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Research Article

Optimal Control Algorithm of Constrained Fuzzy System Integrating Sliding Mode Control and Model Predictive Control

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The sliding mode control and the model predictive control are connected by the value function of the optimal control problem for constrained fuzzy system. New conditions for the existence and stability of a sliding mode are proposed. Those conditions are more general conditions for the existence and stability of a sliding mode. When it is applied to the controller design, the design procedures are different from other sliding mode control (SMC) methods in that only the decay rate of the sliding mode motion is specified. The obtained controllers are state-feedback model predictive control (MPC) and also SMC. From the viewpoint of SMC, sliding mode surface does not need to be specified previously and the sliding mode reaching conditions are not necessary in the controller design. From the viewpoint of MPC, the finite time horizon is extended to the infinite time horizon. The difference with other MPC schemes is that the dependence on the feasibility of the initial point is canceled and the control schemes can be implemented in real time. Pseudosliding mode model predictive controllers are also provided. Closed loop systems are proven to be asymptotically stable. Simulation examples are provided to demonstrate proposed methods.

1. Introduction

The modeling and control of fuzzy systems is a very active research area [1–9]. In the optimal control of fuzzy systems, the value functions for deterministic and uncertain systems satisfy the first order Hamilton-Jacobi-Bellman (HJB) equations and the first order Hamilton-Jacobi-Isaac (HJI) equations, respectively. References [10, 11] tried to solve the optimal control problem of fuzzy systems by linear system methods. References [12, 13] solved HJB equations in the optimal control of constrained fuzzy systems by dynamic programming. References [14, 15] solved HJI equations in the optimal control of uncertain constrained fuzzy systems by differential game. Usually, such equations are difficult to be solved in closed form due to their high nonlinearity. Some numerical methods with convergence proofs [13–21] are proposed to solve the value function. For the development of the above mentioned numerical methods, refer to [13, 14] and the references therein. References [22–25] use the adaptive dynamic programming to approximate the value function by neural networks. The finite difference approximation with sigmoidal transformation (FDAST) algorithms [13–16] is proposed to solve HJB equations arising in receding horizon control (MPC) schemes and HJI equations arising in robust receding horizon control (MPC) schemes, respectively.

Model predictive control (MPC), also known as receding horizon control, is a powerful tool to integrate the control and optimization of constrained nonlinear systems. Many MPC schemes have been developed [13, 16, 26–38]. The development and limitations of traditional MPC schemes are summarized in [13, 16]. Variable structure systems (VSS) first appeared in the late 1950s. Variable structure control (VSC) is an important control method for nonlinear systems [39]. Since VSC has been proposed, it has undergone great development [40]. The dominant role in VSS theory is played by sliding modes, and the core idea of designing VSC algorithms consists of enforcing this type of motion in some manifolds of the system state space into the sliding mode surface. The design of the sliding mode surface generally takes the artificially selected linear equation form $S(x) = Cx = 0$ [41, 42]. This form facilitates the design of variable structure controllers and the discussion of the stability region. However, artificially selected sliding mode surface due to the restriction of the sliding mode reaching condition will limit the stable region of the closed-loop system. Furthermore, the design of sliding mode surface and control is still a difficult problem for constrained uncertain nonlinear systems and often uses the local linearization method. The results obtained are locally stable, and there are no discussions on global stability and semiglobal stability of constrained uncertain nonlinear systems under SMC. The relationship between the optimal control and the sliding mode was not discussed in the literatures.

In this paper, we present the connection between the optimal control and the sliding mode via the value function. We briefly present the FDAST algorithm to solve HJB equations for constrained fuzzy systems. Then, new conditions for the existence and the stability of the sliding mode are proposed. Those conditions are constructed by the optimal value function and the system equation. They integrate the sliding mode existence criteria and the stability criteria for constrained fuzzy systems. This leads to some big variations in the SMC design. Then, sliding mode model predictive control (SMMPC) and pseudosliding mode model predictive control (PSMMPC) schemes are proposed for some kind of constrained fuzzy systems. Those controllers are MPC and also SMC. From the viewpoint of SMC, the sliding mode surface does not need to be specified previously in the SMC design. Sliding mode reaching conditions are not necessary in the controller design, since the closed-loop stability of the trajectory, which cannot reach the sliding mode surface, is guaranteed by MPC schemes. Therefore, it can cancel limitations on the stable zone due to the selection of the sliding mode surface and the sliding mode reaching condition. The closed-loop system is globally stable. From the viewpoint of MPC, those controllers are state-feedback (SF) which is very easy to be adjusted. The finite time horizon is extended to the infinite time horizon which guaranteed the global stability without added terminal penalties and constraints. The value function $V(x)$ is used to design a controller instead of using the optimal control $u^*(t)$ as the current control action. The dependence on the feasibility of the initial point is canceled and the online repeated optimization in MPC can be avoided. Therefore, the control schemes can be implemented in real time.

This paper is organized as follows. Section 2 begins with a description of HJB equations which includes MPC schemes for fuzzy systems. A brief review of the FDAST algorithm is given. Section 3 gives new conditions for the existence and the stability of a sliding mode. Section 4 gives the SMMPC design for some kind of constrained fuzzy systems by using the value function as a design parameter. Those controllers integrate MPC and SMC. Section 5 provides simulations to demonstrate the proposed method. Section 6 concludes the paper with some further remarks.

2. Problem Formulation and Preliminary Results

We consider following constrained fuzzy systems:

$$\begin{aligned} &R^i: \text{IF } \eta_1 \text{ is } F^i_{\eta_1}, \dots, \eta_l \text{ is } F^i_{\eta_l}, \text{ THEN } \dot{x}(t) = f^i(x(t), u(t)), \\ &\text{Subject to } u(t) \in \mathcal{U}_c, \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input, $\mathcal{U}_c$ is a nonempty compact convex subset including the original point 0, $\eta(t) = [\eta_1(t), \dots, \eta_l(t)]^T \in \mathbb{R}^l$ is the premise variable and is some function of $x(t)$ and $u(t)$, R^i ($i = 1, \dots, M$) denotes the ith rule of the fuzzy model, M is the number of fuzzy rules, and $F^i_{\eta_1}, \dots, F^i_{\eta_l}$ are input fuzzy terms in the ith rule.

Assume that $f^i(\cdot,\cdot) : S \times \mathcal{U}_u \to \mathbb{R}^n$ is continuous. The origin $(0, 0)$ is assumed to be the balance point of the global model of system (1); that is, if system (1) is assembled into the global expression

$$\dot{x}(t) = f(x(t), u(t)) \tag{2}$$

by using singleton fuzzifier, product inference, and center-average defuzzifier, then $f(0, 0) = 0$. In the following, when the context is clear, the time label t will be omitted.

Define the cost functional

$$J(x, u) = \int_t^\infty L(x(\tau), u(\tau))\, d\tau, \tag{3}$$

where $L(\cdot,\cdot) > 0$ if and only if $L(0, 0) = 0$.

The value functional in the optimal control of (1) is

$$V(x) = \min_{u(t)\in\mathcal{U}_c} J(x, u). \tag{4}$$

The optimal control problem (1) and (4) satisfies the following HJB equation:

$$\begin{aligned} 0 &= \min_{u(t)\in\mathcal{U}_c} \left[\frac{\partial V(x)}{\partial x} \cdot f(x, u) + L(x, u) \right], \\ 0 &= V(0). \end{aligned} \tag{5}$$

Remark 1. For the detailed derivation of (5), please refer to [14, 15]. Such HJB equation (5) covers MPC of constrained fuzzy systems.

