constant and does not depend on p and q.

To understand the near-field equation better, we define an operator $B : H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \to L^2(\Lambda)$ which maps the boundary data (p, q) to the solution u on Λ. We have the following conclusions about this operator B.

Theorem 6. *Operator B is injective and compact and has dense range in $L^2(\Lambda)$.*

Proof. Let $B(p, q) = 0$; that is, $u = 0$ on Λ; we want to show that $p = 0$ and $q = 0$. From $u = 0$ on Λ, we know that u satisfies problem (19) which has only zero solution on $R^2 \setminus \overline{D}$. By the unique continuation principle, we have that $u = 0$ in $R^2 \setminus \overline{\Gamma}$ and thus $u_\pm|_\Gamma = 0$ and $(\partial u_\pm/\partial\nu)|_\Gamma = 0$. So, from the boundary conditions in (33), we can get $p = 0$ and $q = 0$, which implies that operator B is injective.

Define

$$v_n(x) = \sum_{-n}^{n} c_m J_m(kr)\, e^{im\theta}, \tag{34}$$

where (r, θ) are polar coordinates of x, $r = |x|$, and $\{c_m\} \in l^2$. Clearly v_n satisfies (33) with $p = v_n|_\Gamma$ and $q = (\partial v_n/\partial\nu + i\lambda v_n)|_\Gamma$. Since $v_n|_\Lambda = \sum_{-n}^{n} c_m J_m(kr_\Lambda) e^{im\theta}$, the completeness of the trigonometric sequence in $L^2[0, 2\pi]$ shows that operator B has dense range.

We now show that operator B is compact. Choose a disk $M = \{x \in R^2,\ |x| \leq r < r_\Lambda\}$ such that $\Omega \subset M \subset D$. Using Green's representation formula for u, we can decompose operator B as $B = B_1 B_2$, where $B_2 : H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{1/2}(\partial M) \times H^{-1/2}(\partial M)$ is defined by $B_2(p, q) = (u|_{\partial M}, (\partial u/\partial\nu)|_{\partial M})$ and $B_1 : H^{1/2}(\partial M) \times H^{-1/2}(\partial M) \to L^2(\Lambda)$ is defined by

$$B_1(p_1, q_1) = \int_{\partial M} \left[p_1(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} - q_1(y)\, \Phi(x, y) \right] ds(y), \quad x \in \Lambda. \tag{35}$$

The regularity of the solution to problem (33) implies that operator B_2 is bounded. So, operator B is compact since operator B_1 is compact. □

Theorem 7. *For $x \in \Lambda$, the integral expression $\Phi^L(x)$ is in the range of B if and only if $L \subset \Gamma$.*

Proof. If $L \subset \Gamma$, then $\Phi^L(x)$ is the solution of problem (33) with

$$p = \Phi^L(x)|_\Gamma + \frac{1}{2}\tilde{\beta}^L(x)|_\Gamma,$$
$$q = \left.\frac{\partial \Phi^L(x)}{\partial \nu(x)}\right|_\Gamma + \frac{1}{2}\tilde{\alpha}^L(x)|_\Gamma + i\lambda\left.\left(\Phi^L(x) - \frac{1}{2}\tilde{\beta}^L(x)\right)\right|_\Gamma, \tag{36}$$

where $\tilde{\alpha}^L$ and $\tilde{\beta}^L$ are zero extension of α^L and β^L to the whole boundary $\partial\Omega$. So, $B(p,q) = \Phi^L(x)|_\Lambda$, which implies that $\Phi^L(x)$ is in the range of B.

If $L \not\subset \Gamma$ and $\Phi^L(x)$ is in the range of B, then there exists a solution u with $u_+ = p$ and $\partial u_-/\partial\nu + i\lambda u_- = q$ for $x \in \Gamma$, such that $u|_\Lambda = \Phi^L(x)|_\Lambda$. From [1], we know that this solution has the form

$$u = \int_\Gamma \Phi(x,y)\,\alpha(y)\,ds(y) + \int_\Gamma \frac{\partial\Phi(x,y)}{\partial\nu(y)}\beta(y)\,ds(y), \tag{37}$$

where $\alpha(y) \in \widetilde{H}^{-1/2}(\Gamma)$ and $\beta(y) \in \widetilde{H}^{1/2}(\Gamma)$. Since $u|_\Lambda = \Phi^L(x)|_\Lambda$, the unique continuation principle implies that $u = \Phi^L(x)$ in $R^2 \setminus (\overline{\Gamma} \cup \overline{L})$. Now let $x_0 \in L$, $x_0 \notin \Gamma$, and let $B_\epsilon(x_0)$ be a small ball with center at x_0 such that $B_\epsilon(x_0) \cap \Gamma = \emptyset$. Hence, u is analytic in $B_\epsilon(x_0)$, while $\Phi^L(x)$ has a singularity at x_0 which is a contradiction. This completes the proof of this theorem. □

