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Research Article

Maximum Principle for Optimal Control Problems of Forward-Backward Regime-Switching Systems Involving Impulse Controls

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This paper is concerned with optimal control problems of forward-backward Markovian regime-switching systems involving impulse controls. Here the Markov chains are continuous-time and finite-state. We derive the stochastic maximum principle for this kind of systems. Besides the Markov chains, the most distinguishing features of our problem are that the control variables consist of regular and impulsive controls, and that the domain of regular control is not necessarily convex. We obtain the necessary and sufficient conditions for optimal controls. Thereafter, we apply the theoretical results to a financial problem and get the optimal consumption strategies.

1. Introduction

Maximum principle was first formulated by Pontryagin et al.'s group [1] in the 1950s and 1960s, which focused on the deterministic control system to maximize the corresponding Hamiltonian instead of the optimization problem. Bismut [2] introduced the linear backward stochastic differential equations (BSDEs) as the adjoint equations, which played a role of milestone in the development of this theory. The general stochastic maximum principle was obtained by Peng in [3] by introducing the second order adjoint equations. Pardoux and Peng also proved the existence and uniqueness of solution for nonlinear BSDEs in [4], which has been extensively used in stochastic control and mathematical finance. Independently, Duffie and Epstein introduced BSDEs under economic background, and in [5] they presented a stochastic recursive utility which was a generalization of the standard additive utility with the instantaneous utility depending not only on the instantaneous consumption rate but also on the future utility. Then El Karoui et al. gave the formulation of recursive utilities from the BSDE point of view. As found by [6], the recursive utility process can be regarded as a solution of BSDE. Peng [7] first introduced the stochastic maximum principle for optimal control problems of forward-backward control system as the control domain is convex. Since BSDEs and forward-backward stochastic differential equations (FBSDEs) are involved in a broad range of applications in mathematical finance, economics, and so on, it is natural to study the control problems involving FBSDEs. To establish the necessary optimality conditions, Pontryagin maximum principle is one fundamental research direction for optimal control problems. Rich literature for stochastic maximum principle has been obtained; see [8–12] and the references therein. Recently, Wu [13] established the general maximum principle for optimal controls of forward-backward stochastic systems in which the control domains were nonconvex and forward diffusion coefficients explicitly depended on control variables.

The applications of regime-switching models in finance and stochastic control also have been researched in recent years. Compared to the traditional system based on the diffusion processes, it is more meaningful from the empirical point of view. Specifically, it modulates the system with a continuous-time finite-state Markov chain with each state representing a regime of the system or a level of economic indicator. Based on the switching diffusion model, much work has been done in the fields of option pricing, portfolio management, risk management, and so on. In [14], Crépey focused on the pricing equations in finance. Crépey and

Matoussi [15] investigated the reflected BSDEs with Markov chains. For the controlled problem with regime-switching model, Donnelly studied the sufficient maximum principle in [16]. Using the results about BSDEs with Markov chains in [14, 15], Tao and Wu [17] derived the maximum principle for the forward-backward regime-switching model. Moreover, in [18] the weak convergence of BSDEs with regime switching was studied. For more results of Markov chains, readers can refer to the references therein.

In addition, stochastic impulse control problems have received considerable research attention due to their wide applications in portfolio optimization problems with transaction costs (see [19, 20]) and optimal strategy of exchange rates between different currencies [21, 22]. Korn [23] also investigated some applications of impulse control in mathematical finance. For a comprehensive survey of theory of impulse controls, one is referred to [24]. Wu and Zhang [25] first studied stochastic optimal control problems of forward-backward systems involving impulse controls, in which they assumed the domain of the regular controls was convex and obtained both the maximum principle and sufficient optimality conditions. Later on, in [26] they considered the forward-backward system in which the domain of regular controls was not necessarily convex and the control variable did not enter the diffusion coefficient.

In this paper, we consider a stochastic control system, in which the control system is described by a forward-backward stochastic differential equation, all the coefficients contain Markov chains, and the control variables consist of regular and impulsive parts. This case is more complicated than [17, 25, 26]. We obtain the stochastic maximum principle by using spike variation on the regular control and convex perturbation on the impulsive one. Applying the maximum principle to a financial investment-consumption model, we also get the optimal consumption processes and analyze the effects on consumption by various economic factors.

The rest of this paper is organized as follows. In Section 2, we give preliminaries and the formulation of our problems. A necessary condition in the form of maximum principle is established in Section 3. Section 4 aims to investigate sufficient optimality conditions. An example in finance is studied in Section 5 to illustrate the applications of our theoretical results and some figures are presented to give more explanations. In the end, Section 6 concludes the novelty of this paper.

2. Preliminaries and Problem Formulation

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, P)$ be a complete filtered probability space equipped with a natural filtration $\mathcal{F}_t$ generated by $\{B_s, \alpha_s;\ 0 \le s \le t\}$, $t \in [0,T]$, where $\{B_t\}_{0\le t\le T}$ is a d-dimensional standard Brownian motion defined on the space, $\{\alpha_t,\ 0 \le t \le T\}$ is a finite-state Markov chain with the state space given by $I = \{1, 2, \dots, k\}$, and $T \ge 0$ is a fixed time horizon. The transition intensities are $\lambda(i,j)$ for $i \ne j$ with $\lambda(i,j)$ nonnegative and bounded. $\lambda(i,i) = -\sum_{j\in I\setminus\{i\}} \lambda(i,j)$. For $p \ge 1$, denote by $S^p(\mathbb{R}^n)$ the set of n-dimensional adapted processes $\{\varphi_t,\ 0 \le t \le T\}$ such that $\mathbb{E}[\sup_{0\le t\le T}|\varphi_t|^p] < +\infty$ and denote by $H^p(\mathbb{R}^n)$ the set of n-dimensional adapted processes $\{\psi_t,\ 0 \le t \le T\}$ such that $\mathbb{E}[(\int_0^T |\psi_t|^2 dt)^{p/2}] < +\infty$.

Define $\mathcal{V}$ as the integer-valued random measure on $([0,T]\times I, \mathcal{B}([0,T])\otimes\mathcal{B}_I)$ which counts the jumps $\mathcal{V}_t(j)$ from α to state j between time 0 and t. The compensator of $\mathcal{V}_t(j)$ is $\mathbf{1}_{\{\alpha_t\ne j\}}\lambda(\alpha_t, j)dt$, which means $d\mathcal{V}_t(j) - \mathbf{1}_{\{\alpha_t\ne j\}}\lambda(\alpha_t, j)dt := d\widetilde{\mathcal{V}}_t(j)$ is a martingale (compensated measure). Then the canonical special semimartingale representation for α is given by

$$d\alpha_t = \sum_{j\in I}\lambda(\alpha_t, j)(j - \alpha_t)\,dt + \sum_{j\in I}(j - \alpha_{t-})\,d\widetilde{\mathcal{V}}_t(j). \tag{1}$$

Define $n_t(j) := \mathbf{1}_{\{\alpha_t\ne j\}}\lambda(\alpha_t, j)$. Denote by $\mathcal{M}_\rho$ the set of measurable functions from $(I, \mathcal{B}_I, \rho)$ to $\mathbb{R}$ endowed with the topology of convergence in measure and $|v|_t := \sum_{j\in I}[v(j)^2 n_t(j)]^{1/2} \in \mathbb{R}_+ \cup \{+\infty\}$ the norm of $\mathcal{M}_\rho$; denote by $H^p_{\mathcal{V}}$ the space of $\widetilde{P}$-measurable functions $V : \Omega\times[0,T]\times I \to \mathbb{R}$ such that $\sum_{j\in I}\mathbb{E}[(\int_0^T V_t(j)^2 n_t(j)dt)^{p/2}] < +\infty$.

Let U be a nonempty subset of $\mathbb{R}^k$ and K nonempty convex subset of $\mathbb{R}^n$. Let $\{\tau_i\}$ be a given sequence of increasing $\mathcal{F}_t$-stopping times such that $\tau_i \uparrow +\infty$ as $i \to +\infty$. Denote by $\mathcal{I}$ the class of right continuous processes $\eta(\cdot) = \sum_{i\ge1}\eta_i\mathbf{1}_{[\tau_i,T]}(\cdot)$ such that each η_i is an $\mathcal{F}_{\tau_i}$-measurable random variable. It's worth noting that, the assumption $\tau_i \uparrow +\infty$ implies that at most finitely many impulses may occur on $[0,T]$. Denote by $\mathcal{U}$ the class of adapted processes $v : [0,T]\times\Omega \to U$ such that $\mathbb{E}[\sup_{0\le t\le T}|v_t|^3] < +\infty$ and denote by $\mathcal{K}$ the class of K-valued impulse processes $\eta \in \mathcal{I}$ such that $\mathbb{E}[(\sum_{i\ge1}|\eta_i|)^3] < +\infty$. $\mathcal{A} := \mathcal{U}\times\mathcal{K}$ is called the admissible control set. For notational simplicity, in what follows we focus on the case where all processes are 1-dimensional.

Now we consider the forward regime-switching systems modulated by continuous-time, finite-state Markov chains involving impulse controls. Let $b : [0,T]\times I\times\mathbb{R} \to \mathbb{R}$, $\sigma : [0,T]\times I\times\mathbb{R} \to \mathbb{R}$, and $C : [0,T] \to \mathbb{R}$ be measurable mappings. Given $x \in \mathbb{R}$ and $\eta(\cdot) \in \mathcal{K}$, the system is formulated by

$$\begin{aligned} dx_t &= b(t, \alpha_t, x_t)\,dt + \sigma(t, \alpha_t, x_t)\,dB_t + C_t d\eta_t, \\ x_0 &= x. \end{aligned} \tag{2}$$

The following result is easily obtained.