Remark 2. Throughout the rest of this paper, we use the following notations. For a scalar $a \in \mathbb{R}$, sign function $\text{sign}(\cdot)$ is

$$\text{sign}(a) \triangleq \begin{cases} +1 & \text{if } a \ge 0, \\ -1 & \text{if } a < 0, \end{cases} \tag{6}$$

and the saturation function $\text{sat}(\cdot)$ is

$$\text{sat}(a, \phi) \triangleq \begin{cases} a & \text{if } |a| \le \phi, \\ \text{sign}(a)\,\phi & \text{if } |a| > \phi, \end{cases} \tag{7}$$

where $\phi \in \mathbb{R}$ and $\phi > 0$.

For a vector $a = [a_1, \dots, a_r]^T \in \mathbb{R}^r$ and $\phi = [\phi_1, \dots, \phi_r]^T > 0 \in \mathbb{R}^r$, the vector sign function $\text{Sign}(\cdot)$ is defined as $\text{Sign}(a) = [\text{sign}(a_1), \dots, \text{sign}(a_r)]^T$, the vector saturation function $\text{Sat}(\cdot)$ is defined as $\text{Sat}(a) = [\text{sat}(a_1, \phi_1), \dots, \text{sat}(a_r, \phi_r)]^T$, and $\text{Diag}(a)$ is defined as

$$\text{Diag}(a) \triangleq \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_r \end{bmatrix}. \tag{8}$$

For the function $f(a) : \mathbb{R}^r \rightarrow \mathbb{R}$ and the function vector $g(a) = [g_1(a), \ldots, g_p(a)]$ and $g_i(a) = [g_{1i}(a), \ldots, g_{ri}(a)]^T$ $1 \le i \le p : \mathbb{R}^r \rightarrow \mathbb{R}^r$, $L_g f$ is defined as (Isidori [43])

$$L_g f \triangleq \frac{\partial f(a)}{\partial a} \cdot g(a). \tag{9}$$

3. New Conditions for the Existence and Stability of a Sliding Mode

Since we vary K for the closed-loop system (27) and (36) in simulations, the closed-loop trajectories produce the sliding mode motion. This directly motivates us to consider the reason why the sliding mode motion exists. But we find it impossible to apply the commonly used conditions to judge the sliding mode motion. We first give the two commonly used conditions for the sliding mode existence and then discuss the reason why they cannot work. First, define the sliding mode plane and some notations by

$$S(x,t) = 0, \tag{10}$$

$$\begin{aligned} \lim_{S\to 0^+} f(x,v) &= f^+(x,v^+), \\ \lim_{S\to 0^-} f(x,v) &= f^-(x,v^-). \end{aligned} \tag{11}$$

Then, the two most commonly used conditions for the existence of a sliding mode are

$$\lim_{S\to 0^+} \dot{S} < 0, \qquad \lim_{S\to 0^-} \dot{S} > 0, \tag{12}$$

or

$$\lim_{S\to 0^+} \dot{S} = \nabla S \cdot f^+ < 0, \qquad \lim_{S\to 0^-} \dot{S} = \nabla S \cdot f^- > 0. \tag{13}$$

Generally, $S(x,t)$ is chosen by the designer as

$$S(x) = Cx = 0, \quad C = [c_1, c_2, \ldots, c_n]. \tag{14}$$

The reasons why the commonly used conditions cannot work are as follows.

(i) Since the sliding mode surface is not selected by the designer previously, this means that the sliding mode surface equation $S(x,t) = 0$ is unknown. But in conventional SMC, $S(x,t)$ is a known function selected by the designer. This means the commonly used condition (12) for the existence of a sliding mode does not work.

(ii) Since $S(x,t) = 0$ is unknown, the gradient ∇S is unknown. This means the commonly used condition (13) does not work.

(iii) The stability of the sliding mode motion in the conventional SMC depends on the analysis of the equivalent control and the mean motion on the sliding mode surface $S(x,t) = 0$. Since $S(x,t) = 0$ is unknown, the analysis of the equivalent control and the mean motion can not proceed.

(iv) The controller design and the attraction region in the conventional SMC depend on the reaching condition. Since $S(x,t) = 0$ is unknown, the reaching condition cannot be obtained.

To deal with this encountered situation, the new conditions of the existence and the stability of the sliding mode for general constrained uncertain fuzzy systems are constructed by system (1) and the value function (4).

Theorem 3 (new conditions for the existence and stability of a sliding mode). *Suppose that the system is (1), $V(x)$ is (4), and $S(x) = 0$ is a hyperplane. For $\forall\, \overline{x} \in S(x)$, v is chosen as*

$$\begin{aligned} \nabla V(\overline{x}) \cdot f^+(\overline{x}, v^+) &< 0, \\ \nabla V(\overline{x}) \cdot f^-(\overline{x}, v^-) &< 0. \end{aligned} \tag{15}$$

Define

$$\overline{\zeta} = f^+ - L_{f^+}V(\overline{x}) \frac{\nabla V^T(\overline{x})}{\|\nabla V(\overline{x})\|^2}. \tag{16}$$

For $\delta > 0$, define

$$\zeta = \begin{cases} \overline{\zeta} & if \;\; S(\overline{x} + \delta\overline{\zeta})\big|_{\delta\to 0} \ge 0 \\ -\overline{\zeta} & if \;\; S(\overline{x} + \delta\overline{\zeta})\big|_{\delta\to 0} < 0. \end{cases} \tag{17}$$

The conditions for the existence and the stability of a sliding mode are

$$\zeta^T \cdot f^+ < 0, \qquad \zeta^T \cdot f^- > 0. \tag{18}$$

Proof. First, we need to prove $\nabla V \perp \zeta$. Since

$$\begin{aligned} \nabla V \cdot \overline{\zeta} &= \nabla V \cdot f - L_f V \frac{\nabla V \cdot \nabla V^T}{\|\nabla V\|^2} \\ &= L_f V - L_f V \\ &= 0, \end{aligned} \tag{19}$$

we get

$$\nabla V \cdot \zeta = 0. \tag{20}$$

That is, $\nabla V \perp \zeta$.

From (17) and (20) and referring to Figure 1, the vector ζ is a vector starting at $\overline{x}$ and pointing to the space $S(x) > 0$. For $\forall \tilde{x} \in S(x) < 0$ and $\lim_{S\to 0^-} \tilde{x} \rightarrow \overline{x}$, if (18) is satisfied, the projection of the vector f^- onto the vector ζ has the same direction as the vector ζ. This means that $\tilde{x}$ moves from the inside of the space $S(x) < 0$ toward the sliding mode plane $S(x) = 0$. For $\forall \tilde{x} \in S(x) > 0$ and $\lim_{S\to 0^+} \tilde{x} \rightarrow \overline{x}$, if (18) is satisfied, the projection of the vector f^+ onto the vector ζ has the opposite direction to the vector ζ. This means that $\tilde{x}$ moves from the inside of the space $S(x) > 0$ toward the sliding model plane $S(x) = 0$. From the basic sliding mode principle (i.e., the trajectory of the system in the vicinity of the sliding mode plane moves from the inside toward the sliding mode

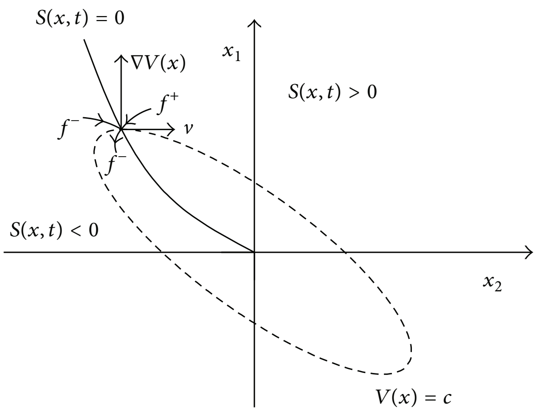

Figure 1: An illustration of sliding mode plane.

plane), we know the hyperplane $S(x)=0$ is the sliding mode plane.

