# An MDADT-Based Approach for $L_{2}$-Gain Analysis of Discrete-Time Switched Delay Systems 

Honglei Xu, ${ }^{1}$ Xiang Xie, ${ }^{1}$ and Lilian Shi ${ }^{2}$<br>${ }^{1}$ School of Energy and Power Engineering, Huazhong University of Science and Technology, Wuhan, Hubei 430074, China<br>${ }^{2}$ School of Engineering, Shaoxing University, Shaoxing, Zhejiang 312000, China<br>Correspondence should be addressed to Xiang Xie; xiang_xie@outlook.com and Lilian Shi; sllian@sina.com

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#### Abstract

We study the $L_{2}$-gain analysis problem for a class of discrete-time switched systems with time-varying delays. A mode-dependent average dwell time (MDADT) approach is applied to analyze the $L_{2}$-gain performance for these discrete-time switched delay systems. Combining a multiple Lyapunov functional method with the MDADT approach, sufficient conditions expressed in form of a set of feasible linear matrix inequalities (LMIs) are established to guarantee the $L_{2}$-gain performance. Finally, a numerical example will be provided to demonstrate the validity and usefulness of the obtained results.


## 1. Introduction

Switched systems consist of a finite number of subsystems and a logical law which orchestrates the switching behaviors between these subsystems. These dynamical systems can mathematically model many practical engineering applications with switching characteristics in a variety of disciplines; see, for example, [1-7].

A constrained switching signal can be regarded as a powerful tool to stabilize and control these switched systems [8-10]. Among them, the average dwell time (ADT) switching is the most common and typical one. It guarantees that the number of types of switching in a finite interval be bounded and the average time between any two types of consecutive switching not be less than a positive constant [11, 12]. In recent years, it has been recognised that ADT is flexible and efficient for dynamics analysis of many switched systems [8, 13-16]. However, the ADT switching's property that the average time interval between any two types of consecutive switching should be greater than a positive number $\tau_{a}$ makes the dwell time independent of the system modes. Hence whether
the dwelling at some classes of subsystems will deteriorate the disturbance attenuation cannot be predicted.

As shown in [17], the minimum of admissible ADT is computed by two mode-independent parameters: the increase coefficient of the Lyapunov-like function and the decay rate of the Lyapunov function, which will cause certain conservativeness. To solve the problem, more recently, a new mode-dependent ADT concept has been introduced in [18]. Two mode-independent parameters can be set in a modedependent manner, which will reduce the conservativeness.

Even though stability analysis for the switched systems with MDADT has been investigated extensively (see, e.g., [17, 18]), how to solve the $L_{2}$-gain problem of the switched systems with MDADT is interesting and worthwhile to study. This has motivated our study in this paper.

The rest of the paper is as follows. In Section 2, we introduce the class of discrete-time switched system, some necessary definitions, and lemmas. In Section 3, sufficient conditions for ensuring $L_{2}$-gain for the discrete-time switched delay system are constructed. In Section 4, a numerical example is presented to illustrate the obtained results. Conclusion remarks are given in Section 5.

## 2. Preliminaries and Problem Statement

Consider a discrete-time switched system with a timevarying delay:

$$
L_{i}:\left\{\begin{array}{l}
x(t+1)=A_{i} x(t)+B_{i} x(t-d(t))+C_{i} w(t),  \tag{1}\\
x_{t_{0}}(l)=x\left(t_{0}+l\right)=\phi(l), \\
z(t)=D_{i} x(t)+E_{i} w(t),
\end{array} l=-d_{M},-d_{M}+1,-d_{M}+2, \ldots, 0,\right.
$$

where $x(t) \in R^{n}$ is the system state, $z(t) \in R^{m}$ is the controlled output, $\phi(l)$ is a vector-valued initial function, $t_{0}$ is the initial time, and $w(t)$ is the disturbance input which belongs to $L_{2}[0,+\infty) . d(t)$ is the time-varying delay and satisfies $0<$ $d_{m}<d(t) \leq d_{M}$, where $d_{m}$ and $d_{M}$ denote the upper and the lower bounds of the delays. $i$ is the switching signal, which takes its values in the finite set $S=\{1, \ldots, M\}$, where $M$ is the number of subsystems. When $t \in\left[t_{i}, t_{i+1}\right), i \in \mathbb{N}$, we call the $i$ th subsystem active. $A_{p}, B_{p}, C_{p}, D_{p}$, and $E_{p}$ are constant matrices with appropriate dimension. When $i=p=$ $1, \ldots, m$, it represents the $p$ th subsystem or $p$ th mode of (1).

To proceed, we need the following definitions and lemmas.

Definition 1 (see [11]). For any $T_{2}>T_{1} \geq 0$ and any switching signal $i, T_{1} \leq t<T_{2}$, let $N_{i}\left(T_{1}, T_{2}\right)$ denote the number of types of switching of $i$ over $\left(T_{1}, T_{2}\right)$. If $N_{i}\left(T_{1}, T_{2}\right) \leq N_{0}+T_{2}-T_{1} / T_{a}$ holds for $N_{0} \geq 0$ and $T_{a}>0$, then $T_{a}$ is the average dwell time and $N_{0}$ is the chatter bound. Without loss of generality, we choose $N_{0}=0$.

Definition 2 (see [18]). For a switching signal $i$ and any $T \geq$ $t \geq 0$, let $N_{i p}(T, t)$ be the switching numbers in which the $p$ th subsystem is activated over the interval $[t, T]$ and let $T_{p}(T, t)$ denote the total running time of the $p$ th subsystem over the interval $[t, T], p \in S$. We say that $i$ has a modedependent average dwell time (MDADT) $\tau_{a p}$ if there exist positive numbers $N_{o p}$ and $\tau_{a p}$ such that

$$
\begin{equation*}
N_{i p}(T, t) \leq N_{o p}+\frac{T_{p}(T, t)}{\tau_{a p}}, \quad \forall T \geq t \geq 0 \tag{2}
\end{equation*}
$$

and we call $N_{o p}$ the mode-dependent chatter bounds. Here, we choose $N_{o p}=0$ as well.