Proposition 1. *Assume that b, σ are Lipschitz with respect to x, $b(\cdot, i, 0)$, $\sigma(\cdot, i, 0) \in H^3(\mathbb{R})$, $\forall i \in I$, and C is a continuous function. Then SDE (2) admits a unique solution $x(\cdot) \in S^3(\mathbb{R})$.*

Given $\zeta \in L^3(\Omega, \mathcal{F}_T, P; \mathbb{R})$ and $\eta(\cdot) \in \mathcal{K}$, consider the following backward regime-switching system modulated by Markov chains α_t involving impulse controls:

$$\begin{aligned} dy_t &= -f(t, \alpha_t, y_t, z_t, W_t(1)n_t(1), \dots, W_t(k)n_t(k))\,dt \\ &\quad + z_t dB_t + \sum_{j\in I} W_t(j)\,d\widetilde{\mathcal{V}}_t(j) - D_t d\eta_t, \\ y_T &= \zeta, \end{aligned} \tag{3}$$

where $f : [0,T] \times I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ and $D : [0,T] \to \mathbb{R}$ are measurable mappings and $W : \Omega \times [0,T] \times I \to \mathbb{R}$ is a measurable function such that $\sum_{j\in I} \mathbb{E}[(\int_0^T W_t(j)^2 n_t(j)dt)^{3/2}] < +\infty$.

Proposition 2. *Assume that $f(t,i,y,z,p)$ is Lipschitz with respect to (y,z,p), $f(\cdot,i,0,0,0) \in H^3(\mathbb{R})$, $\forall i \in I$, and D is a continuous function. Then BSDE* (3) *admits a unique solution $(y(\cdot),z(\cdot),W(\cdot)) \in S^3(\mathbb{R}) \times H^3(\mathbb{R}) \times H^3_{\mathscr{V}}(\mathbb{R})$.*

Proof. Define $A_t := \int_0^t D_s d\eta_s = \sum_{\tau_i \le t} D_{\tau_i}\eta_i$ and $F(t,i,y,z,p) := f(t,i,y-A_t,z,p)$, $\forall i \in I$. It is easy to check that

$$\begin{aligned} &\left|F(t,i,y,z,p) - F(t,i,y',z',p')\right| \\ &\quad \le c_1\left(\left|y-y'\right| + \left|z-z'\right| + \left|p-p'\right|\right). \end{aligned} \tag{4}$$

Since n_t is uniformly bounded, we have

$$\left|\left(W_t(j) - W'_t(j)\right)n_t(j)\right| \le c_2\left|W_t - W'_t\right|_t, \quad \forall j \in I. \tag{5}$$

Here c_1, c_2 are positive constants. Then F is Lipschitz with respect to (y,z,W). We also get that $F(\cdot,i,0,0,0) \in H^3(\mathbb{R})$ and $\mathbb{E}|\zeta + A_T|^3 < +\infty$. Hence, the following BSDE

$$\begin{aligned} dY_t = &-F(t,\alpha_t,Y_t,Z_t,M_t(1)n_t(1),\dots,M_t(k)n_t(k))\,dt \\ &+ Z_t dB_t + \sum_{j\in I} M_t(j)\,d\widetilde{\mathscr{V}}_t(j), \\ &Y_T = \zeta + A_T \end{aligned} \tag{6}$$

admits a unique solution $(Y,Z,M) \in S^3(\mathbb{R}) \times H^3(\mathbb{R}) \times H^3_{\mathscr{V}}(\mathbb{R})$ (see [15, 18] for details). Now define $y_t := Y_t - A_t$, $z_t := Z_t$, and $W_t := M_t$. Then it is easy to check that $(y,z,W) \in S^3(\mathbb{R}) \times H^3(\mathbb{R}) \times H^3_{\mathscr{V}}(\mathbb{R})$ solves BSDE (3).

Let (y^1,z^1,W^1) and (y^2,z^2,W^2) be two solutions of (3). Applying Itô's formula to $(y_s^1 - y_s^2)^2$, $t \le s \le T$ and combining Gronwall's inequality, we get the uniqueness of solution. □

Now, we consider the following stochastic control system:

$$\begin{aligned} dx_t &= b(t,\alpha_t,x_t,v_t)\,dt + \sigma(t,\alpha_t,x_t)\,dB_t + C_t d\eta_t, \\ dy_t & \\ &= -f(t,\alpha_t,x_t,y_t,z_t,W_t(1)n_t(1),\dots,W_t(k)n_t(k),v_t)\,dt \\ &\quad + z_t dB_t + \sum_{j\in I} W_t(j)\,d\widetilde{\mathscr{V}}_t(j) - D_t d\eta_t, \\ &\quad x_0 = x, \qquad y_T = g(x_T), \end{aligned} \tag{7}$$

where $b : [0,T] \times I \times \mathbb{R} \times U \to \mathbb{R}$, $\sigma : [0,T] \times I \times \mathbb{R} \to \mathbb{R}$, $f : [0,T] \times I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^k \times U \to \mathbb{R}$, and $g : \mathbb{R} \to \mathbb{R}$ are deterministic measurable functions and $C : [0,T] \to \mathbb{R}$, $D : [0,T] \to \mathbb{R}$ are continuous functions. In what follows $(W_t(1)n_t(1),\dots,W_t(k)n_t(k))$ will be written as $W_t n_t$ for short. The objective is to maximize, over class $\mathscr{A}$, the cost functional

$$\begin{aligned} J(v(\cdot),\eta(\cdot)) = \mathbb{E}\Bigg\{&\int_0^T h(t,\alpha_t,x_t,y_t,z_t,W_t n_t,v_t)\,dt + \phi(x_T) \\ &+ \gamma(y_0) + \sum_{i\ge 1} l(\tau_i,\eta_i)\Bigg\}, \end{aligned} \tag{8}$$

where $h : [0,T] \times I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^k \times U \to \mathbb{R}$, $\phi : \mathbb{R} \to \mathbb{R}$, $\gamma : \mathbb{R} \to \mathbb{R}$, and $l : [0,T] \times \mathbb{R} \to \mathbb{R}$ are deterministic measurable functions. A control (u,ξ) which solves this problem is called an optimal control.

In what follows, we make the following assumptions.

(H1) b, σ, f, g, h, ϕ, and γ are continuous and continuously differentiable with respect to (x,y,z,p). b, f have linear growth with respect to (x,y,v). l is continuous and continuously differentiable with respect to η.

(H2) The derivatives of b, σ, f, and g are bounded.

(H3) The derivatives of h, ϕ, γ, and l are bounded by $K(1 + |x| + |y| + |z| + |p| + |v|)$, $K(1+|x|)$, $K(1+|y|)$, and $K(1+|\eta|)$, respectively. Moreover, $|h(t,i,0,0,0,0,v)| \le K(1+|v|^3)$ for any (t,v), $i \in I$.

From Propositions 1 and 2, it follows that, under (H1)–(H3), FBSDE (7) admits a unique solution $(x(\cdot),y(\cdot),z(\cdot),W(\cdot)) \in S^3(\mathbb{R}) \times S^3(\mathbb{R}) \times H^3(\mathbb{R}) \times H^3_{\mathscr{V}}(\mathbb{R})$ for any $(v,\eta) \in \mathscr{A}$.

3. Stochastic Maximum Principle

In this section, we will derive the stochastic maximum principle for optimal control problem (7) and (8). We give the necessary conditions for optimal controls.

Let $\xi(\cdot) = \sum_{i\ge 1} \xi_i \mathbf{1}_{[\tau_i,T]}(\cdot)$ and $(u(\cdot),\xi(\cdot)) \in \mathscr{A}$ be an optimal control of this stochastic control problem and let $(x(\cdot),y(\cdot),z(\cdot),W(\cdot))$ be the corresponding trajectory. Now, we introduce the spike variation with respect to $u(\cdot)$ as follows:

$$u^\varepsilon(t) = \begin{cases} v, & \text{if } \tau \le t \le \tau+\varepsilon, \\ u(t), & \text{otherwise}, \end{cases} \tag{9}$$

where $\tau \in [0,T)$ is an arbitrarily fixed time, $\varepsilon > 0$ is a sufficiently small constant, and v is an arbitrary U-valued $\mathscr{F}_\tau$-measurable random variable such that $\mathbb{E}|v|^3 < +\infty$. Let $\eta \in \mathscr{I}$ be such that $\xi + \eta \in \mathscr{K}$. For the reason that domain K is convex, we can check that $\xi^\varepsilon := \xi + \varepsilon\eta$, $0 \le \varepsilon \le 1$, is also an element of $\mathscr{K}$. Let $(x^\varepsilon(\cdot),y^\varepsilon(\cdot),z^\varepsilon(\cdot),W^\varepsilon(\cdot))$ be the trajectory corresponding to $(u^\varepsilon(\cdot),\xi^\varepsilon(\cdot))$. For convenience, we denote $\psi(t) = \psi(t,\alpha_t,x_t,y_t,z_t,W_t n_t,u_t)$, $\psi(u_t^\varepsilon) = \psi(t,\alpha_t,x_t,y_t,z_t,W_t n_t,u_t^\varepsilon)$ for $\psi = b,\sigma,f,h,b_x,b_v,\sigma_x,\sigma_v,f_x,f_y,f_z,f_{w(j)},f_v,h_x,h_y,h_z,h_{w(j)},h_v$, where $f_{w(j)} := f_{W(j)n(j)}$, $h_{w(j)} := h_{W(j)n(j)}$.