From (15) and referring to Figure 1, the projections of f^- and f^+ onto $\nabla V(x)$ have the opposite direction to the vector $\nabla V(x)$. This means that $\overline{x}$ moves from the higher level surface of the value function toward the lower level surface, so the sliding mode motion is stable. That completes the proof. □

Remark 4. It must be emphasized that condition (18) indicates that the sliding mode motion is a common existing motion in the closed-loop system under an appropriate Lyapunov function and control. This is why some closed-loop systems chatter even though they are not designed via SMC methods.

Remark 5. The value function $V(x)$ is one of the candidates of the global Lyapunov function. Other global Lyapunov functions can also play this role if they can be obtained. If the local Lyapunov function is used in the conditions, it limits the stable zone.

Remark 6. Since the value function $V(x)$ is the optimal result of the constrained fuzzy system, condition (18) can deal with constrained fuzzy systems.

Theorem 7 (new sliding mode surface equations). *The hyperplane satisfying one of the equations*

$$\zeta^T \cdot f = 0, \tag{21}$$

$$\|\nabla V\| \|f\| = \|\nabla V \cdot f\|, \tag{22}$$

$$\|\nabla V\| \|f\| = \left\|\frac{dV(t)}{dt}\right\|, \tag{23}$$

$$\|\nabla V\| \|f\| = \|L(x,u)\| \tag{24}$$

and the conditions in Theorem 3 are the sliding mode surface of (1).

Proof. Condition (21) has been proven in Theorem 3. Condition (22) can be obtained by some basic manipulations of (21).

Since

$$\begin{aligned}\overline{\zeta}^T \cdot f &= f^T \cdot f - L_f V \frac{\nabla V \cdot f}{\|\nabla V\|^2} \\ &= \frac{\|\nabla V\|^2 \|f\|^2 - \|\nabla V \cdot f\|^2}{\|\nabla V\|^2} \\ &= 0,\end{aligned} \tag{25}$$

we get

$$\|\nabla V\|^2 \|f\|^2 - \|\nabla V \cdot f\|^2 = 0. \tag{26}$$

The above is (22). Since $\nabla V \cdot f = dV(t)/dt$, we get (23). Since $dV(t)/dt = -L(x,u)$, we get (24). That completes the proof. □

Remark 8. The sliding mode motion on the sliding mode surface, commonly analyzed by the more complicated equivalent control and the mean motion method, has a specified decay ratio. This ratio is $-L(x,u)$, which can be designed by the designer. It has been illustrated in (23) and (24).

Remark 9. Conditions (12) or (13) are only a special case of condition (18) for the sliding mode existence. If $\nabla S/\|\nabla S\| = v/\|v\|$, condition (18) covers conditions (12) or (13) for the sliding mode existence. But condition (18) meanwhile includes the stability.

4. Sliding Mode Robust Receding Horizon Control for Uncertain Constrained Fuzzy Systems

Consider the controlled uncertain constrained fuzzy systems

$$\begin{aligned}&R^i: \text{IF } x_1 \text{ is } F^i_{x_1} \cdots x_n \text{ is } F^i_{x_n}, \text{ THEN } \dot{x}(t) = A_i x(t) + B_i u(t), \\ &\text{Subject to } u(t) \in \mathscr{U}_c,\ \mathscr{U}_c \triangleq \{u(t) : |u(t)| \le \phi_u,\ \phi_u > 0\},\end{aligned} \tag{27}$$

where definitions of $x(t)$ and $u(t)$ refer to (1) and A_i and B_i are matrices with proper dimensions.

The value function of system (27) is defined as

$$V(x) = \min_{u \in \mathscr{U}_c} \int_t^\infty x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau)\, d\tau, \tag{28}$$

where $Q \in \mathbb{R}^n \times \mathbb{R}^n$ and $R \in \mathbb{R}^m \times \mathbb{R}^m$ are positive-definite, symmetric weighting matrices.

The setup of the SMC design in this paper is different from conventional ones. First, choose the sliding mode motion's decay rate $-L(x,u)$. Then construct the global Lyapunov function, that is, the value function $V(x)$, for constrained fuzzy systems. Third, solve $V(x)$. Finally, construct sliding mode control $u(x)$. Coincidentally, this procedure is compatible with MPC schemes.

For those states outside the sliding mode surface, which can be forced into the sliding mode surface, they move along the sliding mode surface to the origin. For those states, which cannot reach the sliding mode surface, they can be stabilized by some form of the optimal control. Thus, the reaching condition and the attraction region are no longer necessary in the SMC design. This idea is illustrated in the following controller design.

4.1. Sliding Mode Model Predictive Control (SMMPC). Based on Theorems 3 and 7, the sliding mode model predictive controller is designed for the fuzzy system (27) according to the steps listed below.

Step 1. Choose the needed decay rate of the sliding mode motion

$$\rho \leq -\lambda \|x\|^2, \tag{29}$$

where $\lambda > 0$ is the decay constant.

Step 2. Choose positive-definite, symmetric matrices $Q \in \mathbb{R}^n \times \mathbb{R}^n$ and $R \in \mathbb{R}^m \times \mathbb{R}^m$, where

$$\lambda_{\min}(Q) \geq \lambda. \tag{30}$$

Here $\lambda_{\min}(Q)$ is the smallest eigenvalue of Q.

Step 3. Design the running cost

$$L(x,u) = xQx + uRu. \tag{31}$$

Step 4. Construct the value function (28) according to model predictive control scheme with the infinite horizon:

$$V(x) = \min_{u\in\mathcal{U}_c} \int_t^\infty L(x,u)\, d\tau. \tag{32}$$

Step 5. Solve $V(x)$ by the FDAST algorithm.

Step 6. Construct the sliding mode model predictive controller by $V(x)$.

For a concise expression, define

$$f_F(x,u) = \sum_{i=1}^{M} \Theta_i(x)\left(A_i x + B_i u\right), \tag{33}$$

where $\Theta_i(x) = \prod_{j=1}^n \mu^i(x_j) / \sum_{i=1}^M (\prod_{j=1}^n \mu^i(x_j))$ and $\mu^i(x_j)$ is the fuzzy membership of $F_{x_j}^i$. Further, let

$$\begin{aligned} f_a(x(t)) &= \sum_{i=1}^{N} \Theta_i(x(t)) A_i x(t), \\ g_a(x(t)) &= \sum_{i=1}^{N} \Theta_i(x(t)) B_i. \end{aligned} \tag{34}$$

Then

$$\dot{x}(t) = f_a(x) + g_a(x) u(t). \tag{35}$$

Theorem 10. *System (27) under the state-feedback sliding mode model predictive controller (36) is stable:*

$$u = -\mathrm{Sat}\left(\frac{1}{2} K R^{-1} \left(L_{g_a} V\right)^T + \eta \|x\| \,\mathrm{Sign}\left(L_{g_a} V\right)^T\right), \tag{36}$$

where $K = \mathrm{Diag}([k_1, \ldots, k_m]^T) \geq I_m$ *and* $\eta = \mathrm{Diag}([\eta_1, \ldots, \eta_m]^T) \geq 0_m$.

Proof. The first step is to show whether the value function $V(x)$, obtained by solving the optimal control problem (28), can work as a Lyapunov function. Since

$$V(x) = \min_{u\in\mathcal{U}_c} \int_t^\infty x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau)\, d\tau, \tag{37}$$

we get that $V(x) > 0$, $V(x) \to +\infty$ when $x \to \infty$ and $V(0) = 0$. From the above, $V(x)$ can work as a Lyapunov function.