Definition 3. For $\gamma>0$, the switched delay system (1) is said to have $L_{2}$-gain property, if, under zero initial condition $\phi(l)=$ $0, l \in\left[t_{0}-d_{M}, t_{0}\right]$, it holds that

$$
\begin{equation*}
\int_{0}^{\infty} z^{T}(s) z(s) d s \leq \gamma^{2} \int_{0}^{\infty} w^{T}(s) w(s) d s \tag{3}
\end{equation*}
$$

Lemma 4. For any given matrices $X, Y \in R^{n \times n}$, it holds that

$$
\begin{equation*}
X^{T} Y+Y^{T} X \leq \delta X^{T} X+\delta^{-1} Y^{T} Y \tag{4}
\end{equation*}
$$

Lemma 5 (see [6]). Let $A, D, E, F$, and $P$ be real matrices of appropriate dimensions with $P>0$ and $F$ satisfying $F^{T} F \leq I$. Then for any scalar $\varepsilon>0$ satisfying $P^{-1}-\varepsilon^{-1} D D^{T}>0$, one has

$$
\begin{align*}
& (A+D F E)^{T} P(A+D F E) \\
& \quad \leq A^{T}\left(P^{-1}-\varepsilon^{-1} D D^{T}\right)^{-1} A+\varepsilon E^{T} E \tag{5}
\end{align*}
$$

Lemma 6 (Schur complement). Let $M, P$, and $Q$ be given matrices such that $Q>0$. Then

$$
\begin{align*}
{\left[\begin{array}{cc}
P & M \\
* & -Q
\end{array}\right]<0 \Longleftrightarrow }  \tag{6}\\
P+M Q^{-1} M^{T}<0 .
\end{align*}
$$

Lemma 7 (see [13]). Let $\phi(k) \in R^{n}$ be a vector-valued function. If there exist any matrices $R>0, G_{1}, G_{2}$, and a scalar $d \geq 0$, then the following inequality

$$
\begin{align*}
& -\sum_{s=k-d}^{k-1} N^{T}(s) R N(s) \\
& \quad \leq \eta^{T}(k)\left[\begin{array}{cc}
G_{1}+G_{1}^{T} & -G_{1}^{T}+G_{2} \\
* & -G_{2}-G_{2}^{T}
\end{array}\right] \eta(k)  \tag{7}\\
& \quad+\eta^{T}(k)\left[\begin{array}{c}
G_{1}^{T} \\
G_{2}^{T}
\end{array}\right] d R^{-1}\left[\begin{array}{ll}
G_{1} & G_{2}
\end{array}\right] \eta(k)
\end{align*}
$$

holds, where $N(s)=\phi(s+1)-\phi(s)$ and $\eta(t)=\left[\begin{array}{c}\phi(t) \\ \phi(t-d)\end{array}\right]$.

## 3. $L_{2}$-Gain Analysis

Firstly, we will introduce two important lemmas for the $L_{2}-$ gain analysis of the switched delay system (1). The first lemma will provide the decay estimation of the Lyapunov functional $V_{i}(t)$ along the trajectory of the switched delay system without disturbances.

Lemma 8. Consider the switched delay system (1) with $w(t)=$ 0 . For given positive integers $d_{M}, d_{m}$, and $\lambda_{i}$, suppose that there exist matrices $G_{1}, G_{2}, \Omega_{1}, \Omega_{2}$, and $\Omega_{3}$ such that
(i)

$$
\begin{equation*}
\Omega_{3} \leq 0 . \tag{8}
\end{equation*}
$$

(ii)

$$
\begin{equation*}
\Omega_{1}-\Omega_{2} \Omega_{3}^{-1} \Omega_{2}^{T} \leq 0, \tag{9}
\end{equation*}
$$

where

$$
\begin{align*}
\Omega_{1}= & A_{i}^{T} P_{i} A_{i}-P_{i}+\lambda_{i}^{-2} Q_{i}+\left(d_{M}-d_{m}\right) \lambda_{i}^{-2} Q_{i} \\
& +\lambda_{i}^{-2} d_{M}\left[\lambda_{i}^{2} A_{i}^{T} R_{i} A_{i}-\lambda_{i} R_{i} A_{i}-\lambda_{i} A_{i}^{T} R_{i}+R_{i}\right] \\
& +\lambda_{i}^{-2}\left(G_{1}+G_{1}^{T}+d_{M} G_{1}^{T} R_{i}^{-1} G_{1}\right), \\
\Omega_{2}= & A_{i}^{T} P_{i} B_{i}+d_{M}\left(A_{i}^{T} R_{i} B_{i}-\lambda_{i}^{-1} R_{i} B_{i}\right)  \tag{10}\\
& +\lambda_{i}^{-2}\left(-G_{1}^{T}+G_{2}+d_{M} G_{1}^{T} R_{i}^{-1} G_{2}\right), \\
\Omega_{3}= & B_{i}^{T} P_{i} B_{i}-\lambda_{i}^{-2\left(1+d_{M}\right)} Q_{i}+d_{M} B_{i}^{T} R_{i} B_{i} \\
& +\lambda_{i}^{-2}\left(-G_{2}-G_{2}^{T}+d_{M} G_{2}^{T} R_{i}^{-1} G_{2}\right)
\end{align*}
$$

with $P_{i}, Q_{i}$, and $R_{i}$ being symmetric positive definite matrices; then the Lyapunov functional $V_{i}(t)$ along the trajectory of the switched delay system (1) will satisfy

$$
\begin{equation*}
V_{i}(t) \leq \lambda_{i}^{-2\left(t-t_{0}\right)} V_{i}\left(t_{0}\right) \tag{11}
\end{equation*}
$$