Introduce the following FBSDE which is called the variational equation:

$$\begin{aligned} dx_t^1 &= \left[b_x(t)x_t^1 + b(u_t^\varepsilon) - b(t)\right]dt + \sigma_x(t)x_t^1 dB_t \\ &\quad + \varepsilon C_t d\eta_t, \\ dy_t^1 &= -\Bigg[f_x(t)x_t^1 + f_y(t)y_t^1 + f_z(t)z_t^1 \\ &\quad + \sum_{j\in I} f_{w(j)}(t)P_t(j)n_t(j) + f(u_t^\varepsilon) - f(t)\Bigg]dt \\ &\quad + z_t^1 dB_t + \sum_{j\in I} P_t(j)\,d\widetilde{\mathscr{V}}_t(j) - \varepsilon D_t d\eta_t, \\ x_0^1 &= 0, \qquad y_T^1 = g_x(x_T)x_T^1. \end{aligned} \tag{10}$$

Obviously, this FBSDE admits a unique solution $(x^1, y^1, z^1, P) \in S^3(\mathbb{R}) \times S^3(\mathbb{R}) \times H^3(\mathbb{R}) \times H^3_{\mathscr{V}}(\mathbb{R})$.

We have the following lemma. In what follows, we denote by c a positive constant which can be different from line to line.

Lemma 3. *Consider*

$$\sup_{0\le t\le T} \mathbb{E}\left|x_t^1\right|^3 \le c\varepsilon^3, \tag{11}$$

$$\begin{aligned} &\sup_{0\le t\le T} \mathbb{E}\left|y_t^1\right|^3 + \mathbb{E}\left[\left(\int_0^T \left|z_t^1\right|^2 dt\right)^{3/2}\right] \\ &\quad + \sum_{j\in I}\mathbb{E}\left[\left(\int_0^T \left|P_t(j)\right|^2 n_t(j)\,dt\right)^{3/2}\right] \le c\varepsilon^3. \end{aligned} \tag{12}$$

Proof. By the boundedness of (b_x, σ_x) and using Hölder's inequality, we have

$$\begin{aligned} &\sup_{0\le t\le r} \mathbb{E}\left|x_t^1\right|^3 \\ &\quad \le c\int_0^r \left[\sup_{0\le s\le t} \mathbb{E}\left|x_s^1\right|^3\right] dt \\ &\quad + c\mathbb{E}\left(\int_0^T \left|b(u_t^\varepsilon) - b(t)\right| dt\right)^3 + c\varepsilon^3\mathbb{E}\left(\int_0^T \left|C_t\right| d\eta_t\right)^3, \end{aligned} \tag{13}$$

$\forall 0 \le r \le T$. Noting the definition of $u^\varepsilon(\cdot)$, we get

$$\begin{aligned} &\mathbb{E}\left(\int_0^T \left|b(u_t^\varepsilon) - b(t)\right| dt\right)^3 \\ &\quad = \mathbb{E}\left(\int_\tau^{\tau+\varepsilon} \left|b(t, \alpha_t, x_t, v) - b(t)\right| dt\right)^3 \\ &\quad \le \varepsilon^2\mathbb{E}\int_\tau^{\tau+\varepsilon} \left|b(t, \alpha_t, x_t, v) - b(t)\right|^3 dt \\ &\quad \le c\varepsilon^3\left(1 + \sup_{0\le t\le T}\mathbb{E}\left[\left|x_t\right|^3 + \left|u_t\right|^3 + |v|^3\right]\right) \\ &\quad \le c\varepsilon^3. \end{aligned} \tag{14}$$

Here we apply Hölder's inequality for $p = 3$, $q = 3/2$, and the growth condition of b in (H1). Since C_t is bounded on $[0, T]$, then (11) is obtained by applying Gronwall's inequality.

By the result of Section 5 in [6] and noting that the predictable covariation of $\widetilde{\mathscr{V}}_t(j)$ is

$$d\left\langle \widetilde{\mathscr{V}}_t(j), \widetilde{\mathscr{V}}_t(j)\right\rangle_t = n_t(j)\,dt, \tag{15}$$

we obtain

$$\begin{aligned} &\sup_{0\le t\le T} \mathbb{E}\left|y_t^1\right|^3 + \mathbb{E}\left[\left(\int_0^T \left|z_t^1\right|^2 dt\right)^{3/2}\right] \\ &\quad + \sum_{j\in I}\mathbb{E}\left[\left(\int_0^T \left|P_t(j)\right|^2 n_t(j)\,dt\right)^{3/2}\right] \\ &\quad \le c\mathbb{E}\left|g_x(x_T)x_T^1\right|^3 + c\mathbb{E}\left(\int_0^T \left|f_x(t)x_t^1 + f(u_t^\varepsilon) - f(t)\right| dt\right)^3 \\ &\quad + c\varepsilon^3\mathbb{E}\left(\int_0^T \left|D_t\right| d\eta_t\right)^3. \end{aligned} \tag{16}$$

On the one hand, since g_x is bounded, by (11), we have

$$\mathbb{E}\left|g_x(x_T)x_T^1\right|^3 \le c\varepsilon^3. \tag{17}$$

On the other hand, since f_x is bounded, using the basic inequality and (11), we have

$$\begin{aligned} &\mathbb{E}\left(\int_0^T \left|f_x(t)x_t^1 + f(u_t^\varepsilon) - f(t)\right| dt\right)^3 \\ &\quad \le c\varepsilon^3 + c\mathbb{E}\left(\int_0^T \left|f(u_t^\varepsilon) - f(t)\right| dt\right)^3. \end{aligned} \tag{18}$$

From the growth condition of f in (H1) and the same technique as above, it follows that

$$\mathbb{E}\left(\int_0^T \left|f(u_t^\varepsilon) - f(t)\right| dt\right)^3 \le c\varepsilon^3. \tag{19}$$

Besides, D_t is bounded on $[0, T]$; then (12) is obtained. The proof is complete. □

Denote $\widehat{x}_t = x_t^\varepsilon - x_t - x_t^1$, $\widehat{y}_t = y_t^\varepsilon - y_t - y_t^1$, $\widehat{z}_t = z_t^\varepsilon - z_t - z_t^1$, and $\widehat{W}_t = W_t^\varepsilon - W_t - P_t$, and then we have the following.

Lemma 4. *Consider*

$$\sup_{0\le t\le T}\mathbb{E}\left|\hat{x}_t\right|^2 \le C_\varepsilon \varepsilon^2, \quad (20)$$

$$\sup_{0\le t\le T}\mathbb{E}\left|\hat{y}_t\right|^2 + \mathbb{E}\left[\int_0^T \left|\hat{z}_t\right|^2 dt\right] + \sum_{j\in I}\mathbb{E}\left[\int_0^T \left|\widehat{W}_t(j)\right|^2 n_t(j)\,dt\right] \le C_\varepsilon \varepsilon^2, \quad (21)$$

where $C_\varepsilon \to 0$ *as* $\varepsilon \to 0$.

Proof. It is easy to check that $\hat{x}$ satisfies

$$\begin{aligned} d\hat{x}_t &= \left[\Lambda_1(t) + \Lambda_2(t)\right]dt + \left[\Xi_1(t) + \Xi_2(t)\right]dB_t, \\ \hat{x}_0 &= 0, \end{aligned} \quad (22)$$

where

$$\begin{aligned} \Lambda_1(t) &:= b\left(t,\alpha_t,x_t^\varepsilon,u_t^\varepsilon\right) - b\left(t,\alpha_t,x_t + x_t^1,u_t^\varepsilon\right), \\ \Lambda_2(t) &:= b\left(t,\alpha_t,x_t + x_t^1,u_t^\varepsilon\right) - b\left(u_t^\varepsilon\right) - b_x(t)\,x_t^1, \\ \Xi_1(t) &:= \sigma\left(t,\alpha_t,x_t^\varepsilon\right) - \sigma\left(t,\alpha_t,x_t + x_t^1\right), \\ \Xi_2(t) &:= \sigma\left(t,\alpha_t,x_t + x_t^1\right) - \sigma(t) - \sigma_x(t)\,x_t^1. \end{aligned} \quad (23)$$

Then we have

$$\begin{aligned} \sup_{0\le t\le r}\mathbb{E}\left|\hat{x}_t\right|^2 \le{}& c\mathbb{E}\left(\int_0^r \left|\Lambda_1(t) + \Lambda_2(t)\right|dt\right)^2 \\ &+ c\mathbb{E}\int_0^r \left|\Xi_1(t) + \Xi_2(t)\right|^2 dt \end{aligned} \quad (24)$$

$\forall 0 \le r \le T$. Since $\Lambda_1(t) = \int_0^1 b_x(t,\alpha_t,x_t + x_t^1 + \lambda\hat{x}_t,u_t^\varepsilon)\,d\lambda\,\hat{x}_t$, by the boundedness of b_x, we have $|\Lambda_1(t)| \le c|\hat{x}_t|$. Further we get

$$\mathbb{E}\left(\int_0^r \left|\Lambda_1(t)\right|dt\right)^2 \le c\mathbb{E}\int_0^r \left|\hat{x}_t\right|^2 dt. \quad (25)$$

On the other hand, since $\Lambda_2(t) = \int_0^1 [b_x(t,\alpha_t,x_t + \lambda x_t^1,u_t^\varepsilon) - b_x(t)]\,d\lambda\,x_t^1$, we have

$$\int_0^r \left|\Lambda_2(t)\right|dt \le \int_0^T \left|\Lambda_2(t)\right|dt \le I_1 + I_2, \quad (26)$$

where

$$\begin{aligned} I_1 &:= \int_\tau^{\tau+\varepsilon}\left|\int_0^1 \left[b_x\left(t,\alpha_t,x_t + \lambda x_t^1,v\right) - b_x(t)\right]d\lambda\,x_t^1\right|dt, \\ I_2 &:= \int_0^T\left|\int_0^1 \left[b_x\left(t,\alpha_t,x_t + \lambda x_t^1,u_t\right) - b_x(t)\right]d\lambda\,x_t^1\right|dt. \end{aligned} \quad (27)$$