It is obvious that the control law (36) satisfies the constraint $|u| \leq \phi_u$.

Define $S(x) = \|\nabla V\| \|f_a + g_a u\| - \|L_{f_a} V + L_{g_a} V u\| = 0$. f_F^+ and f_F^- are defined in (11) where $f_F = f_a + g_a u$. $S(x) = 0$ satisfying (15)–(18) is the sliding mode surface. Consider now the closed-loop system of (27) and (36). Evaluating the time derivative of the Lyapunov function along the closed-loop trajectory, we obtain

$$\begin{aligned} \dot{V} &= L_{f_a} V + L_{g_a} V u \\ &= L_{f_a} V - L_{g_a} V \,\mathrm{Sat}\left(\frac{1}{2} K R^{-1} \left(L_{g_a} V\right)^T + \eta \|x\| \mathrm{Sign}\left(L_{g_a} V\right)^T\right). \end{aligned} \tag{38}$$

From Theorems 3 and 7, the hyperplane $S(x)$ satisfying (21) is the sliding mode plane. Now we classify the points in the state space into two subsets. One is the point belonging to the sliding mode plane. The other is the points outside of the sliding mode plane.

For the subset $x \in S(x) = 0$ (i.e., x is on the sliding mode plane), since the sliding mode plane cannot be expressed analytically, the commonly used methods such as the equivalent control method cannot work. But we can directly use the Lyapunov method (39) to judge its stability:

$$\dot{V} = \lim_{\delta t_1 + \delta t_2 \to 0} \frac{\nabla V(x) f_F^+ \delta t_1 + \nabla V(x) f_F^- \delta t_2}{\delta t_1 + \delta t_2}. \tag{39}$$

Noticing that $V(x)$ is the value function, we have

$$\begin{aligned} \dot{V} &= \lim_{\delta t_1 + \delta t_2 \to 0} \frac{-x^T Q x \delta t_1 - u^T R u \delta t_1 - x^T Q x \delta t_2 - u^T R u \delta t_2}{\delta t_1 + \delta t_2}, \\ &= -x^T Q x - u^T R u, \\ &< 0. \end{aligned} \tag{40}$$

Since $K \geq I_m$, let

$$K = I_m + \delta_K, \quad \delta_K \geq 0. \tag{41}$$

For the subset $x \notin S(x) = 0$, the following two cases are considered. For the case $|(1/2)KR^{-1}(L_{g_a}V)^T + \eta\|x\|\mathrm{Sign}(L_{g_a}V)^T| \le \phi_u$, we have

$$\begin{aligned}
\dot{V}(t) &= L_{f_a}V - \frac{1}{2}KL_{g_a}VR^{-1}\left(L_{g_a}V\right)^T \\
&\quad - \eta\|x\| L_{g_a}V\mathrm{Sign}\left(L_{g_a}V\right)^T \\
&= L_{f_a}V - \frac{1}{2}\left(I_n + \delta_K\right)L_{g_a}VR^{-1}\left(L_{g_a}V\right)^T - \eta\|x\|\left|L_{g_a}V\right| \\
&= L_{f_a}V + L_{g_a}Vu^* - \frac{1}{2}\delta_K L_{g_a}VR^{-1}\left(L_{g_a}V\right)^T \\
&\quad - \eta\|x\|\left|L_{g_a}V\right| \\
&= -x^TQx - u^{*T}Ru^* \\
&\quad - \frac{1}{2}\delta_K L_{g_a}VR^{-1}\left(L_{g_a}V\right)^T - \eta\|x\|\left|L_{g_a}V\right| \\
&\le -x^TQx - \frac{1}{4}L_{g_a}VR^{-1}\left(L_{g_a}V\right)^T - \eta\|x\|\left|L_{g_a}V\right| \\
&< 0.
\end{aligned} \tag{42}$$

Before studying another case, first define

$$\chi = \frac{1}{2}KR^{-1}\left(L_{g_a}V\right)^T + \eta\|x\|\mathrm{Sign}\left(L_{g_a}V\right)^T \tag{43}$$

and index sets

$$j \triangleq \{i; \chi_i > \phi_{ui}\}, \qquad j^c \triangleq i \setminus j. \tag{44}$$

Define two parts of the function vector h by

$$\begin{aligned}
[h]_j &\triangleq \begin{cases} h_i & \text{if } h_i > \phi_{ui} \\ 0 & \text{if } h_i \le \phi_{ui}, \end{cases} \\
[h]_{j^c} &\triangleq h - [h]_j.
\end{aligned} \tag{45}$$

For the case $|[\chi]_j| > [\phi_u]_j$ and noticing that u^* is the solution to the optimization problem (28), we have

$$\phi_u - |u^*| \ge 0. \tag{46}$$

Substituting (41) and (46) into (38), we have

$$\begin{aligned}
\dot{V} &= L_{f_a}V \\
&\quad + L_{g_a}V\Bigg([u^*]_j - L_{g_a}V\mathrm{Sat}\Bigg(\Bigg[\frac{1}{2}KR^{-1}\left(L_{g_a}V\right)^T \\
&\qquad\qquad + \eta\|x\|\mathrm{Sign}\left(L_{g_a}V\right)^T\Bigg]_{j^c}\Bigg) \\
&\qquad\qquad - \mathrm{Diag}\left(\mathrm{Sign}\left(\left[L_{g_a}V\right]_j\right)\right)[\phi_u]_j - [u^*]_j\Bigg) \\
&= L_{f_a}V + L_{g_a}V[u^*]_{j^c} - \frac{1}{2}\left[\delta_K L_{g_a}VR^{-1}\left(L_{g_a}V\right)^T\right]_{j^c} \\
&\quad - \eta\|x\|\left|\left[L_{g_a}V\right]_{j^c}\right| + L_{g_a}V[u^*]_j \\
&\quad - L_{g_a}V\mathrm{Diag}\left(\mathrm{Sign}\left(\left[L_{g_a}V\right]_j\right)\right)[\phi_u]_j - L_{g_a}V[u^*]_j \\
&= L_{f_a}V + L_{g_a}Vu^* - \frac{1}{2}\left[\delta_K L_{g_a}VR^{-1}\left(L_{g_a}V\right)^T\right]_{j^c} \\
&\quad - \eta\|x\|\left|\left[L_{g_a}V\right]_{j^c}\right| - \left|\left[L_{g_a}V\right]_j\right|[\phi_u]_j - L_{g_a}V[u^*]_j \\
&= -x^TQx - u^{*T}Ru^* - \frac{1}{2}\left[\delta_K L_{g_a}VR^{-1}\left(L_{g_a}V\right)^T\right]_{j^c} \\
&\quad - \eta\|x\|\left|\left[L_{g_a}V\right]_{j^c}\right| - \left|\left[L_{g_a}V\right]_j\right|\left([\phi_u]_j - \left|[u^*]_j\right|\right) \\
&\le -x^TQx - \frac{1}{4}L_{g_a}VR^{-1}\left(L_{g_a}V\right)^T \\
&< 0.
\end{aligned} \tag{47}$$

Summarizing all the cases, we have

$$\dot{V} < 0. \tag{48}$$

Then the closed-loop system is asymptotically stable. That completes the proof. □

Remark 11. The stability of the closed-loop system does not depend on whether the state can reach the sliding mode plane or not. From the whole closed-loop system view, no matter what states can reach or not reach the sliding mode surface, the closed-loop system is stable if and only if the state on the sliding mode plane is stable and the state outside the sliding mode plane is stable. The reaching conditions of the sliding mode plane, which are often used in the stability analysis and the controller design of SMC, do not need to be considered. From another view, the optimal result has integrated the sliding mode motion and the global stability.