Proof. Choose the following Lyapunov functional candidate:

$$
\begin{equation*}
V_{i}(t)=V_{i_{1}}(t)+V_{i_{2}}(t)+V_{i_{3}}(t)+V_{i_{4}}(t) . \tag{12}
\end{equation*}
$$

Here,

$$
\begin{align*}
& V_{i_{1}}(t)=x^{T}(t) P_{i} x(t), \\
& V_{i_{2}}(t)=\sum_{s=t-d(t)}^{t-1} \lambda_{i}^{2(s-t)} x^{T}(s) Q_{i} x(s), \\
& V_{i_{3}}(t)=\sum_{\theta=-d_{M}+2}^{-d_{m}+1} \sum_{s=t-1+\theta}^{t-1} \lambda_{i}^{2(s-t)} x^{T}(s) Q_{i} x(s),  \tag{13}\\
& V_{i_{4}}(t)=\sum_{\theta=-d_{M}+1}^{0} \sum_{s=t-1+\theta}^{t-1} \lambda_{i}^{2(s-t)} y^{T}(s) R_{i} y(s),
\end{align*}
$$

where $P_{i}, Q_{i}$, and $R_{i}$ are symmetric positive definite matrices, $\lambda_{i}>1$ is a given constant, and $y(s)=\lambda_{i} x(s+1)-x(s)$. Next, we will estimate the difference of $V_{i}(t)$ along the trajectory of the switched delay system (1):

$$
\begin{aligned}
\Delta V_{i_{1}}(t) & =V_{i_{1}}(t+1)-V_{i_{1}}(t) \\
& =x^{T}(t+1) P_{i} x(t+1)-x^{T}(t) P_{i} x(t)
\end{aligned}
$$

$$
\begin{align*}
= & x^{T}(t) A_{i}^{T} P_{i} A_{i} x(t) \\
& +x^{T}(t-d(t)) B_{i}^{T} P_{i} A_{i} x(t) \\
& +x^{T}(t) A_{i}^{T} P_{i} B_{i} x(t-d(t)) \\
& +x^{T}(t-d(t)) B_{i}^{T} P_{i} B_{i} x(t-d(t)) \\
& -x^{T}(t) P_{i} x(t) . \tag{14}
\end{align*}
$$

Then, we have

$$
\begin{align*}
& \Delta V_{i_{1}}(t)=\left[\begin{array}{c}
x(t) \\
x(t-d(t))
\end{array}\right]^{T} \\
& \quad .\left[\begin{array}{cc}
A_{i}^{T} P_{i} A_{i}-P_{i} & A_{i}^{T} P_{i} B_{i} \\
B_{i}^{T} P_{i} A_{i} & B_{i}^{T} P_{i} B_{i}
\end{array}\right]\left[\begin{array}{c}
x(t) \\
x(t-d(t))
\end{array}\right],  \tag{15}\\
& \Delta V_{i_{2}}(t)=V_{i_{2}}(t+1)-V_{i_{2}}(t) \leq V_{i_{2}}(t+1)-\lambda_{i}^{-2} V_{i_{2}}(t) \\
& \quad=\sum_{s=t+1-d(t+1)}^{t} \lambda_{i}^{2(s-t-1)} x^{T}(s) Q_{i} x(s) \\
& \quad-\sum_{s=t-d(t)}^{t-1} \lambda_{i}^{2(s-t-1)} x^{T}(s) Q_{i} x(s)=\lambda_{i}^{-2} x^{T}(t) Q_{i} x(t)  \tag{16}\\
& \quad+\sum_{s=t+1-d(t+1)}^{t-1} \lambda_{i}^{2(s-t-1)} x^{T}(s) Q_{i} x(s) \\
& \quad-\sum_{s=t-d(t)}^{t-1} \lambda_{i}^{2(s-t-1)} x^{T}(s) Q_{i} x(s) .
\end{align*}
$$

Since the delay $d(t)$ satisfies $0<d_{m}<d(t) \leq d_{M}$, we can consider the following two cases.

When $d_{m}>1$, it holds that

$$
\begin{align*}
& \sum_{s=t+1-d(t+1)}^{t-1} \lambda_{i}^{2(s-t-1)} x^{T}(s) Q_{i} x(s) \\
& \leq \sum_{s=t+1-d_{m}}^{t-1} \lambda_{i}^{2(s-t-1)} x^{T}(s) Q_{i} x(s) \\
& \quad+\sum_{s=t+1-d(t+1)}^{t-d_{m}} \lambda_{i}^{2(s-t-1)} x^{T}(s) Q_{i} x(s)  \tag{17}\\
& \leq \sum_{s=t+1-d(t)}^{t-1} \lambda_{i}^{2(s-t-1)} x^{T}(s) Q_{i} x(s) \\
& \quad+\sum_{s=t+1-d_{M}}^{t-d_{m}} \lambda_{i}^{2(s-t-1)} x^{T}(s) Q_{i} x(s)
\end{align*}
$$