Since b_x is bounded, by Lemma 3 we get

$$\begin{aligned} &\mathbb{E}\left|I_1\right|^2 \\ &\le \varepsilon\int_\tau^{\tau+\varepsilon}\mathbb{E}\left[\left|\int_0^1 \left[b_x\left(t,\alpha_t,x_t + \lambda x_t^1,v\right) - b_x(t)\right]d\lambda\,x_t^1\right|^2\right]dt \\ &\le c\varepsilon^2\sup_{0\le t\le T}\mathbb{E}\left|x_t^1\right|^2 \\ &\le c\varepsilon^4. \end{aligned} \quad (28)$$

For I_2, by Hölder's inequality, Lemma 3, and the dominated convergence theorem, it follows that

$$\begin{aligned} &\mathbb{E}\left|I_2\right|^2 \\ &\le \mathbb{E}\left\{\int_0^T\left|\int_0^1 \left[b_x\left(t,\alpha_t,x_t + \lambda x_t^1,u_t\right) - b_x(t)\right]d\lambda\right|^2 dt \right. \\ &\qquad \left. \cdot \int_0^T \left|x_t^1\right|^2 dt\right\} \\ &\le \left\{\mathbb{E}\left(\int_0^T\left|\int_0^1 \left[b_x\left(t,\alpha_t,x_t + \lambda x_t^1,u_t\right) - b_x(t)\right]d\lambda\right|^2 dt\right)^3\right\}^{1/3} \\ &\quad \times \left\{\mathbb{E}\left(\int_0^T \left|x_t^1\right|^2 dt\right)^{3/2}\right\}^{2/3} \\ &\le C_\varepsilon \varepsilon^2. \end{aligned} \quad (29)$$

Then we get

$$\mathbb{E}\left(\int_0^r \left|\Lambda_2(t)\right|dt\right)^2 \le 2\mathbb{E}\left(\left|I_1\right|^2 + \left|I_2\right|^2\right) \le C_\varepsilon \varepsilon^2 \quad (30)$$

and obtain

$$\mathbb{E}\left(\int_0^r \left|\Lambda_1(t) + \Lambda_2(t)\right|dt\right)^2 \le C_\varepsilon \varepsilon^2 + c\mathbb{E}\int_0^r \left|\hat{x}_t\right|^2 dt. \quad (31)$$

In the same way, we have

$$\mathbb{E}\int_0^r \left|\Xi_1(t) + \Xi_2(t)\right|^2 dt \le C_\varepsilon \varepsilon^2 + c\mathbb{E}\int_0^r \left|\hat{x}_t\right|^2 dt. \quad (32)$$

From (24), (31), and (32) it follows that

$$\sup_{0\le t\le r}\mathbb{E}\left|\hat{x}_t\right|^2 \le C_\varepsilon \varepsilon^2 + c\int_0^r \left[\sup_{0\le s\le t}\mathbb{E}\left|\hat{x}_s\right|^2\right]dt. \quad (33)$$

Finally, applying Gronwall's inequality implies (20). □

To get estimate (21), for simplicity, we introduce

$$\begin{aligned} \Theta_t &= \left(t,\alpha_t,x_t + \lambda x_t^1,y_t + \lambda y_t^1,z_t + \lambda z_t^1,\left(W_t + \lambda P_t\right)n_t\right), \\ \Sigma_t &= \left(t,\alpha_t,x_t + x_t^1 + \lambda\hat{x}_t,y_t + y_t^1 + \lambda\hat{y}_t,z_t + z_t^1 + \lambda\hat{z}_t,\right. \\ &\qquad \left.\left(W_t + P_t + \lambda\widehat{W}_t\right)n_t\right). \end{aligned} \quad (34)$$

It is easy to check that $(\hat{y}, \hat{z}, \widehat{W})$ satisfies

$$d\hat{y}_t = -\left[f_1(t) + f_2(t)\right]dt + \hat{z}_t dB_t + \sum_{j\in I}\widehat{W}_t(j)\,d\widetilde{\mathscr{V}}_t(j),$$
$$\hat{y}_T = G_1 + G_2, \tag{35}$$

where

$$\begin{aligned} f_1(t) &:= f\left(t, \alpha_t, x_t^\varepsilon, y_t^\varepsilon, z_t^\varepsilon, W_t^\varepsilon n_t, u_t^\varepsilon\right) \\ &\quad - f\left(t, \alpha_t, x_t + x_t^1, y_t + y_t^1, z_t + z_t^1, \left(W_t + P_t\right)n_t, u_t^\varepsilon\right), \\ f_2(t) &:= f\left(t, \alpha_t, x_t + x_t^1, y_t + y_t^1, z_t + z_t^1, \left(W_t + P_t\right)n_t, u_t^\varepsilon\right) \\ &\quad - f\left(u_t^\varepsilon\right) - f_x(t)\,x_t^1 - f_y(t)\,y_t^1 - f_z(t)\,z_t^1 \\ &\quad - \sum_{j\in I} f_{w(j)}(t)\,P_t(j)\,n_t(j), \\ G_1 &:= g\left(x_T^\varepsilon\right) - g\left(x_T + x_T^1\right), \\ G_2 &:= g\left(x_T + x_T^1\right) - g\left(x_T\right) - g_x\left(x_T\right)x_T^1. \end{aligned} \tag{36}$$

Similar to the proof above, we have

$$\begin{aligned} f_1(t) &= \int_0^1 f_x\left(\Sigma_t, u_t^\varepsilon\right)d\lambda\,\hat{x}_t + \int_0^1 f_y\left(\Sigma_t, u_t^\varepsilon\right)d\lambda\,\hat{y}_t \\ &\quad + \int_0^1 f_z\left(\Sigma_t, u_t^\varepsilon\right)d\lambda\,\hat{z}_t \\ &\quad + \sum_{j\in I}\int_0^1 f_{w(j)}\left(\Sigma_t, u_t^\varepsilon\right)d\lambda\,\widehat{W}_t(j)\,n_t(j), \\ f_2(t) &= \int_0^1 \left[f_x\left(\Theta_t, u_t^\varepsilon\right) - f_x(t)\right]d\lambda\,x_t^1 \\ &\quad + \int_0^1 \left[f_y\left(\Theta_t, u_t^\varepsilon\right) - f_y(t)\right]d\lambda\,y_t^1 \\ &\quad + \int_0^1 \left[f_z\left(\Theta_t, u_t^\varepsilon\right) - f_z(t)\right]d\lambda\,z_t^1 \\ &\quad + \sum_{j\in I}\int_0^1 \left[f_{w(j)}\left(\Theta_t, u_t^\varepsilon\right) - f_{w(j)}(t)\right]d\lambda\,P_t(j)\,n_t(j), \\ G_1 &= \int_0^1 g_x\left(x_T + x_T^1 + \lambda\hat{x}_T\right)d\lambda\,\hat{x}_T, \\ G_2 &= \int_0^1 \left[g_x\left(x_T + \lambda x_T^1\right) - g_x\left(x_T\right)\right]d\lambda\,x_T^1. \end{aligned} \tag{37}$$

Then for BSDE (35), by the estimates of BSDEs, we obtain

$$\begin{aligned} &\sup_{0\le t\le T}\mathbb{E}\left|\hat{y}_t\right|^2 + \mathbb{E}\left[\int_0^T \left|\hat{z}_t\right|^2 dt\right] + \sum_{j\in I}\mathbb{E}\left[\int_0^T \left|\widehat{W}_t(j)\right|^2 n_t(j)\,dt\right] \\ &\le c\mathbb{E}\Bigg[\left|\int_0^1 g_x\left(x_T + x_T^1 + \lambda\hat{x}_T\right)d\lambda\,\hat{x}_T\right|^2 \\ &\quad + \left|\int_0^1 \left[g_x\left(x_T + \lambda x_T^1\right) - g_x\left(x_T\right)\right]d\lambda\,x_T^1\right|^2 \\ &\quad + \left(\int_0^T \left|\int_0^1 f_x\left(\Sigma_t, u_t^\varepsilon\right)d\lambda\,\hat{x}_t\right|dt\right)^2 \\ &\quad + \left(\int_0^T \left|\int_0^1 \left[f_x\left(\Theta_t, u_t^\varepsilon\right) - f_x(t)\right]d\lambda\,x_t^1\right|dt\right)^2 \\ &\quad + \left(\int_0^T \left|\int_0^1 \left[f_y\left(\Theta_t, u_t^\varepsilon\right) - f_y(t)\right]d\lambda\,y_t^1\right|dt\right)^2 \\ &\quad + \left(\int_0^T \left|\int_0^1 \left[f_z\left(\Theta_t, u_t^\varepsilon\right) - f_z(t)\right]d\lambda\,z_t^1\right|dt\right)^2 \\ &\quad + \left(\int_0^T \left|\sum_{j\in I}\int_0^1 \left[f_{w(j)}\left(\Theta_t, u_t^\varepsilon\right) - f_{w(j)}(t)\right]d\lambda\,P_t(j) \times n_t(j)\right|dt\right)^2\Bigg]. \end{aligned} \tag{38}$$

Applying Hölder's inequality, Cauchy-Schwartz inequality, the dominated convergence theorem, Lemma 3, and (20) and noting the boundedness of n_t, we obtain (21).

Now, we are ready to state the variational inequality.