4.2. Pseudosliding Mode Model Predictive Controller (PSMMPC). To keep the features of the optimal control outside the vicinity of the sliding mode surface, a boundary layer (BL) $|S| \le \Phi$ is introduced.

Theorem 12. *Consider system (27) under the following bounded nonlinear feedback controller:*

$$u = \begin{cases} -\operatorname{Sat}\left(\frac{1}{2}KR^{-1}L_{g_a}V\right) & if\ S(x(t)) > \Phi, \\ -\operatorname{Sat}\left(\frac{1}{2}KR^{-1}L_{g_a}V + \eta\,\|x\|\operatorname{Sign}\left(L_{g_a}V\right)^T\right) & \\ \qquad\qquad if\ S(x(t)) \le \Phi, \end{cases} \tag{49}$$

where $\Phi > 0$ is a very small positive number. Then the closed-loop system is stable.

Proof. When $S(x(t)) \le \Phi$, the analysis procedures of closed-loop stability have been given in Theorem 10. When $S(x(t)) > \Phi$, the analysis procedures of closed-loop stability are the special cases of Theorem 10 when $\eta = 0$. That completes the proof. □

Remark 13. Commonly in literatures, the boundary layer $|S(x)| \le \Phi$ is introduced to eliminate chattering caused by the sliding mode control. But from Theorem 12, the obtained controller (49) is still a sliding mode controller. The chattering problem can be solved by the second introduction of the boundary layer and the system can be stabilized to a specified degree ϵ (ϵ is the boundary layer's width). But motivated by the principle of MPC, we have a more appreciable controller form to avoid chattering without stabilizing errors in the following content.

Since controllers (36) and (49) involved infinite times fast switches in the sliding plane, they cannot be directly applied to practical systems except in the theoretical analysis and in the simulation. Here we briefly recall the basic MPC principle to introduce the no-chattering SMMPC/PSMMPC. In general, convectional MPC schemes are formulated as solving online a finite horizon open-loop optimal control problem subject to system dynamics and constraints and repeating this procedure at the new sample time. In the sense of conventional MPC schemes, the closed-loop control $u(\cdot)$ is defined as

$$u_{\mathrm{MPC}}(\delta) = u^*(\delta; x(t), t, T), \quad \delta \in [t, t+\Delta], \tag{50}$$

where $u^*(\delta; x(t), t, T)$ is a solution to the open-loop optimal control problem. Updating with the new measurement of $x(t+\Delta)$, the optimization will be solved again to find a new input profile. The exact state-feedback control in conventional MPC is

$$u_{\mathrm{MPC}}(x) = \lim_{\Delta\to 0} u^*(\delta; x(t), t, T), \quad \delta \in [t, t+\Delta]. \tag{51}$$

However, from the viewpoint of computation, $u_{\mathrm{MPC}}(x)$ requires infinite times optimization on the infinite numbers of discrete points in the time interval $[t, t+\Delta]$ and only makes sense in the theoretic analysis. For a numerical implementation, the input profile is generally parameterized in a step-shaped manner [29]. This means that the applicable version of conventional MPC is $u_{\mathrm{MPC}}(\delta)$.

Define controllers (36) and (49) as $u(x,\psi)$ where $\psi = \{K,\eta\}$ and $\psi = \{K,\eta,\Phi\}$, respectively. The applicable version of $u(x,\psi)$ in the vicinity of the sliding mode plane should have a similar form to $u_{\mathrm{MPC}}(\delta)$. This idea is expressed in Theorem 14.

Theorem 14. *Define controllers (36) and (49) as $u(x,\psi)$ where $\psi = \{K,\eta\}$ and $\psi = \{K,\eta,\Phi\}$, respectively. Define $t_\Phi = \{t : S(x(t)) \le \Phi\}$. No-chattering PSMMPC has the following form:*

$$u(x,\psi) = \begin{cases} u(x,\psi) & if\ S(x(t)) > \Phi, \\ u(x,\psi,\delta) = u(x(\delta_i),\psi) & \\ \qquad if\ S(x(t)) \le \Phi, \end{cases} \tag{52}$$

where $\delta_i = i\delta$, $i = 0,1,2,\ldots,\infty$ and $\delta_i \in t_\Phi$. Under the same conditions of Theorems 10 and 12, when $\delta \to 0$, closed-loop system (27) and (52) is stable.

Proof. Let

$$u(x,\psi,\delta) = u(x(\delta_i),\psi). \tag{53}$$

For $\forall\delta_i \in t_\Phi$ and $S(x(t)) \le \Phi$, it is obvious that

$$u(x,\psi) = \lim_{\delta\to 0} u(x,\psi,\delta). \tag{54}$$

That completes the proof. □

Remark 15. Theorem 14 means that if we select a suitable small time interval δ, the controller (52) can keep the stability property of (36) and (49) and at the same time avoid chattering in the sliding mode plane.

Remark 16. Control strategies defined in (36), (49), and (52) are not directly implementable since they involve the exact solution of the value function $V(x)$. Generally speaking, the value function and the optimal control for constrained fuzzy systems cannot be solved analytically except for some special demos. If numerical methods are used to solve $V(x)$, the solution is discrete. Since $V(x)$ is involved in the controller design, an implementable strategy could be that $V(x)$ is approximated by some continuous interpolation of $V(x)$. The interpolation procedures can refer to the corresponding part of [13, 16], so they are omitted.

5. Simulation

In this section, we apply the FDAST algorithm to solve HJB equations and use the value function to implement SMMPC/PSMMPC for constrained fuzzy systems.

Example 1. The nonlinear system in [29, 35] is

$$\begin{aligned} \dot{x}_1 &= x_2 + u(\mu + (1-\mu)x_1), \\ \dot{x}_2 &= x_1 + u(\mu - 4(1-\mu)x_2), \\ |u(t)| &\le 4, \end{aligned} \tag{55}$$

where $\mu = 0.6$.

System (55) can be modeled by the following T-S type constrained fuzzy system in the zone $x \in [-1,1] \times [-1,1]$:

$$\begin{aligned} R^i &: \text{IF } x_1 \text{ is } F^i_{x_1},\ x_2 \text{ is } F^i_{x_2}, \\ &\quad \text{THEN } \dot{x} = A_i x + B_i u. \end{aligned} \tag{56}$$

The zone $x \in [-1,1] \times [-1,1]$ is divided into four cells: C_{11}, C_{12}, C_{21}, and C_{22}, where $C_{11} \in [0,1] \times [0,1]$, $C_{12} \in [0,1] \times [-1,0]$, $C_{21} \in [-1,0] \times [0,1]$, and $C_{22} \in [-1,0] \times [-1,0]$. The triangle membership functions on $F^i_{x_1}$ and $F^i_{x_2}$ are

$$\begin{aligned}
\mu^1(x_1) &= \begin{cases} x_1 & x_1 \in [0,1] \\ 0 & x_1 \notin [0,1], \end{cases} \\
\mu^2(x_1) &= \begin{cases} 1 - x_1 & x_1 \in [0,1] \\ 1 + x_1 & x_1 \in [-1,0) \\ 0 & x_1 \notin [-1,1], \end{cases} \\
\mu^3(x_1) &= \begin{cases} -x_1 & x_1 \in [-1,0] \\ 0 & x_1 \notin [-1,0], \end{cases} \\
\mu^1(x_2) &= \begin{cases} x_2 & x_2 \in [0,1] \\ 0 & x_2 \notin [0,1], \end{cases} \\
\mu^2(x_2) &= \begin{cases} 1 - x_2 & x_2 \in [0,1] \\ 1 + x_2 & x_2 \in [-1,0) \\ 0 & x_2 \notin [-1,1], \end{cases} \\
\mu^3(x_2) &= \begin{cases} -x_2 & x_2 \in [-1,0] \\ 0 & x_2 \notin [-1,0]. \end{cases}
\end{aligned} \tag{57}$$