When $d_{m}=1$,

$$
\begin{align*}
& \sum_{s=t+1-d(t+1)}^{t-1} \lambda_{i}^{2(s-t-1)} x^{T}(s) Q_{i} x(s) \\
& \leq \sum_{s=t+1-d(t)}^{t-1} \lambda_{i}^{2(s-t-1)} x^{T}(s) Q_{i} x(s)  \tag{18}\\
& \quad+\sum_{s=t+1-d_{M}}^{t-d_{m}} \lambda_{i}^{2(s-t-1)} x^{T}(s) Q_{i} x(s)
\end{align*}
$$

is satisfied as well.
So from (16) and (17) we can obtain

$$
\begin{align*}
\Delta V_{i_{2}}(t) \leq & \lambda_{i}^{-2} x^{T}(t) Q_{i} x(t) \\
& +\sum_{s=t+1-d(t)}^{t-1} \lambda_{i}^{2(s-t-1)} x^{T}(s) Q_{i} x(s) \\
& +\sum_{s=t+1-d_{M}}^{t-d_{m}} \lambda_{i}^{2(s-t-1)} x^{T}(s) Q_{i} x(s) \\
& -\sum_{s=t-d(t)}^{t-1} \lambda_{i}^{2(s-t-1)} x^{T}(s) Q_{i} x(s)  \tag{19}\\
= & \lambda_{i}^{-2} x^{T}(t) Q_{i} x(t) \\
& -\lambda_{i}^{2(-1-d(t))} x^{T}(t-d(t)) Q_{i} x(t-d(t)) \\
& +\sum_{s=t+1-d_{M}}^{t-d_{m}} \lambda_{i}^{2(s-t-1)} x^{T}(s) Q_{i} x(s)
\end{align*}
$$

Since $-\lambda_{i}^{2(-1-d(t))} \leq-\lambda_{i}^{2\left(-1-d_{M}\right)}$, we get

$$
\begin{align*}
\Delta V_{i_{2}}(t) \leq & {\left[\begin{array}{c}
x(t) \\
x(t-d(t))
\end{array}\right]^{T} } \\
& \cdot\left[\begin{array}{cc}
\lambda_{i}^{-2} Q_{i} & 0 \\
0 & -\lambda_{i}^{2\left(-1-d_{M}\right)} Q_{i}
\end{array}\right]\left[\begin{array}{c}
x(t) \\
x(t-d(t))
\end{array}\right]  \tag{20}\\
& +\sum_{s=t+1-d_{M}}^{t-d_{m}} \lambda_{i}^{2(s-t-1)} x^{T}(s) Q_{i} x(s)
\end{align*}
$$

$$
\begin{align*}
\Delta V_{i_{4}}(t) \leq & \lambda_{i}^{-2} d_{M}\left[\begin{array}{c}
x(t) \\
x(t-d(t))
\end{array}\right]^{T}\left[\begin{array}{cc}
\left(\lambda_{i} A_{i}^{T}-I\right) R_{i}\left(\lambda_{i} A_{i}-I\right) & \lambda_{i}\left(\lambda_{i} A_{i}^{T}-I\right) R_{i} B_{i} \\
B_{i}^{T} R_{i}\left(\lambda_{i} A_{i}-I\right) \lambda_{i} & \lambda_{i}^{2} B_{i}^{T} R_{i} B_{i}
\end{array}\right]\left[\begin{array}{c}
x(t) \\
x(t-d(t))
\end{array}\right]  \tag{24}\\
& -\sum_{s=t-d_{M}}^{t-1} \lambda_{i}^{2(s-t-1)} y^{T}(s) R_{i} y(s)
\end{align*}
$$

where we apply the transformation $\phi(s)=\lambda_{i}^{(s-t-1)} x(s)$. Then we have $\lambda_{i}^{s-t-1} y(s)=\phi(s+1)-\phi(s)$; by Lemma 7 we continue to have

$$
\begin{gathered}
-\sum_{s=t-d_{M}}^{t-1} \lambda_{i}^{2(s-t-1)} y^{T}(s) R_{i} y(s) \leq\left[\begin{array}{c}
\phi(t) \\
\phi(t-d(t))
\end{array}\right]^{T} \\
\cdot\left[\begin{array}{cc}
G_{1}+G_{1}^{T} & -G_{1}^{T}+G_{2} \\
* & -G_{2}-G_{2}^{T}
\end{array}\right]\left[\begin{array}{c}
\phi(t) \\
\phi(t-d(t))
\end{array}\right]
\end{gathered}
$$

$$
\begin{align*}
& +\left[\begin{array}{c}
\phi(t) \\
\phi(t-d(t))
\end{array}\right]^{T} \\
& \cdot\left[\begin{array}{c}
G_{1}^{T} \\
G_{2}^{T}
\end{array}\right] d_{M} R_{i}^{-1}\left[\begin{array}{ll}
G_{1} & G_{2}
\end{array}\right]\left[\begin{array}{c}
\phi(t) \\
\phi(t-d(t))
\end{array}\right] \tag{25}
\end{align*}
$$

Due to the fact that $\phi(t)=\lambda_{i}^{-1} x(t), \phi(t-d(t))=\lambda_{i}^{-(d(t)+1)} x(t-$ $d(t)) \leq \lambda_{i}^{-1} x(t-d(t))$, it holds that