Lemma 5. *The following variational inequality holds:*

$$\begin{aligned} &\mathbb{E}\left[\int_0^T \left(h_x(t)\,x_t^1 + h_y(t)\,y_t^1 + h_z(t)\,z_t^1 + \sum_{j\in I} h_{w(j)}(t)\,P_t(j)\,n_t(j) + h\left(u_t^\varepsilon\right) - h(t)\right)dt\right] \\ &\quad + \mathbb{E}\left[\phi_x\left(x_T\right)x_T^1 + \gamma_y\left(y_0\right)y_0^1 + \varepsilon\sum_{i\ge 1} l_\xi\left(\tau_i, \xi_i\right)\eta_i\right] \le o(\varepsilon). \end{aligned} \tag{39}$$

Proof. From the optimality of $(u(\cdot), \xi(\cdot))$, we have

$$J\left(u^\varepsilon(\cdot), \xi^\varepsilon(\cdot)\right) - J\left(u(\cdot), \xi(\cdot)\right) \le 0. \tag{40}$$

By Lemmas 3 and 4, we have

$$\begin{aligned}
&\mathbb{E}\left[\phi\left(x_T^{\varepsilon}\right)-\phi\left(x_T\right)\right] \\
&\quad=\mathbb{E}\left[\phi\left(x_T^{\varepsilon}\right)-\phi\left(x_T+x_T^1\right)\right]+\mathbb{E}\left[\phi\left(x_T+x_T^1\right)-\phi\left(x_T\right)\right] \\
&\quad=\mathbb{E}\left[\phi_x\left(x_T\right)x_T^1\right]+o\left(\varepsilon\right), \\
&\mathbb{E}\left[\gamma\left(y_0^{\varepsilon}\right)-\gamma\left(y_0\right)\right] \\
&\quad=\mathbb{E}\left[\gamma\left(y_0^{\varepsilon}\right)-\gamma\left(y_0+y_0^1\right)\right]+\mathbb{E}\left[\gamma\left(y_0+y_0^1\right)-\gamma\left(y_0\right)\right] \\
&\quad=\mathbb{E}\left[\gamma_y\left(y_0\right)y_0^1\right]+o\left(\varepsilon\right).
\end{aligned} \tag{41}$$

Similarly, we obtain

$$\mathbb{E}\left[\sum_{i\geq 1}l\left(\tau_i,\xi_i^{\varepsilon}\right)-\sum_{i\geq 1}l\left(\tau_i,\xi_i\right)\right]=\varepsilon\mathbb{E}\left[\sum_{i\geq 1}l_{\xi}\left(\tau_i,\xi_i\right)\eta_i\right]+o\left(\varepsilon\right). \tag{42}$$

Next, we aim to get the first term of (39). For convenience, we introduce two notations as follows:

$$\begin{aligned}
H_1:=\mathbb{E}\Bigg[\int_0^T&\left(h\left(t,\alpha_t,x_t^{\varepsilon},y_t^{\varepsilon},z_t^{\varepsilon},W_t^{\varepsilon}n_t,u_t^{\varepsilon}\right)\right. \\
&-h\left(t,\alpha_t,x_t+x_t^1,y_t+y_t^1,z_t\right. \\
&\left.\left.+\,z_t^1,\left(W_t+P_t\right)n_t,u_t^{\varepsilon}\right)\right)dt\Bigg], \\
H_2:=\mathbb{E}\Bigg[\int_0^T&\Bigg(h\Bigg(t,\alpha_t,x_t+x_t^1,y_t+y_t^1, \\
&z_t+z_t^1,\left(W_t+P_t\right)n_t,u_t^{\varepsilon}\Bigg) \\
&-h\left(u_t^{\varepsilon}\right)-h_x\left(t\right)x_t^1-h_y\left(t\right)y_t^1 \\
&-\,h_z\left(t\right)z_t^1-\sum_{j\in I}h_{w(j)}\left(t\right)P_t\left(j\right)n_t\left(j\right)\Bigg)dt\Bigg].
\end{aligned} \tag{43}$$

Applying the same technique to the proof of Lemma 4, we obtain

$$H_1\sim H_2=o\left(\varepsilon\right). \tag{44}$$

Hence

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T\left(h\left(t,\alpha_t,x_t^{\varepsilon},y_t^{\varepsilon},z_t^{\varepsilon},W_t^{\varepsilon}n_t,u_t^{\varepsilon}\right)-h\left(t\right)\right)dt\right] \\
&\quad=\mathbb{E}\Bigg[\int_0^T\Bigg(h_x\left(t\right)x_t^1+h_y\left(t\right)y_t^1+h_z\left(t\right)z_t^1 \\
&\qquad+\sum_{j\in I}h_{w(j)}\left(t\right)P_t\left(j\right)n_t\left(j\right)+h\left(u_t^{\varepsilon}\right)-h\left(t\right)\Bigg)dt\Bigg] \\
&\quad+o\left(\varepsilon\right).
\end{aligned} \tag{45}$$

Thus, variational inequality (39) follows from (41)–(45).

Let us introduce the following adjoint equations:

$$\begin{aligned}
dp_t=&\left[f_y\left(t\right)p_t-h_y\left(t\right)\right]dt+\left[f_z\left(t\right)p_t-h_z\left(t\right)\right]dB_t \\
&+\sum_{j\in I}\left[f_{w(j)}\left(t-\right)p_{t-}-h_{w(j)}\left(t-\right)\right]d\widetilde{\mathscr{V}}_t\left(j\right),
\end{aligned} \tag{46}$$

$$p_0=-\gamma_y\left(y_0\right),$$

$$\begin{aligned}
-dq_t=&\left[b_x\left(t\right)q_t+\sigma_x\left(t\right)k_t-f_x\left(t\right)p_t+h_x\left(t\right)\right]dt \\
&-k_tdB_t-\sum_{j\in I}M_t\left(j\right)d\widetilde{\mathscr{V}}_t\left(j\right),
\end{aligned} \tag{47}$$

$$q_T=-g_x\left(x_T\right)p_T+\phi_x\left(x_T\right),$$

where $\varphi_{w(j)}(t-) := \varphi_{w(j)}(t,\alpha_{t-},x_{t-},y_{t-},z_t,W_tn_{t-},u_{t-})$ for $\varphi = f,h$. It is easy to check that SDE (46) admits a unique solution $p(\cdot) \in S^3(\mathbb{R})$. Besides, the generator of BSDE (47) does not contain $M_t(j)$. Therefore, the Lipschitz condition is satisfied obviously. Hence (47) admits a unique solution $(q(\cdot),k(\cdot),M(\cdot)) \in S^3(\mathbb{R})\times H^3(\mathbb{R})\times H^3_{\mathscr{V}}(\mathbb{R})$. Now we establish the stochastic maximum principle. □

Theorem 6. *Let assumptions (H1)–(H3) hold. Suppose $(u(\cdot),\xi(\cdot))$ is an optimal control, $(x(\cdot),y(\cdot),z(\cdot),W(\cdot))$ is the corresponding trajectory, and $(p(\cdot),q(\cdot),k(\cdot),M(\cdot))$ is the solution of adjoint equations (46) and (47). Then, $\forall v \in U, \eta(\cdot) \in \mathscr{K}$, it holds that*

$$\begin{aligned}
&H\left(t,\alpha_t,x_t,y_t,z_t,W_t,v,p_t,q_t,k_t\right) \\
&\quad-H\left(t,\alpha_t,x_t,y_t,z_t,W_t,u_t,p_t,q_t,k_t\right)\leq 0,\quad a.e.,\ a.s.,
\end{aligned} \tag{48}$$

$$\mathbb{E}\left[\sum_{i\geq 1}\left(l_{\xi}\left(\tau_i,\xi_i\right)+q_{\tau_i}C_{\tau_i}-p_{\tau_i}D_{\tau_i}\right)\left(\eta_i-\xi_i\right)\right]\leq 0, \tag{49}$$

where $H:[0,T]\times I\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathscr{M}_{\rho}\times U\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is the Hamiltonian defined by

$$\begin{aligned}
&H\left(t,\alpha_t,x,y,z,W,v,p,q,k\right) \\
&\quad=-f\left(t,\alpha_t,x,y,z,Wn_t,v\right)p+b\left(t,\alpha_t,x,v\right)q+\sigma\left(t,\alpha_t,x\right)k \\
&\qquad+h\left(t,\alpha_t,x,y,z,Wn_t,v\right),
\end{aligned} \tag{50}$$

where $Wn_t = (W(1)n_t(1),\ldots,W(k)n_t(k))$.

Proof. Applying Itô's formula to $p_t y_t^1 + q_t x_t^1$ and combining with Lemma 5, we obtain

$$\begin{aligned}&\mathbb{E}\left[\int_0^T \left(H\left(t,\alpha_t,x_t,y_t,z_t,W_t,u_t^{\varepsilon},p_t,q_t,k_t\right)\right.\right.\\&\qquad\left.\left.-H\left(t,\alpha_t,x_t,y_t,z_t,W_t,u_t,p_t,q_t,k_t\right)\right)dt\right]\\&\quad+\varepsilon\mathbb{E}\left[\sum_{i\geq 1}\left(l_\xi\left(\tau_i,\xi_i\right)+q_{\tau_i}C_{\tau_i}-p_{\tau_i}D_{\tau_i}\right)\theta_i\right]\leq o\left(\varepsilon\right),\end{aligned}\tag{51}$$

where $\theta \in \mathscr{I}$ such that $\xi + \theta = \eta \in \mathscr{K}$. Then it follows that

$$\begin{aligned}&\varepsilon^{-1}\mathbb{E}\left[\int_\tau^{\tau+\varepsilon} \left(H\left(t,\alpha_t,x_t,y_t,z_t,W_t,v,p_t,q_t,k_t\right)\right.\right.\\&\qquad\left.\left.-H\left(t,\alpha_t,x_t,y_t,z_t,W_t,u_t,p_t,q_t,k_t\right)\right)dt\right]\\&\quad+\mathbb{E}\left[\sum_{i\geq 1}\left(l_\xi\left(\tau_i,\xi_i\right)+q_{\tau_i}C_{\tau_i}-p_{\tau_i}D_{\tau_i}\right)\theta_i\right]\leq 0.\end{aligned}\tag{52}$$

Letting $\varepsilon \to 0$, we obtain

$$\begin{aligned}&\mathbb{E}\left[H\left(\tau,\alpha_\tau,x_\tau,y_\tau,z_\tau,W_\tau,v,p_\tau,q_\tau,k_\tau\right)\right.\\&\quad\left.-H\left(\tau,\alpha_\tau,x_\tau,y_\tau,z_\tau,W_\tau,u_\tau,p_\tau,q_\tau,k_\tau\right)\right]\\&\quad+\mathbb{E}\left[\sum_{i\geq 1}\left(l_\xi\left(\tau_i,\xi_i\right)+q_{\tau_i}C_{\tau_i}-p_{\tau_i}D_{\tau_i}\right)\theta_i\right]\leq 0,\\&\qquad\text{a.e. } \tau\in[0,T].\end{aligned}\tag{53}$$