Parameters in fuzzy rules are shown in Table 1 and are as follows:

$$\begin{gathered}
A_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
A_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A_5 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad A_6 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \\
A_7 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A_8 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A_9 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \\
A_{10} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A_{11} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
A_{13} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad A_{14} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad A_{15} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \\
A_{16} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.6 \\ -1 \end{bmatrix},
\end{gathered}$$

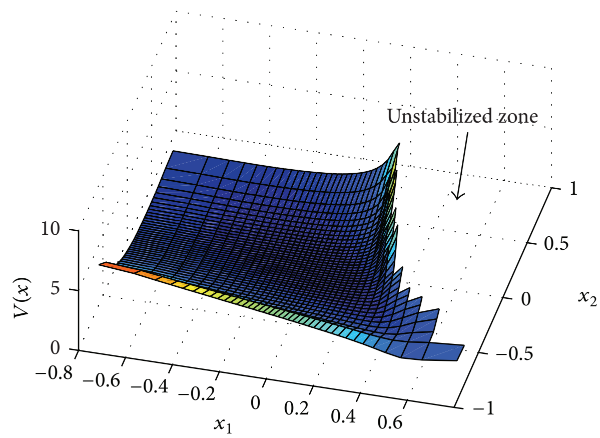

FIGURE 2: $V(x)$ in $x \in [-0.77, 0.77] \times [-0.77, 0.77]$ with the indicated unstabilized zone.

$$\begin{gathered}
B_3 = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0.6 \\ 0.6 \end{bmatrix}, \quad B_5 = \begin{bmatrix} 1 \\ 2.2 \end{bmatrix}, \\
B_6 = \begin{bmatrix} 0.6 \\ 2.2 \end{bmatrix}, \quad B_7 = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix}, \quad B_8 = \begin{bmatrix} 0.6 \\ 0.6 \end{bmatrix}, \\
B_9 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad B_{10} = \begin{bmatrix} 0.6 \\ -1 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} 0 \\ 0.6 \end{bmatrix}, \\
B_{12} = \begin{bmatrix} 0.6 \\ 0.6 \end{bmatrix}, \quad B_{13} = \begin{bmatrix} 0 \\ 2.2 \end{bmatrix}, \quad B_{14} = \begin{bmatrix} 0.6 \\ 2.2 \end{bmatrix}, \\
B_{15} = \begin{bmatrix} 0 \\ 0.6 \end{bmatrix}, \quad B_{16} = \begin{bmatrix} 0.6 \\ 0.6 \end{bmatrix}.
\end{gathered} \tag{58}$$

The decay constant λ of the sliding mode motion is set to $\lambda = 1$. Then, choose $P = I_2$ and $R = 1$. Now suppose the optimal zone is $x \in [-0.77, 0.77] \times [-0.77, 0.77]$. Choose $\alpha = 2$, $\beta = 4.8$, $h = 0.05$, $P = I_2$, and $R = 1$ in optimization. Figures 2 and 3 show the numerical solution to $V(x)$ with the indicated unstabilized zone in $x \in [-0.77, 0.77] \times [-0.77, 0.77]$ and its contour, respectively.

Choose $\hbar^{j_i}_{x_i} = x^{j_i+1}_i - x^{j_i}_i = \Psi^{-1}(y^{j_i+1}_i) - \Psi^{-1}(y^{j_i}_i)$ to implement $\partial V(x)/\partial x$. Initial states are $x(0) = [0.5, -0.2]^T$, $x(0) = [0.5, 0.1]^T$, $x(0) = [-0.5, 0.5]^T$, and $x(0) = [-0.5, -0.5]^T$.

Figure 4 shows closed-loop trajectories under SMMPC (36) with $K = 1$ and $\eta = 0$. Figure 5 shows closed-loop trajectories under SMMPC (36) with $K = 10$ and $\eta = 0$. Comparing Figure 4 with Figure 5, it is easy to identify the sliding mode surface labeled in Figure 5.

Figure 6 shows closed-loop trajectories under SMMPC (36) with $K = 1$ and $\eta = 0.5$. Figure 7 shows closed-loop trajectories under PSMMPC (52) with $K = 1$, $\eta = 0.5$, $\Phi = 0.001$, and $\delta = 0.01$. Comparing Figure 6 with Figure 7, it is shown that the chattering is avoided in Figure 7.

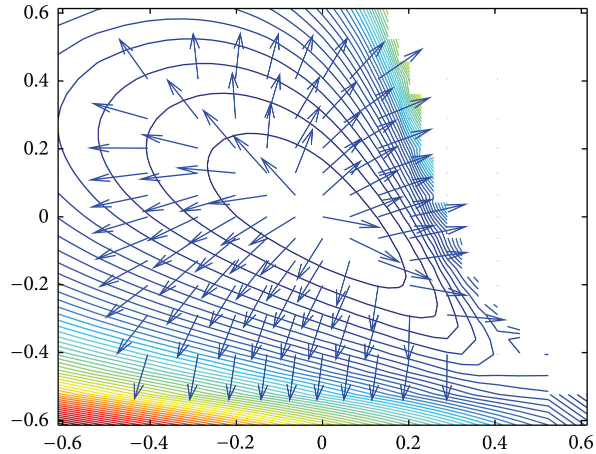

FIGURE 3: Contour and gradient field of $V(x)$.

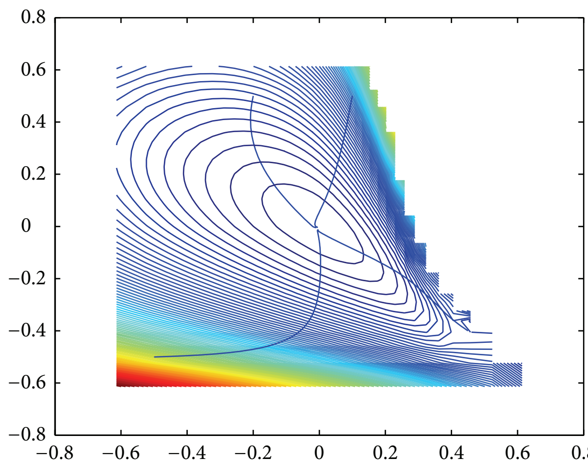

FIGURE 4: Closed-loop trajectories under SMMPC (36) with $K = 1$.

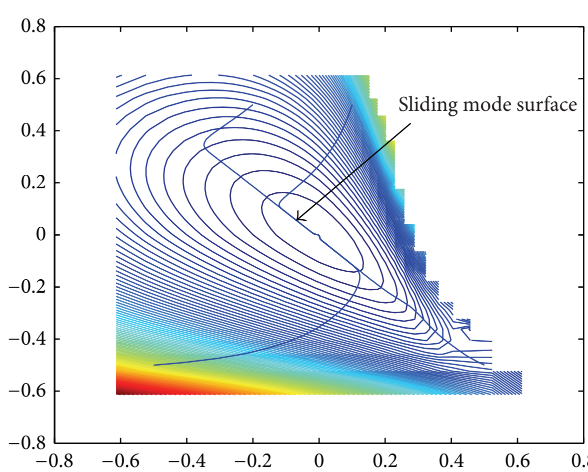

FIGURE 5: Closed-loop trajectories under SMMPC (36) with $K = 10$.