$$
\begin{align*}
\Delta V_{i_{4}}(t) \leq & \lambda_{i}^{-2} d_{M}\left[\begin{array}{c}
x(t) \\
x(t-d(t))
\end{array}\right]^{T}\left[\begin{array}{cc}
\left(\lambda_{i} A_{i}^{T}-I\right) R_{i}\left(\lambda_{i} A_{i}-I\right) & \lambda_{i}\left(\lambda_{i} A_{i}^{T}-I\right) R_{i} B_{i} \\
B_{i}^{T} R_{i}\left(\lambda_{i} A_{i}-I\right) \lambda_{i} & \lambda_{i}^{2} B_{i}^{T} R_{i} B_{i}
\end{array}\right]\left[\begin{array}{c}
x(t) \\
x(t-d(t))
\end{array}\right] \\
& +\lambda_{i}^{-2}\left[\begin{array}{c}
x(t) \\
x(t-d(t))
\end{array}\right]^{T}\left[\begin{array}{cc}
G_{1}+G_{1}^{T} & -G_{1}^{T}+G_{2} \\
* & -G_{2}-G_{2}^{T}
\end{array}\right]\left[\begin{array}{c}
x(t) \\
x(t-d(t))
\end{array}\right]  \tag{26}\\
& +\lambda_{i}^{-2}\left[\begin{array}{c}
x(t) \\
x(t-d(t))
\end{array}\right]^{T}\left[\begin{array}{c}
G_{1}^{T} \\
G_{2}^{T}
\end{array}\right] d_{M} R_{i}^{-1}\left[\begin{array}{ll}
G_{1} & G_{2}
\end{array}\right]\left[\begin{array}{c}
x(t) \\
x(t-d(t))
\end{array}\right]
\end{align*}
$$

Let $\xi(t)=\left[\begin{array}{c}x(t) \\ x(t-d(t))\end{array}\right]$; then we add (15), (20), (22), and (26) together to yield

$$
\begin{equation*}
\Delta V_{i} \leq \xi^{T}(t) \Omega \xi(t) \tag{27}
\end{equation*}
$$

where $\Omega=\left[\begin{array}{cc}\Omega_{1} & \Omega_{2} \\ * & \Omega_{3}\end{array}\right]$,

$$
\begin{aligned}
\Omega_{1}= & A_{i}^{T} P_{i} A_{i}-P_{i}+\lambda_{i}^{-2} Q_{i}+\left(d_{M}-d_{m}\right) \lambda_{i}^{-2} Q_{i} \\
& +\lambda_{i}^{-2} d_{M}\left[\lambda_{i}^{2} A_{i}^{T} R_{i} A_{i}-\lambda_{i} R_{i} A_{i}-\lambda_{i} A_{i}^{T} R_{i}+R_{i}\right] \\
& +\lambda_{i}^{-2}\left(G_{1}+G_{1}^{T}+d_{M} G_{1}^{T} R_{i}^{-1} G_{1}\right), \\
\Omega_{2}= & A_{i}^{T} P_{i} B_{i}+d_{M}\left(A_{i}^{T} R_{i} B_{i}-\lambda_{i}^{-1} R_{i} B_{i}\right) \\
& +\lambda_{i}^{-2}\left(-G_{1}^{T}+G_{2}+d_{M} G_{1}^{T} R_{i}^{-1} G_{2}\right), \\
\Omega_{3}= & B_{i}^{T} P_{i} B_{i}-\lambda_{i}^{-2\left(1+d_{M}\right)} Q_{i}+d_{M} B_{i}^{T} R_{i} B_{i} \\
& +\lambda_{i}^{-2}\left(-G_{2}-G_{2}^{T}+d_{M} G_{2}^{T} R_{i}^{-1} G_{2}\right) .
\end{aligned}
$$

By (8) and (9) and Lemma 6, we can obtain

$$
\Omega=\left[\begin{array}{cc}
\Omega_{1} & \Omega_{2}  \tag{29}\\
* & \Omega_{3}
\end{array}\right] \leq 0
$$

It follows from (27) and (29) that

$$
\begin{equation*}
V_{i}(t+1) \leq \lambda_{i}^{-2} V_{i}(t) \tag{30}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
V_{i}(t) \leq \lambda_{i}^{-2} V_{i}(t-1) \leq \cdots \leq \lambda_{i}^{-2\left(t-t_{0}\right)} V_{i}\left(t_{0}\right) \tag{31}
\end{equation*}
$$

This completes the proof.

Lemma 9. For given constants $\lambda_{i}$ and $\gamma_{0}$, suppose that there exist matrices $\Xi_{1}, \Xi_{2}$, and $\Xi_{3}$ such that
(i)

$$
\begin{equation*}
\Xi_{3} \leq 0 \tag{32}
\end{equation*}
$$

$$
\begin{equation*}
\Xi_{1}-\Xi_{2} \Xi_{3}^{-1} \Xi_{2}^{T} \leq 0 \tag{33}
\end{equation*}
$$

and $\gamma_{0}>0, \varepsilon_{1}>0$, and $\varepsilon_{2}>0$ satisfying

$$
\begin{equation*}
\gamma_{0}^{2} I \geq \varepsilon_{1}^{-1} I+\varepsilon_{2}^{-1} I+C_{i}^{T} P_{i} C_{i}+d_{M} C_{i}^{T} R_{i} C_{i}+E_{i}^{T} E_{i} ; \tag{34}
\end{equation*}
$$

then along the trajectory of system (1), one has

$$
\begin{equation*}
V_{i}(t+1) \leq \lambda_{i}^{-2} V_{i}(t)+\gamma_{0}^{2} w^{T}(t) w(t)-Z^{T}(t) Z(t) \tag{35}
\end{equation*}
$$

where

$$
\begin{align*}
& \Xi_{1}=\Omega_{1}+\varepsilon_{1} \varphi_{1_{i}}^{T} \varphi_{1_{i}}+D_{i}^{T} D_{i} \\
& \Xi_{2}=\Omega_{2} \\
& \Xi_{3}=\Omega_{3}+\varepsilon_{2} \varphi_{2_{i}}^{T} \varphi_{2_{i}}  \tag{36}\\
& \varphi_{1_{i}}=C_{i}^{T} P_{i} A_{i}+d_{M}\left(C_{i}^{T} R_{i} A_{i}-\lambda_{i}^{-1} C_{i}^{T} R_{i}\right)+E_{i}^{T} D_{i}
\end{align*}
$$