By choosing $v = u_\tau$ we get (49). Setting $\eta \equiv \xi$, then for any $v \in \mathscr{F}_\tau$ we have

$$\begin{aligned}&\mathbb{E}\left[H\left(\tau,\alpha_\tau,x_\tau,y_\tau,z_\tau,W_\tau,v,p_\tau,q_\tau,k_\tau\right)\right.\\&\quad\left.-H\left(\tau,\alpha_\tau,x_\tau,y_\tau,z_\tau,W_\tau,u_\tau,p_\tau,q_\tau,k_\tau\right)\right]\leq 0,\quad \text{a.e.}\end{aligned}\tag{54}$$

Let $v_\tau = v\mathbf{1}_A + u_\tau\mathbf{1}_{A^c}$ for $A \in \mathscr{F}_\tau$ and $v \in U$. Obviously $v_\tau \in \mathscr{F}_\tau$ and $\mathbb{E}|v_\tau|^3 < +\infty$. Then it follows that for any $A \in \mathscr{F}_\tau$

$$\begin{aligned}&\mathbb{E}\left\{\left[H\left(\tau,\alpha_\tau,x_\tau,y_\tau,z_\tau,W_\tau,v,p_\tau,q_\tau,k_\tau\right)\right.\right.\\&\quad\left.\left.-H\left(\tau,\alpha_\tau,x_\tau,y_\tau,z_\tau,W_\tau,u_\tau,p_\tau,q_\tau,k_\tau\right)\right]\mathbf{1}_A\right\}\leq 0,\quad \text{a.e.,}\end{aligned}\tag{55}$$

which implies

$$\begin{aligned}&\mathbb{E}\left\{\left[H\left(\tau,\alpha_\tau,x_\tau,y_\tau,z_\tau,W_\tau,v,p_\tau,q_\tau,k_\tau\right)\right.\right.\\&\quad\left.\left.-H\left(\tau,\alpha_\tau,x_\tau,y_\tau,z_\tau,W_\tau,u_\tau,p_\tau,q_\tau,k_\tau\right)\right]\mid\mathscr{F}_\tau\right\}\\&=H\left(\tau,\alpha_\tau,x_\tau,y_\tau,z_\tau,W_\tau,v,p_\tau,q_\tau,k_\tau\right)\\&\quad-H\left(\tau,\alpha_\tau,x_\tau,y_\tau,z_\tau,W_\tau,u_\tau,p_\tau,q_\tau,k_\tau\right)\leq 0,\quad \text{a.e., a.s.}\end{aligned}\tag{56}$$

The proof is complete. □

4. Sufficient Optimality Conditions

In this section, we add additional assumptions to obtain the sufficient conditions for optimal controls. Let us introduce the following.

(H4) The control domain U is a convex body in $\mathbb{R}$. The measurable functions b, f, and l are locally Lipschitz with respect to v, and their partial derivatives are continuous with respect to (x, y, z, W, v).

Theorem 7. *Let (H1)–(H4) hold. Suppose that the functions $\phi(\cdot)$, $\gamma(\cdot)$, $\eta \to l(t,\eta)$, and $H(t,\alpha_t,\cdot,\cdot,\cdot,\cdot,\cdot,p_t,q_t,k_t)$ are concave and $(p(\cdot),q(\cdot),k(\cdot),M(\cdot))$ is the solution of adjoint equations* (46) *and* (47) *corresponding to control $(u(\cdot),\xi(\cdot)) \in \mathscr{A}$. Moreover, assume that $y_T^{v,\eta}$ is of the special form $y_T^{v,\eta} = K(\alpha_T)x_T^{v,\eta} + \zeta$, $\forall(v,\eta) \in \mathscr{A}$, where K is a deterministic measurable function and $\zeta \in L^3(\Omega,\mathscr{F}_T,P;\mathbb{R})$. Then (u,ξ) is an optimal control if it satisfies* (48) *and* (49).

Proof. Let $(x_t^{v,\eta}, y_t^{v,\eta}, z_t^{v,\eta}, W_t^{v,\eta})$ be the trajectory corresponding to $(v,\eta) \in \mathscr{A}$. By the concavity of ϕ, γ and $\eta \to l(t,\eta)$, we derive

$$\begin{aligned}&J\left(v,\eta\right)-J\left(u,\xi\right)\\&\leq\mathbb{E}\left[\int_0^T\left(h\left(t,\alpha_t,x_t^{v,\eta},y_t^{v,\eta},z_t^{v,\eta},W_t^{v,\eta}n_t,v_t\right)-h\left(t\right)\right)dt\right]\\&\quad+\mathbb{E}\left[\phi_x\left(x_T^{u,\xi}\right)\left(x_T^{v,\eta}-x_T^{u,\xi}\right)\right]\\&\quad+\mathbb{E}\left[\gamma_y\left(y_0^{u,\xi}\right)\left(y_0^{v,\eta}-y_0^{u,\xi}\right)\right]\\&\quad+\mathbb{E}\left[\sum_{i\geq 1}l_\xi\left(\tau_i,\xi_i\right)\left(\eta_i-\xi_i\right)\right].\end{aligned}\tag{57}$$

Define

$$\mathscr{H}^{v,\eta}\left(t\right):=H\left(t,\alpha_t,x_t^{v,\eta},y_t^{v,\eta},z_t^{v,\eta},W_t^{v,\eta},v_t,p_t,q_t,k_t\right).\tag{58}$$

Applying Itô's formula to $(x_t^{v,\eta} - x_t^{u,\xi})q_t + (y_t^{v,\eta} - y_t^{u,\xi})p_t$ and noting $q_T = -K(\alpha_T)p_T + \phi_x(x_T^{u,\xi})$, we obtain

$$\begin{aligned}&J\left(v,\eta\right)-J\left(u,\xi\right)\\&\leq\mathbb{E}\left[\int_0^T\left(\mathscr{H}^{v,\eta}\left(t\right)-\mathscr{H}^{u,\xi}\left(t\right)-\mathscr{H}_x^{u,\xi}\left(t\right)\left(x_t^{v,\eta}-x_t^{u,\xi}\right)\right.\right.\\&\qquad-\mathscr{H}_y^{u,\xi}\left(t\right)\left(y_t^{v,\eta}-y_t^{u,\xi}\right)-\mathscr{H}_z^{u,\xi}\left(t\right)\left(z_t^{v,\eta}-z_t^{u,\xi}\right)\\&\qquad\left.\left.-\sum_{j\in I}\mathscr{H}_{w(j)}^{u,\xi}\left(t\right)\left(W_t^{v,\eta}\left(j\right)-W_t^{u,\xi}\left(j\right)\right)n_t\left(j\right)\right)dt\right]\\&\quad+\mathbb{E}\left[\sum_{i\geq 1}\left(l_\xi\left(\tau_i,\xi_i\right)+q_{\tau_i}C_{\tau_i}-p_{\tau_i}D_{\tau_i}\right)\left(\eta_i-\xi_i\right)\right]\\&:=\Gamma_1+\Gamma_2.\end{aligned}\tag{59}$$

By (48) and Lemma 2.3 of Chapter 3 in [27], we have

$$0 \in \partial_v \mathscr{H}^{u,\xi}(t). \tag{60}$$

By Lemma 2.4 of Chapter 3 in [27], we further conclude that

$$\left(\mathscr{H}_x^{u,\xi}(t), \mathscr{H}_y^{u,\xi}(t), \mathscr{H}_z^{u,\xi}(t), \mathscr{H}_W^{u,\xi}(t), 0\right) \in \partial_{x,y,z,W,v} \mathscr{H}^{u,\xi}(t). \tag{61}$$

Finally, by the concavity of $H(t, \alpha_t, \cdot, \cdot, \cdot, \cdot, \cdot, p_t, q_t, k_t)$ and (49), we obtain $\Gamma_1 \leq 0, \Gamma_2 \leq 0$. Thus, it follows that $J(v, \eta) - J(u, \xi) \leq 0$. We complete the proof. □

5. Application in Finance

This section is devoted to studying an investment and consumption model under the stochastic recursive utility arising from financial markets, which naturally motivates the study of the problem (7) and (8).

5.1. An Example in Finance. In a financial market, suppose there are two kinds of securities which can be invested: a bond, whose price $S_0(t)$ is given by

$$dS_0(t) = r_t S_0(t)\,dt, \quad S_0(0) > 0, \tag{62}$$

and a stock, whose price is

$$dS_1(t) = S_1(t)\left(\mu_t dt + \sigma_t dB_t\right), \quad S_1(0) > 0. \tag{63}$$

Here, $\{B_t\}$ is the standard Brownian motion and r_t, μ_t, and σ_t are bounded deterministic functions. For the sake of rationality, we assume $\mu_t > r_t, \sigma_t^2 \geq \delta > 0$. Here, δ stands for a positive constant, which ensures that σ_t is nondegenerate. In reality, in order to get stable profit and avoid risk of bankruptcy, many small companies and individual investors usually make a plan at the beginning of a year or a period, in which the weight invested in stock was fixed. Denote by π_t the weight invested in stock which is called the portfolio strategy. It means no matter how much the wealth x_t is, the portfolio strategy π_t is fixed, which is a bounded deterministic function with respect to t. Then the wealth dynamics are given as

$$\begin{aligned} dx_t &= \left[r_t x_t + (\mu_t - r_t)\pi_t x_t - c_t\right] dt \\ &\quad + \sigma_t \pi_t x_t dB_t - \theta d\eta_t, \\ x_0 &= x > 0, \end{aligned} \tag{64}$$

where $\theta \geq 0$, $c_t \geq 0$, and $\eta_t = \sum_{i\geq 1} \eta_i \mathbf{1}_{[\tau_i, T]}(t)$. Here, c_t is a continuous consumption process, η_t is a piecewise consumption process, and θ is a weight factor. Not only in the mode of continuous consumption, but also in reality society, one consumes piecewise. Hence our setting of consumption process is practical.