To further understand the near-field operator F, we define function w_g by

$$w_g(x) := \int_\Lambda \Phi(x,y)\,g(y)\,ds(y), \quad x \in R^2 \setminus \Lambda \tag{38}$$

and define an operator $S : L^2(\Lambda) \to H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)$ given by

$$Sg = \left(w_g(x)|_\Gamma, \left(\frac{\partial w_g(x)}{\partial\nu} + i\lambda w_g\right)(x)|_\Gamma\right). \tag{39}$$

Then by superposition we have the following relation:

$$Fg = -B(Sg). \tag{40}$$

Theorem 8. *Operator S is bounded and injective and has dense range in $H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)$.*

Proof. From the definition of operator S, we know that S is bounded. To prove that S is injective, we let $Sg = 0$ and want to prove that $g = 0$.

It is easy to check that w_g defined in (38) satisfies problem (22).

By using the same arguments as that in proving that F is injective in Theorem 6, we have $g = 0$; that is, operator S is injective.

Next, we will show that S has dense range in $H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)$. Let $\chi = (p,q) \in \widetilde{H}^{-1/2}(\Gamma)\times\widetilde{H}^{1/2}(\Gamma)$ such that $\langle Sg, \chi\rangle = 0$; here $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\widetilde{H}^{-1/2}(\Gamma)\times\widetilde{H}^{1/2}(\Gamma)$ and $H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)$. This means that

$$\begin{aligned}
&\int_\Gamma\int_\Lambda \Phi(x,z)\,g(z)\,ds(z)\,\overline{p(x)}ds(x) \\
&\quad + \int_\Gamma\int_\Lambda \left(\frac{\partial\Phi(x,z)}{\partial\nu(x)} + i\lambda\Phi(x,z)\right) g(z)\,ds(z)\,\overline{q(x)}ds(x) \\
&= \int_\Lambda\int_\Gamma \Phi(x,z)\,\overline{p(x)}ds(x)\,g(z)\,ds(z) \\
&\quad + \int_\Lambda\int_\Gamma \left(\frac{\partial\Phi(x,z)}{\partial\nu(x)} + i\lambda\Phi(x,z)\right)\overline{q(x)}ds(x)\,g(z)\,ds(z) \\
&= 0,
\end{aligned} \tag{41}$$

for all $g \in L^2(\Lambda)$. Then we have

$$\int_\Gamma \Phi(x,z)\,\overline{p(x)}ds(x) + \int_\Gamma \left(\frac{\partial\Phi(x,z)}{\partial\nu(x)} + i\lambda\Phi(x,z)\right)\overline{q(x)}ds(x) = 0, \quad z \in \Lambda. \tag{42}$$

If we define

$$v(z) = \int_\Gamma \Phi(x,z)\,\overline{p(x)}ds(x) + \int_\Gamma \left(\frac{\partial\Phi(x,z)}{\partial\nu(x)} + i\lambda\Phi(x,z)\right)\overline{q(x)}ds(x), \quad z \in R^2 \setminus \overline{\Gamma}, \tag{43}$$

then $v(z)$ satisfies problem (19). The same analysis as before shows that $v(z) = 0$ for $z \in R^2 \setminus \overline{\Gamma}$. Therefore, by the jump relationships of single potential and double potential across the crack Γ, we get

$$\overline{q} = \left.(v_+ - v_-)\right|_\Gamma = 0, \tag{44}$$

and then

$$\overline{p} = \left.\left(\frac{\partial v_+}{\partial\nu} - \frac{\partial v_-}{\partial\nu}\right)\right|_\Gamma = 0. \tag{45}$$

So, we have shown that $\chi = (0,0)$, which implies that operator S has dense range in $H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)$. □

We are now in the position to give the main result of this paper.

Theorem 9. *Assume that Γ is an oriented nonintersecting piecewise smooth arc without cups. Then, if F is the near-field* *operator corresponding to the scattering problem (6), then one has the following results:*

(1) *If $L \subset \Gamma$, then for every $\epsilon > 0$ there exists a solution $g_\epsilon^L \in L^2(\Lambda)$ satisfying*

$$\left\| F g_\epsilon^L + \Phi^L \right\|_{L^2(\Lambda)} < \epsilon. \quad (46)$$