$$
\varphi_{2_{i}}=C_{i}^{T} P_{i} B_{i}+d_{M} C_{i}^{T} R_{i} B_{i} .
$$

Proof. Using Lemma 8 and (1), we have

$$
\begin{align*}
& V_{i}(t+1)-\lambda_{i}^{-2} V_{i}(t)+Z^{T}(t) Z(t)-\gamma_{0}^{2} w^{T}(t) w(t) \\
& \quad \leq \xi^{T}(t) \Omega \xi(t)+x^{T}(t) \\
& \quad \cdot\left[A_{i}^{T} P_{i} C_{i}+d_{M}\left(A_{i}^{T} R_{i} C_{i}-\lambda_{i}^{-1} R_{i} C_{i}\right)+D_{i}^{T} E_{i}\right] \\
& \quad \cdot w(t)+w^{T}(t) \\
& \quad \cdot\left[C_{i}^{T} P_{i} A_{i}+d_{M}\left(C_{i}^{T} R_{i} A_{i}-\lambda_{i}^{-1} C_{i}^{T} R_{i}\right)+E_{i}^{T} D_{i}\right]  \tag{37}\\
& \quad \cdot x(t)+x^{T}(t-d(t))\left[B_{i}^{T} P_{i} C_{i}+d_{M} B_{i}^{T} R_{i} C_{i}\right] w(t) \\
& \quad+w^{T}(t)\left[C_{i}^{T} P_{i} B_{i}+d_{M} C_{i}^{T} R_{i} B_{i}\right] x(t-d(t)) \\
& \quad+x^{T}(t) D_{i}^{T} D_{i} x(t)+w^{T}(t) \\
& \quad \cdot\left(C_{i}^{T} P_{i} C_{i}+d_{M} C_{i}^{T} R_{i} C_{i}+E_{i}^{T} E_{i}-\gamma_{0}^{2} I\right) w(t)
\end{align*}
$$

Based on Lemmas 4 and 5, it holds that

$$
\begin{aligned}
& x^{T}(t) {\left[A_{i}^{T} P_{i} C_{i}+d_{M}\left(A_{i}^{T} R_{i} C_{i}-\lambda_{i}^{-1} R_{i} C_{i}\right)+D_{i}^{T} E_{i}\right] } \\
& \cdot w(t)+w^{T}(t) \\
& \cdot {\left[C_{i}^{T} P_{i} A_{i}+d_{M}\left(C_{i}^{T} R_{i} A_{i}-\lambda_{i}^{-1} C_{i}^{T} R_{i}\right)+E_{i}^{T} D_{i}\right] } \\
& \cdot x(t) \leq \varepsilon_{1} x^{T}(t) \varphi_{1_{i}}^{T} \varphi_{1_{i}} x(t)+\varepsilon_{1}^{-1} w^{T}(t) w(t) . \\
& x^{T}(t-d(t))\left[B_{i}^{T} P_{i} C_{i}+d_{M} B_{i}^{T} R_{i} C_{i}\right] w(t)+w^{T}(t) \\
& \cdot {\left[C_{i}^{T} P_{i} B_{i}+d_{M} C_{i}^{T} R_{i} B_{i}\right] x(t-d(t)) } \\
& \quad \leq \varepsilon_{2} x^{T}(t-d(t)) \varphi_{2_{i}}^{T} \varphi_{2_{i}} x(t-d(t))+\varepsilon_{2}^{-1} w^{T}(t) \\
& \cdot w(t) .
\end{aligned}
$$

Then, it follows from (35) and (38) that

$$
\begin{align*}
& V_{i}(t+1)-\lambda_{i}^{-2} V_{i}(t)+Z^{T}(t) Z(t)-\gamma_{0}^{2} w^{T}(t) w(t) \\
& \quad \leq \xi^{T}(t) \\
& \quad .\left[\begin{array}{cc}
\Omega_{1}+\varepsilon_{1} \varphi_{1_{i}}^{T} \varphi_{1_{i}}+D_{i}^{T} D_{i} & \Omega_{2} \\
* & \Omega_{3}+\varepsilon_{2} \varphi_{2_{i}}^{T} \varphi_{2_{i}}
\end{array}\right] \xi(t)  \tag{39}\\
& \quad+w^{T}(t)\left[\varepsilon_{1}^{-1} I+\varepsilon_{2}^{-1} I+C_{i}^{T} P_{i} C_{i}+d_{M} C_{i}^{T} R_{i} C_{i}\right. \\
& \left.\quad+E_{i}^{T} E_{i}-\gamma_{0}^{2} I\right] w(t) .
\end{align*}
$$

Combining (32), (33) with (34) will lead to

$$
\begin{equation*}
V_{i}(t+1) \leq \lambda_{i}^{-2} V_{i}(t)+\gamma_{0}^{2} w^{T}(t) w(t)-Z^{T}(t) Z(t) \tag{40}
\end{equation*}
$$

This completes the proof.

Now, our $L_{2}$-gain analysis results can be presented as follows.