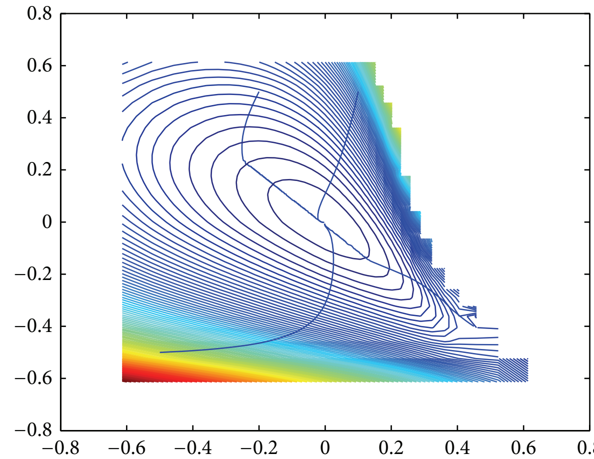

FIGURE 6: Closed-loop trajectories under SMMPC (36) with $K = 1$ and $\eta = 0.5$.

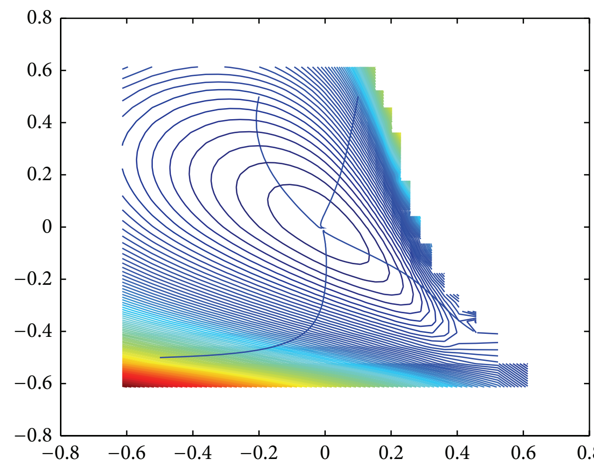

FIGURE 7: Closed-loop trajectories under PSMMPC (52) with $K = 1$, $\eta = 0.5$, $\Phi = 0.001$, and $\delta = 0.01$.

TABLE 1: Parameters in the ith fuzzy rule.

x_1	x_2	A_i	B_i	x_1	x_2	A_i	B_i
	In the cell C_{11}				In the cell C_{12}		
$F_{x_1}^1$	$F_{x_2}^1$	A_1	B_1	$F_{x_1}^1$	$F_{x_2}^3$	A_5	B_5
$F_{x_1}^2$	$F_{x_2}^1$	A_2	B_2	$F_{x_1}^2$	$F_{x_2}^3$	A_6	B_6
$F_{x_1}^1$	$F_{x_2}^2$	A_3	B_3	$F_{x_1}^1$	$F_{x_2}^2$	A_7	B_7
$F_{x_1}^2$	$F_{x_2}^2$	A_4	B_4	$F_{x_1}^2$	$F_{x_2}^2$	A_8	B_8
	In the cell C_{21}				In the cell C_{22}		
$F_{x_1}^2$	$F_{x_2}^1$	A_9	B_9	$F_{x_1}^3$	$F_{x_2}^3$	A_{13}	B_{13}
$F_{x_1}^3$	$F_{x_2}^1$	A_{10}	B_{10}	$F_{x_1}^2$	$F_{x_2}^3$	A_{14}	B_{14}
$F_{x_1}^2$	$F_{x_2}^2$	A_{11}	B_{11}	$F_{x_1}^3$	$F_{x_2}^2$	A_{15}	B_{15}
$F_{x_1}^3$	$F_{x_2}^2$	A_{12}	B_{12}	$F_{x_1}^2$	$F_{x_2}^2$	A_{16}	B_{16}

Example 2. The inverted pendulum system in [3] can be formulated as

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= \frac{g\sin x_1 - ml\sigma x_2^2\cos x_1\sin x_1}{l\left((4/3) - m\sigma\cos^2 x_1\right)}\\ &\quad + \frac{\sigma\cos x_1}{l\left((4/3) - m\sigma\cos^2 x_1\right)}u,\\ |u(t)| &\le 40,\end{aligned} \tag{59}$$

where $g = 9.8\,\mathrm{m/s^2}$ is the acceleration due to gravity, $\sigma = 1/(m_c + m)$, $m_c = 1\,\mathrm{kg}$ is the mass of cart, $m = 0.1\,\mathrm{kg}$ is the mass of pole, $l = 0.5\,\mathrm{m}$ is the half length of pole, and u is the applied force (control).

According to the study in [3], the nonlinear system can be described by the following T-S type fuzzy system:

$$\begin{aligned}\dot{x} = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2}&\mu^i(z_1)\,\mu^j(z_2)\,\mu^k(z_3)\\ &\cdot\mu^l(z_4)\left(A_{ijkl}x + B_{ijkl}u\right).\end{aligned} \tag{60}$$

The fuzzy rules are

$$\begin{aligned}R^{ijkl}&: \text{IF } z_1 \text{ is } F^i_{z_1},\ z_2 \text{ is } F^j_{z_2},\ z_3 \text{ is } F^k_{z_3},\ z_4 \text{ is } F^l_{z_4},\\ &\quad \text{THEN } \dot{x} = A_{ijkl}x + B_{ijkl}u.\end{aligned} \tag{61}$$

Here, $F^i_{z_1}$, $F^j_{z_2}$, $F^k_{z_3}$, and $F^l_{z_4}$ are fuzzy sets and

$$\begin{aligned}z_1 &= \frac{1}{4l/3 - aml\cos^2(x_1)}, \qquad z_2 = \sin(x_1),\\ z_3 &= x_2\sin(2x_1), \qquad z_4 = \cos(x_1).\end{aligned} \tag{62}$$

The membership functions on fuzzy sets are

$$\begin{aligned}\mu^1(z_1) &= \frac{z_1 - q_2}{q_1 - q_2}, \qquad \mu^2(z_1) = \frac{q_1 - z_1}{q_1 - q_2},\\ \mu^1(z_2) &= \frac{z_2 - (2/\pi)\sin^{-1}(z_2)}{1 - (2/\pi)\sin^{-1}(z_2)},\\ \mu^2(z_2) &= \frac{\sin^{-1}(z_2) - z_2}{1 - (2/\pi)\sin^{-1}(z_2)},\\ \mu^1(z_3) &= \frac{z_3 - c_2}{c_1 - c_2}, \qquad \mu^2(z_1) = \frac{c_1 - z_3}{c_1 - c_2},\\ \mu^1(z_4) &= \frac{z_4 - d_2}{d_1 - d_2}, \qquad \mu^2(z_4) = \frac{d_1 - z_4}{d_1 - d_2},\end{aligned} \tag{63}$$

and for the values of $d_1, d_2, q_1, q_2, c_1, c_2, A_{ijkl}$, and B_{ijkl}, please refer to [3] pages 14–23.

The decay constant λ of the sliding mode motion is set to $\lambda = 1$. Then, choose $P = I_2$ and $R = 1$. Now suppose

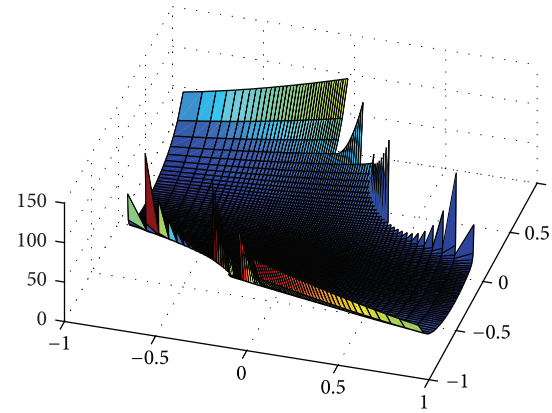

Figure 8: $V(x)$ in $x \in [-0.9, 0.9] \times [-0.9, 0.9]$.