(2) *If $L \not\subset \Gamma$, then for every $\epsilon > 0$ and $\delta > 0$ there exists a function $g_{\epsilon,\delta}^L \in L^2(\Lambda)$ such that*

$$\begin{gathered} \left\| F g_{\epsilon,\delta}^L + \Phi^L \right\|_{L^2(\Lambda)} < \epsilon + \delta, \\ \lim_{\delta \to 0} \left\| g_{\epsilon,\delta}^L \right\|_{L^2(\Lambda)} = \infty. \end{gathered} \quad (47)$$

Proof. If $L \subset \Gamma$, by using Theorem 7, there exists $\chi = (p, q) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ such that $B\chi = \Phi^L(x)$ for $x \in \Lambda$. From Theorem 8, for every $\epsilon_0 > 0$, there exists a function $g_{\epsilon_0}^L \in L^2(\Lambda)$ such that

$$\left\| S g_{\epsilon_0}^L - \chi \right\|_{H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)} < \epsilon_0. \quad (48)$$

By using Theorem 6, operator B is bounded and we have

$$\left\| B S g_{\epsilon_0}^L - B\chi \right\|_{L^2(\Lambda)} < c_1 \epsilon_0, \quad (49)$$

where c_1 is a constant; that is,

$$\left\| F g_\epsilon^L + \Phi^L \right\|_{L^2(\Lambda)} < \epsilon, \quad (50)$$

where $\epsilon = c_1 \epsilon_0$.

Next, we assume that $L \not\subset \Gamma$. In this case, by Theorem 7, $\Phi^L(x)$ for $x \in \Lambda$ is not in the range of B. But from Theorem 6 we know that operator B has dense range in $L^2(\Lambda)$. Hence, for every $\delta > 0$, we can construct a unique Tikhonov regularized solution $\chi^{\rho,L} \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ of $B\chi = \Phi^L$, such that

$$\left\| B\chi^{\rho,L} - \Phi^L \right\|_{L^2(\Lambda)} < \delta, \quad (51)$$

where ρ is the regularization parameter (chosen by a regular regularization strategy, e.g., the Morozov discrepancy principle). Then we have $\|\chi^{\rho,L}\|_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} \to \infty$ as $\rho \to 0$. By Theorem 8, S has dense range, so for $\epsilon > 0$ sufficiently small there exists $g_{\epsilon,\rho}^L$ such that

$$\left\| S g_{\epsilon,\rho}^L - \chi^{\rho,L} \right\|_{H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)} < \frac{\epsilon}{c_1}. \quad (52)$$

Combining (51) and (52), we obtain that for every $\epsilon > 0$ and $\delta > 0$ there exists $g_{\epsilon,\rho}^L \in L^2(\Lambda)$ such that

$$\begin{aligned} \left\| F g_{\epsilon,\rho}^L + \Phi^L \right\|_{L^2(\Lambda)} &= \left\| B S g_{\epsilon,\rho}^L - \Phi^L \right\|_{L^2(\Lambda)} \\ &< \left\| B S g_{\epsilon,\rho}^L - B\chi^{\rho,L} \right\|_{L^2(\Lambda)} \\ &\quad + \left\| B\chi^{\rho,L} - \Phi^L \right\|_{L^2(\Lambda)} < \epsilon + \delta. \end{aligned} \quad (53)$$

Since $\lim_{\delta \to 0} \rho(\delta) = 0$, we have that $\lim_{\delta \to 0} \|\chi^{\rho,L}\|_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} \to \infty$. From (52), we have that $\lim_{\delta \to 0} \|S g_{\epsilon,\delta}^L\|_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} \to \infty$. By the definition of operator S given by (39), we have that $\lim_{\delta \to 0} \|g_{\epsilon,\delta}^L\|_{L^2(\Lambda)} \to \infty$. Then we complete the proof of this theorem. □

Remark 10. In numerical analysis, we can choose some suitable smooth arcs as a set such as $\mathscr{L}$ and then consider near-field equation

$$F g(x) = \Phi^L(x), \quad L \in \mathscr{L}. \quad (54)$$

If $L \subset \Gamma$, we can find a bounded solution to the near-field equation (30) with discrepancy ϵ, whereas if $L \not\subset \Gamma$, then there exists solution of the near-field equation (with discrepancy $\epsilon + \delta$) with arbitrary large norm in the limit as $\delta \to 0$. Then the arc can be characterized by the behavior of this solution. But how to determine the surface impedance is a problem we need to study further.

Remark 11. Applying reciprocity gap functional method to reconstruct a crack, we need to know the near-field Cauchy data u and $\partial u/\partial \nu$ of the total field (see [6]). Qin and Colton used a method that may be called a modified linear sampling method to recover a cavity by using the near-field data (see [2, 7]). We combine these two methods to recover a crack (which has empty inner product) by using the measurement of near-field data u. In the process of recovering the crack, the near-field equation that we introduced is different.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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