Theorem 10. For given constants $\lambda_{i}$ and $\gamma_{0}$, suppose that there exist matrices $\Xi_{1}, \Xi_{2}$, and $\Xi_{3}$ such that
(i)

$$
\begin{equation*}
\Xi_{3} \leq 0 \tag{41}
\end{equation*}
$$

(ii)

$$
\begin{equation*}
\Xi_{1}-\Xi_{2} \Xi_{3}^{-1} \Xi_{2}^{T} \leq 0 \tag{42}
\end{equation*}
$$

and $\gamma_{0}>0, \varepsilon_{1}>0$, and $\varepsilon_{2}>0$ satisfying

$$
\begin{equation*}
\gamma_{0}^{2} I \geq \varepsilon_{1}^{-1} I+\varepsilon_{2}^{-1} I+C_{i}^{T} P_{i} C_{i}+d_{M} C_{i}^{T} R_{i} C_{i}+E_{i}^{T} E_{i} . \tag{43}
\end{equation*}
$$

Then the switched delay system (1) has a $L_{2}$-gain with MDADT $\tau_{a p}>\tau_{a p}^{*}=\ln \mu_{p} / 2 \ln \lambda_{p}$, where $\mu_{p} \geq 1$ satisfying (35) and $\varphi_{1_{i}}$, $\varphi_{2_{i}}, \Xi_{1}, \Xi_{2}$, and $\Xi_{3}$ are defined in Lemma 9.

Proof. Choose the Lyapunov functional candidate (12). From (41) and (42) and Lemma 9, we have

$$
\begin{equation*}
V_{i}(t+1) \leq \lambda_{i}^{-2} V_{i}(t)+\gamma_{0}^{2} w^{T}(t) w(t)-Z^{T}(t) Z(t) . \tag{44}
\end{equation*}
$$

Let $\Gamma(t)=\gamma_{0}^{2} w^{T}(t) w(t)-Z^{T}(t) Z(t)$. From (35), since $t_{i-1}=$ $t_{i}-1$, we have

$$
\begin{align*}
& V_{\sigma(t)}(t) \leq \lambda_{\sigma\left(t_{i}\right)}^{-2\left(t-t_{i}\right)} V_{\sigma\left(t_{i}\right)}\left(t_{i}\right)+\sum_{j=t_{i}}^{t-1} \lambda_{\sigma\left(t_{i}\right)}^{-2(t-j-1)} \Gamma(j) \\
& \quad \leq \mu_{\sigma\left(t_{i}\right)} \lambda_{\sigma\left(t_{i}\right)}^{-2\left(t-t_{i}\right)} V_{\sigma\left(t_{i-1}\right)}\left(t_{i}\right)+\sum_{j=t_{i}}^{t-1} \lambda_{\sigma\left(t_{i}\right)}^{-2(t-j-1)} \Gamma(j) \\
& \quad \leq \mu_{\sigma\left(t_{i}\right)} \lambda_{\sigma\left(t_{i}\right)}^{-2\left(t-t_{i}\right)}\left\{\lambda_{\sigma\left(t_{i-1}\right)}^{-2\left(t_{i}-t_{i-1}\right)} V_{\sigma\left(t_{i-1}\right)}\left(t_{i-1}\right)\right. \\
& \left.\quad+\sum_{j=t_{i-1}}^{t_{i}-1} \lambda_{\sigma\left(t_{i-1}\right)}^{-2\left(t_{i}-j-1\right)} \Gamma(j)\right\}+\sum_{j=t_{i}}^{t-1} \lambda_{\sigma\left(t_{i}\right)}^{-2(t-j-1)} \Gamma(j) \leq \cdots \\
& \quad \leq\left(\prod_{s=1}^{i} \mu_{\sigma\left(t_{s}\right)}\right) \cdot \exp \left(-2 \sum_{s=1}^{i} \ln \lambda_{\sigma\left(t_{s-1}\right)}\left(t_{s}-t_{s-1}\right)\right)  \tag{45}\\
& \quad \cdot V_{\sigma\left(t_{0}\right)}\left(t_{0}\right)+\sum_{k=1}^{i}\left[\left(\prod_{s=k}^{i} \mu_{\sigma\left(t_{s}\right)}\right)\right. \\
& \left.\quad \cdot \exp \left(-2 \sum_{s=k}^{i} \ln \lambda_{\sigma\left(t_{s-1}\right)}\left(t_{s}-t_{s-1}\right)\right)\right] \Gamma\left(t_{k-1}\right) \\
& \quad=\exp \left[\sum_{s=1}^{i}\left(\ln \mu_{\sigma\left(t_{s}\right)}-2 \ln \lambda_{\sigma\left(t_{s-1}\right)}\left(t_{s}-t_{s-1}\right)\right)\right] \\
& \quad \cdot V_{\sigma\left(t_{0}\right)}\left(t_{0}\right) \\
& \quad+\sum_{k=1}^{i}\left\{\exp \left[\sum_{s=k}^{i}\left(\ln \mu_{\sigma\left(t_{s}\right)}-2 \ln \lambda_{\sigma\left(t_{s-1}\right)}\left(t_{s}-t_{s-1}\right)\right)\right]\right\} \\
& \quad \cdot \Gamma\left(t_{k-1}\right)
\end{align*}
$$

which combined with Definition 2 and the MDADT scheme $N_{\sigma p}(T, t) \leq T_{p}(T, t) / \tau_{a p}$ yields

$$
\begin{aligned}
& V_{\sigma(t)}(t) \leq \exp \left[\sum_{p=1}^{m}\left(\ln \mu_{p}^{N_{\sigma p}\left(t, t_{0}\right)}-2 \ln \lambda_{p} T_{p}\left(t, t_{0}\right)\right)\right] \\
& \quad \cdot V_{\sigma\left(t_{0}\right)}\left(t_{0}\right) \\
& \quad+\sum_{k=1}^{i}\left\{\exp \left[\sum_{p=1}^{m}\left(\ln \mu_{p}^{N_{\sigma p}\left(t, t_{k-1}\right)}-2 \ln \lambda_{p} T_{p}\left(t, t_{k-1}\right)\right)\right]\right\} \\
& \quad \cdot \Gamma\left(t_{k-1}\right) \leq e^{-\beta T_{p}\left(t, t_{0}\right)} V_{\sigma\left(t_{0}\right)}\left(t_{0}\right)+\sum_{k=1}^{i} e^{-\beta T_{p}\left(t, t_{k-1}\right)} \Gamma\left(t_{k-1}\right)
\end{aligned}
$$