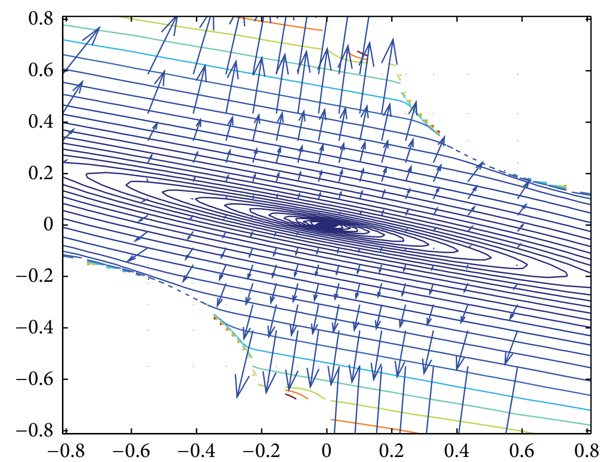

Figure 9: Contour and gradient field of $V(x)$.

that the optimal zone is $x \in [-0.9, 0.9] \times [-0.9, 0.9]$. Choose $\alpha = 2$, $\beta = 4$, and $h = 0.025$ in optimization. Figures 8 and 9 show $V(x)$ in $x \in [-0.9, 0.9] \times [-0.9, 0.9]$ and its contour, respectively. The unstabilized zones are indicated in Figure 9.

Initial states are $x(0) = [0.4, 0]^T$, $x(0) = [-0.4, 0]^T$, $x(0) = [0, 0.4]^T$, and $x(0) = [0, -0.4]^T$. Figure 10 shows closed-loop trajectories under SMMPC (36) with $K = 1$ and $\eta = 0$. Figure 11 shows closed-loop trajectories under SMMPC (36) with $K = 2$ and $\eta = 0$. The sliding mode surface is indicated in Figure 11.

Figure 12 shows closed-loop trajectories under SMMPC (36) with $K = 1$ and $\eta = 1$. Figure 13 shows closed-loop trajectories under PSMMPC (52) with $K = 1$, $\eta = 1$, $\Phi = 0.001$, and $\delta = 0.01$. The chattering is avoided in Figure 13.

6. Conclusion

In this paper, the optimal control problem of constrained fuzzy systems is connected to the sliding mode control by the optimal value function. New conditions for the existence and the global stability of a sliding mode are proposed for constrained fuzzy systems. The new conditions not only cover current conditions for the existence of a sliding mode but

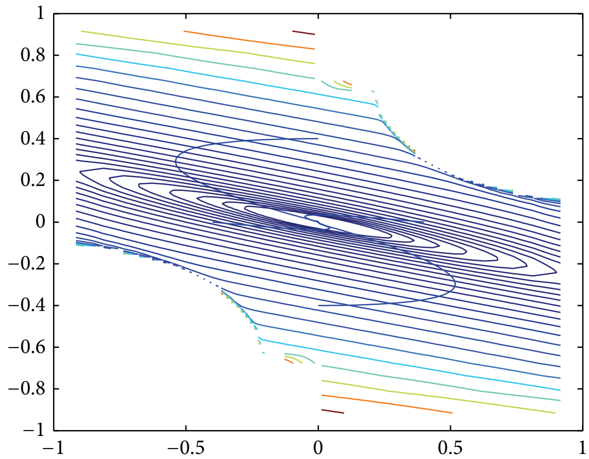

Figure 10: Closed-loop trajectories under SFMPC (36) with $K = 1$.

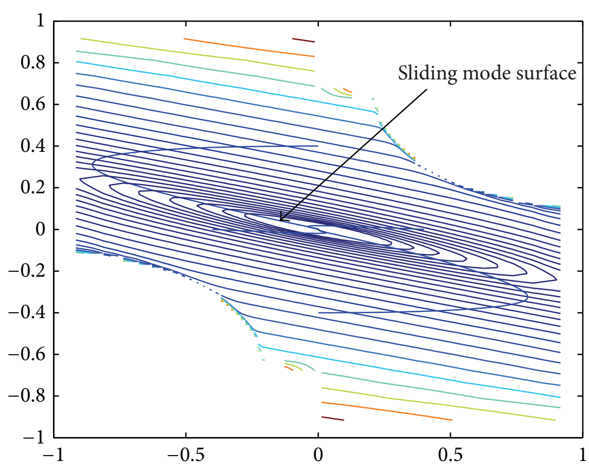

Figure 11: Closed-loop trajectories under SFMPC (36) with $K = 2$.

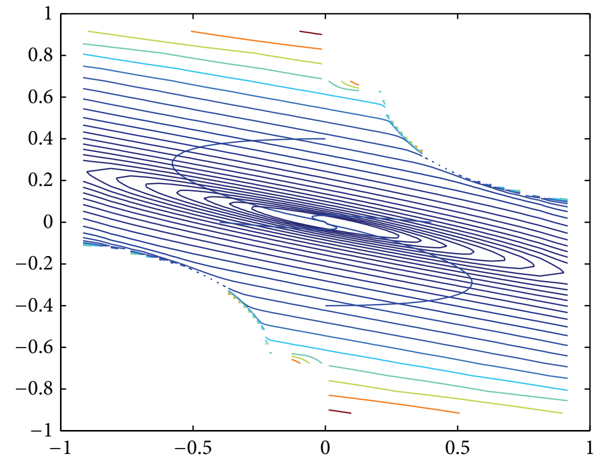

Figure 12: Closed-loop trajectories under SMMPC (36) with $K = 1$ and $\eta = 1$.

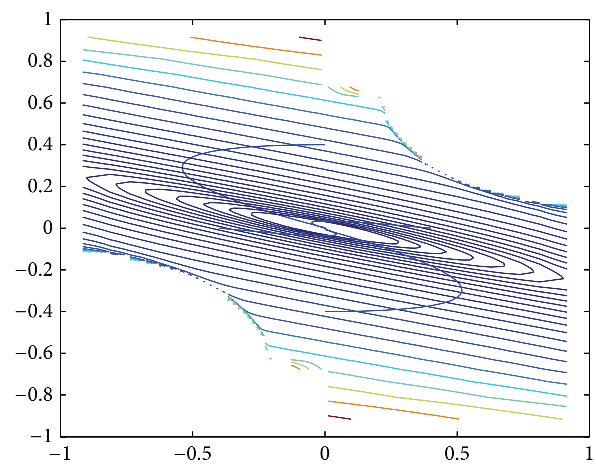

Figure 13: Closed-loop trajectories under PSMMPC (52) with $K = 1$, $\eta = 1$, $\Phi = 0.001$, and $\delta = 0.01$.

also integrate the stability. They can be used in the SMMPC and PSMMPC designs. In the design procedure, the value function solved by the FDAST algorithm is used as a design parameter. Those controllers are state-feedback MPC and also SMC. From the viewpoint of SMC, the setup is different from other SMC. The sliding mode surface need not be selected previously. Only the decay rate needs to be specified. Sliding mode reaching conditions are not necessary in the SMC design. The obtained controllers are globally stable. Therefore, they can avoid limitations on the stable zone due to selections of the sliding mode surface and sliding mode reaching conditions. From the viewpoint of MPC, they are infinite time horizon MPC schemes. The terminal constraints and penalties, which are manually added to guarantee the stability, are removed. The closed-loop stability does not depend on the feasibility of the initial point. Those control schemes are real time.

Current researches focus on a further reduction on the computational burden in optimization, on adaptive control schemes for constrained fuzzy systems, and on controller structures of more general fuzzy systems.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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