Besides, if the macroeconomic conditions are also taken into account in this model, above model has obvious imperfections because it lacks the flexibility to describe the changing stochastically of investment environment. One can modulate the uncertainty of the economic situation by a continuous-time finite-state Markov chain. Then the wealth is formulated by a switching process as

$$\begin{aligned} dx_t &= \left[r(t,\alpha_t) x_t + (\mu(t,\alpha_t) - r(t,\alpha_t))\pi(t,\alpha_t) x_t - c_t\right] dt \\ &\quad + \sigma(t,\alpha_t)\pi(t,\alpha_t) x_t dB_t - \theta d\eta_t, \\ x_0 &= x, \qquad \alpha_0 = i. \end{aligned} \tag{65}$$

Let U be a nonempty subset of $\{\mathbb{R}_+ \cup 0\}$ and K a nonempty convex subset of $\{\mathbb{R}_+ \cup 0\}$. Suppose $\{\mathscr{F}_t\}$ is the natural filtration generated by the Brownian motion and the Markov chains, c_t is an $\mathscr{F}_t$-progressively measurable process satisfying

$$c_t \in U, \text{ a.s., a.e.,} \quad \mathbb{E}\int_0^T |c_t|^3 dt < +\infty, \tag{66}$$

$\{\tau_i\}$ is a fixed sequence of increasing $\mathscr{F}_t$-stopping times, and each η_i is an $\mathscr{F}_{\tau_i}$-measurable random variable satisfying

$$\eta_i \in K, \text{ a.s.,} \quad \mathbb{E}\left(\sum_{i\geq 1} |\eta_i|\right)^2 < +\infty. \tag{67}$$

We consider the following stochastic recursive utility, which is described by a BSDE with the Markov chain α_t:

$$\begin{aligned} -dy_t &= \left[b(t,\alpha_t) x_t + f(t,\alpha_t) y_t + g(t,\alpha_t) z_t - c_t\right] dt \\ &\quad - z_t dB_t - \sum_{j\in I} W_t(j)\, d\widetilde{\mathscr{V}}_t(j) - \zeta d\eta_t, \\ y_T &= x_T, \end{aligned} \tag{68}$$

where $I = 1, 2, \ldots, k$, $\zeta \geq 0$. The recursive utility is meaningful in economics and theory. Details can be found in Duffie and Epstein [5] and El Karoui et al. [6].

Define the associated utility functional as

$$J(c(\cdot), \eta(\cdot)) = \mathbb{E}\left[\int_0^T L e^{-\beta t} \frac{(c_t)^{1-R}}{1-R} dt + \frac{S}{2}\sum_{i\geq 1} \eta_i^2 + H y_0\right], \tag{69}$$

where L, S, and H are positive constants, β is a discount factor, and $\beta \in (0,1)$ is also called Arrow-Pratt index of risk aversion (see, e.g., Karatzas and Shreve [28]). To get the explicit solution, we also assume $b(t, \alpha_t) \geq 0$. The first and second terms in (69) measure the total utility from $c(\cdot)$ and $\eta(\cdot)$, while the third term characterizes the initial reserve y_0. It is natural to desire to maximize the expected utility functional representing cumulative consumption and the recursive utility y_0, which means to find $(c(\cdot), \eta(\cdot))$ satisfying (66) and (67), respectively, to maximize $J(c(\cdot), \eta(\cdot))$ in (69).

We solve the problem by the maximum principle derived in Section 3. The Hamiltonian corresponding to this model is

$$\begin{aligned} & H(t,\alpha_t,x,y,z,c,p,q,k) \\ & = -p\left[b(t,\alpha_t)x + f(t,\alpha_t)y + g(t,\alpha_t)z - c\right] \\ & \quad + q\left[r(t,\alpha_t)x + (\mu(t,\alpha_t) - r(t,\alpha_t))\pi(t,\alpha_t)x - c\right] \\ & \quad + k\sigma(t,\alpha_t)\pi(t,\alpha_t)x + Le^{-\beta t}\frac{(c_t)^{1-R}}{1-R}, \end{aligned} \tag{70}$$

where (p,q,k,M) is the solution of the following adjoint equations:

$$\begin{gathered} dp_t = f(t,\alpha_t)p_t dt + g(t,\alpha_t)p_t dB_t, \\ p_0 = -H, \end{gathered} \tag{71}$$

$$\begin{aligned} -dq_t = {} & \left[(r(t,\alpha_t) + (\mu(t,\alpha_t) - r(t,\alpha_t))\pi(t,\alpha_t))q_t \right. \\ & \left. + \sigma(t,\alpha_t)\pi(t,\alpha_t)k_t - b(t,\alpha_t)p_t\right]dt \\ & - k_t dB_t - \sum_{j\in I} M_t(j)\, d\widetilde{\mathcal{V}}_t(j), \\ q_T = {} & -p_T. \end{aligned} \tag{72}$$

From (71) it is easy to obtain that

$$\begin{aligned} p_t = -H\exp\left\{\int_0^t\left[f(s,\alpha_s) - \frac{1}{2}g^2(s,\alpha_s)\right]ds \right. \\ \left. + \int_0^t g(s,\alpha_s)\,dB_s\right\} < 0. \end{aligned} \tag{73}$$

To solve (72), we introduce the dual process

$$\begin{aligned} d\Lambda_s = {} & \left[r(s,\alpha_s)(1-\pi(s,\alpha_s)) + \mu(s,\alpha_s)\pi(s,\alpha_s)\right]\Lambda_s ds \\ & + \sigma(s,\alpha_s)\pi(s,\alpha_s)\Lambda_s dB_s, \\ & \Lambda_t = 1, \quad (s \ge t). \end{aligned} \tag{74}$$

Actually, (74) is solved by

$$\begin{aligned} \Lambda_s = \exp\left\{\int_t^s \left[r(\tau,\alpha_\tau)(1-\pi(\tau,\alpha_\tau)) + \mu(\tau,\alpha_\tau)\pi(\tau,\alpha_\tau) \right.\right. \\ \left. - \frac{1}{2}\sigma^2(\tau,\alpha_\tau)\pi^2(\tau,\alpha_\tau)\right]d\tau \\ \left. + \int_t^s \sigma(\tau,\alpha_\tau)\pi(\tau,\alpha_\tau)\,dB_\tau\right\} > 0. \end{aligned} \tag{75}$$

Applying Itô's formula to $\Lambda_s q_s$ and taking conditional expectation with respect to $\mathcal{F}_t$, we obtain

$$\begin{aligned} q_t = {} & \mathbb{E}\left[-p_T\Lambda_T - \int_t^T b(s,\alpha_s)\,p_s\Lambda_s ds \mid \mathcal{F}_t\right] \\ = {} & H\mathbb{E}\left[\exp\left\{\int_0^T\left[f(\tau,\alpha_\tau) + r(\tau,\alpha_\tau)(1-\pi(\tau,\alpha_\tau)) \right.\right.\right. \\ & + \mu(\tau,\alpha_\tau)\pi(\tau,\alpha_\tau) - \frac{1}{2}g^2(\tau,\alpha_\tau) \\ & \left. - \frac{1}{2}\sigma^2(\tau,\alpha_\tau)\pi^2(\tau,\alpha_\tau)\right]d\tau \\ & - \int_0^t\left[r(\tau,\alpha_\tau)(1-\pi(\tau,\alpha_\tau)) \right. \\ & + \mu(\tau,\alpha_\tau)\pi(\tau,\alpha_\tau) \\ & \left. - \frac{1}{2}\sigma^2(\tau,\alpha_\tau)\pi^2(\tau,\alpha_\tau)\right]d\tau \\ & + \int_0^T\left[g(\tau,\alpha_\tau) + \sigma(\tau,\alpha_\tau)\pi(\tau,\alpha_\tau)\right]dB_\tau \\ & \left. - \int_0^t \sigma(\tau,\alpha_\tau)\pi(\tau,\alpha_\tau)\,dB_\tau\right\} + \int_t^T b(s,\alpha_s) \\ & \times\exp\left\{\int_0^s\left[f(\tau,\alpha_\tau) + r(\tau,\alpha_\tau)(1-\pi(\tau,\alpha_\tau)) \right.\right. \\ & + \mu(\tau,\alpha_\tau)\pi(\tau,\alpha_\tau) - \frac{1}{2}g^2(\tau,\alpha_\tau) \\ & \left. - \frac{1}{2}\sigma^2(\tau,\alpha_\tau)\pi^2(\tau,\alpha_\tau)\right]d\tau \\ & - \int_0^t\left[r(\tau,\alpha_\tau)(1-\pi(\tau,\alpha_\tau)) \right. \\ & + \mu(\tau,\alpha_\tau)\pi(\tau,\alpha_\tau) \\ & \left. - \frac{1}{2}\sigma^2(\tau,\alpha_\tau)\pi^2(\tau,\alpha_\tau)\right]d\tau \\ & + \int_0^s\left[g(\tau,\alpha_\tau) + \sigma(\tau,\alpha_\tau)\pi(\tau,\alpha_\tau)\right]dB_\tau \\ & \left.\left. - \int_0^t \sigma(\tau,\alpha_\tau)\pi(\tau,\alpha_\tau)\,dB_\tau\right\} ds \mid \mathcal{F}_t\right]. \end{aligned} \tag{76}$$

Note that $b(t,\alpha_t) \ge 0$; then we have $q_t > 0$. Thus, by Theorem 6 we get the optimal consumption processes $(c^*(\cdot),\eta^*(\cdot))$ for the regime-switching investment-consumption problem (65)–(69) as follows:

$$\begin{aligned} c_t^* &= \left(\frac{L}{q_t - p_t}\right)^{1/R} e^{-\beta t/R}, \quad \text{a.e., a.s.,} \\ \eta_i^* &= \frac{\theta q_{\tau_i} - \zeta p_{\tau_i}}{S}, \quad \forall i \ge 1, \ \text{a.s,} \end{aligned} \tag{77}$$

where (p_t, q_t) is given by (73) and (76), respectively.