where $\beta=\sum_{p=1}^{m}\left(2 \ln \lambda_{p}-\ln \mu_{p} / \tau_{\sigma p}\right)>0$.
Under zero initial condition, from (46), one obtains

$$
\begin{equation*}
0 \leq V_{\sigma(t)}(t) \leq \sum_{k=1}^{i} e^{-\beta T_{p}\left(t, t_{0}\right)} \Gamma\left(t_{k-1}\right) \tag{47}
\end{equation*}
$$

which implies that

$$
\begin{align*}
& \sum_{k=1}^{i} e^{-\beta T_{p}\left(t, t_{k-1}\right)} Z^{T}\left(t_{k-1}\right) Z\left(t_{k-1}\right)  \tag{48}\\
& \quad \leq \sum_{k=1}^{i} e^{-\beta T_{p}\left(t, t_{k-1}\right)} \gamma_{0}^{2} w^{T}\left(t_{k-1}\right) w\left(t_{k-1}\right) .
\end{align*}
$$

Then, we multiply both sides by $e^{-\beta T_{p}\left(t_{k-1}, t_{0}\right)}$ to get

$$
\begin{align*}
& \sum_{k=1}^{i} e^{-\beta T_{p}\left(t, t_{0}\right)} Z^{T}\left(t_{k-1}\right) Z\left(t_{k-1}\right) \\
& \quad \leq \sum_{k=1}^{i} e^{-\beta T_{p}\left(t, t_{0}\right)} \gamma_{0}^{2} w^{T}\left(t_{k-1}\right) w\left(t_{k-1}\right) \tag{49}
\end{align*}
$$

Thus,

$$
\sum_{k=0}^{i} Z^{T}\left(t_{k}\right) Z\left(t_{k}\right) \leq \sum_{k=0}^{i} \gamma_{0}^{2} w^{T}\left(t_{k}\right) w\left(t_{k}\right)
$$

This completes the proof.

## 4. A Numerical Example

Consider the switched delay system (1) with the following specifications:

$$
\begin{align*}
& A_{1}=\left[\begin{array}{cc}
-0.2 & 0.3 \\
0.1 & -0.5
\end{array}\right] \\
& B_{1}=\left[\begin{array}{cc}
0.4 & 0 \\
0.1 & -0.5
\end{array}\right] \\
& A_{2}=\left[\begin{array}{cc}
-0.1 & 1 \\
0 & -0.6
\end{array}\right] \\
& B_{2}=\left[\begin{array}{cc}
-0.7 & 0.1 \\
1 & 0.2
\end{array}\right]  \tag{51}\\
& C_{1}=\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right] \\
& C_{2}=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right] \\
& D_{1}=[1,1] \\
& D_{2}=[0,1] \\
& E_{1}=E_{2}=[0.2,0.8]
\end{align*}
$$

and $d(t)=\sin (t \pi / 2)+1$, so that $d_{M}=2, d_{m}=0$. The disturbance input is defined as

$$
w(t)= \begin{cases}1, & 0<t \leq 20  \tag{52}\\ 0, & t>20\end{cases}
$$

Let $\mu_{1}=\mu_{2}=12$; by the LMI Control Toolbox and Theorem 10, we obtain

$$
\begin{align*}
& P_{1}=\left[\begin{array}{ll}
12.5739 & 5.0613 \\
5.0613 & 4.8703
\end{array}\right], \\
& Q_{1}=\left[\begin{array}{ll}
2.6676 & 1.2445 \\
1.2445 & 0.9452
\end{array}\right], \\
& R_{1}=\left[\begin{array}{cc}
4.4703 & -0.6996 \\
-0.6996 & 9.3862
\end{array}\right],  \tag{53}\\
& P_{2}=\left[\begin{array}{ll}
12.3445 & 18.3565 \\
18.3565 & 30.2222
\end{array}\right], \\
& Q_{2}=\left[\begin{array}{ll}
2.3941 & 3.7907 \\
3.7907 & 6.1097
\end{array}\right], \\
& R_{2}=\left[\begin{array}{ll}
16.1813 & 18.8134 \\
18.8134 & 22.4268
\end{array}\right],
\end{align*}
$$

and $\lambda_{1}=27.2485, \lambda_{2}=38.7807$, where $\tau_{a 1}^{*}=\ln \mu_{1} / 2 \ln \lambda_{1}=$ 0.3759 and $\tau_{a 2}^{*}=\ln \mu_{2} / 2 \ln \lambda_{2}=0.3397$. Now, we choose


Figure 1: State trajectories of the switched delay system (1) under MDADT switching.
the switching periods $\tau_{a 1}=2, \tau_{a 2}=1$ and take the initial state condition $\psi(l)=[1 ; 2]$ for all $l=-2,-1,0$. Then the numerical simulations can be shown in Figure 1.

It can be seen from Figure 1 that under the designed MDADT switching signals the switched delay system can achieve better dynamics performance and disturbance tolerance capability, which shows the potentiality of our results in practice.

## 5. Conclusions

In this paper, the problem of $L_{2}$-gain analysis for discretetime switched systems with MDADT switching has been investigated. By combining with the multiple Lyapunov function method, sufficient conditions are established to ensure $L_{2}$-gain performance for discrete-time switched delay system, and the admissible MDADT switching signals are also designed accordingly. Finally, a numerical example is given to demonstrate the usefulness of the obtained results.

## Competing Interests

The authors declare that they have no competing interests.

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