5.2. Numerical Simulation. In this part, we calculate the optimal consumption functions explicitly according to (71)–(77) in the case that all coefficients are constants and discuss the relationship between consumption and some financial parameters, which can further illustrate our results obtained in this paper. We only consider the optimal regular consumption process $c^*(\cdot)$ and in this case the Markov chain $\alpha_t \equiv \alpha$ has two states $\{1, -1\}$. Here α_t will not change from 0 to T. Further we fix $[H, \beta, L, R] = [0.1, 0.5, 2, 0.2]$ and $T = 1$ year throughout this part.

5.2.1. The Relationship between $c^*(t)$ and r. As $\alpha = 1$, we set

$$[r1, r2, r3, f(\alpha), g(\alpha), \pi(\alpha), \sigma(\alpha), b(\alpha), \mu] = [0.02, 0.03, 0.04, 0.1, 0.1, 0.5, 0.2, 0.2, 0.05]. \tag{78}$$

From Figure 1, we find that the higher the risk-free interest rate is, the lower the optimal consumption is. It coincides with the financial behaviors in reality. As the risk-free interest rate r grows higher, the investors can gain more profits via deposit. Consequently, the desire of consumption is declined.

As $\alpha = -1$, we set

$$[r1, r2, r3, f(\alpha), g(\alpha), \pi(\alpha), \sigma(\alpha), b(\alpha), \mu] = [0.02, 0.03, 0.04, 0.05, 0.05, 0.4, 0.3, 0.15, 0.05]. \tag{79}$$

Figure 2 shows the influence of risk-free interest rate on the optimal consumption function as $\alpha = -1$. Same as Figure 1, when the risk-free interest rate gets higher, the optimal consumption becomes smaller. From Figures 1 and 2, we also find that under different strategies of government's macrocontrol (different α), the optimal consumption has different values and changes trends with respect to t, even for the same risk-free interest rate r. It is natural because α affects some parameters in this model such as f, g, π, σ, and b.

5.2.2. The Relationship between $c^*(t)$ and μ. The following two figures show the relationships between the optimal consumption function and appreciation rate of stock. First, for $\alpha = 1$, we fix

$$[\mu1, \mu2, \mu3, f(\alpha), g(\alpha), \pi(\alpha), \sigma(\alpha), b(\alpha), r] = [0.05, 0.06, 0.07, 0.1, 0.1, 0.5, 0.2, 0.2, 0.02]. \tag{80}$$

From Figure 3, we can see that the higher the appreciation rate of stock is, the lower the optimal consumption is. It is also reasonable since a higher appreciation rate of stock μ inspires investors to put more money into stock market and thereby reduce the consumption. For $\alpha = -1$, we fix

$$[\mu1, \mu2, \mu3, f(\alpha), g(\alpha), \pi(\alpha), \sigma(\alpha), b(\alpha), r] = [0.05, 0.06, 0.07, 0.05, 0.05, 0.4, 0.3, 0.15, 0.02]. \tag{81}$$

Figure 4 also presents the same influence of appreciation rate on the optimal consumption function as $\alpha = -1$. In addition, Figures 3 and 4 enhance us to understand that the

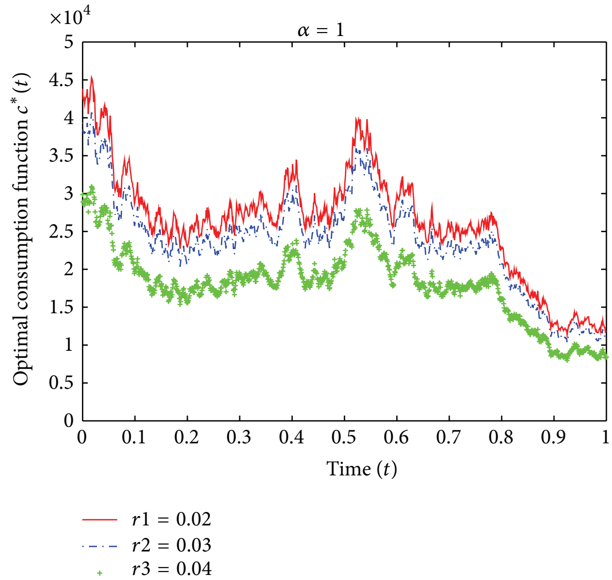

Figure 1: The relationship between $c^*(t)$ and r as $\alpha = 1$.

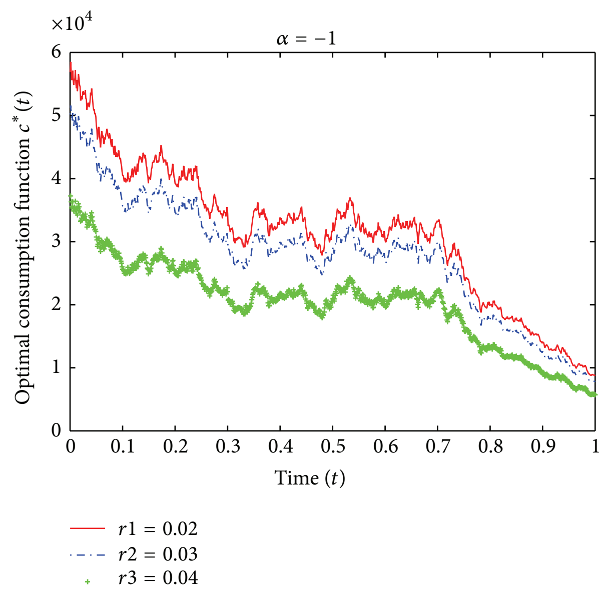

Figure 2: The relationship between $c^*(t)$ and r as $\alpha = -1$.

optimal consumption has different values and changes trends with respect to t for the same appreciation rate μ by considering different strategies of government's macrocontrol.

Based on Figures 1–4, we analyze the relationships between the optimal consumption function and the risk-free interest rate, the appreciation rate of stock, and the government's macrocontrol, which are quite important and applicable in financial problems.

6. Conclusion

In this paper, we consider the optimal control problem of forward-backward Markovian regime-switching systems

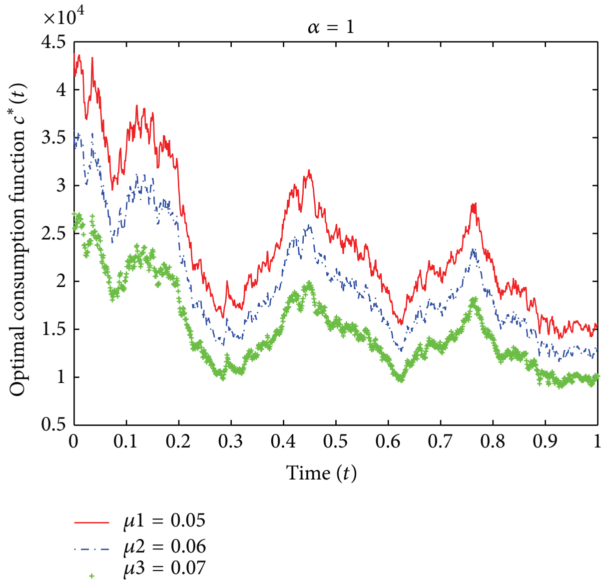

FIGURE 3: The relationship between $c^*(t)$ and μ as $\alpha = 1$.

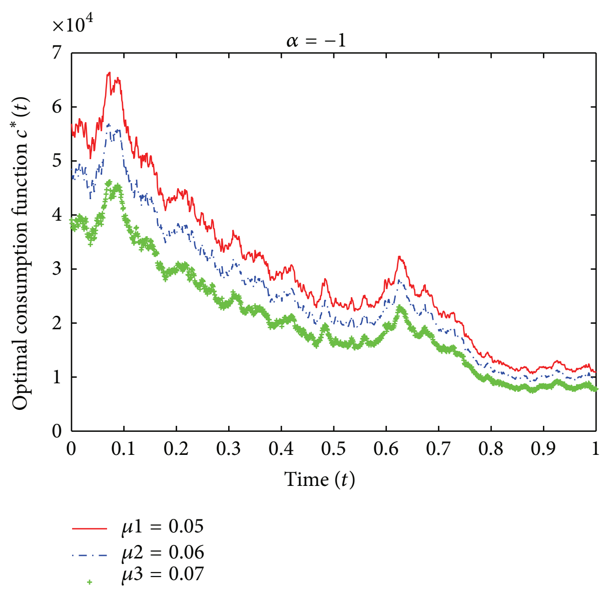

FIGURE 4: The relationship between $c^*(t)$ and μ as $\alpha = -1$.

involving impulse controls. The control system is described by FBSDEs involving impulse controls and modulated by continuous-time, finite-state Markov chains. Based on both spike and convex variation techniques, we establish the maximum principle and sufficient optimality conditions for optimal controls. Here, the regular control does not enter in the diffusion term of the forward system. In the future, we may focus on the cases that the diffusion coefficient contains controls, fully coupled forward-backward Markovian regime-switching system involving impulse controls, and game problems in this framework. It is worth pointing out that if the domain of regular control is not convex and the control enters in the forward diffusion coefficient, it will be more complicated and bring some difficulties immediately by applying spike variation. Based on the methods and results of [13], we hope to further research for such kind of control problems and investigate more applications in reality.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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