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Research Article

Limit Cycle Analysis in a Class of Hybrid Systems

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Hybrid systems are those that inherently combine discrete and continuous dynamics. This paper considers the hybrid system model to be an extension of the discrete automata associating a continuous evolution with each discrete state. This model is called the hybrid automaton. In this work, we achieve a mathematical formulation of the steady state and we show a way to obtain the initial conditions region to reach a specific limit cycle for a class of uncoupled and coupled continuous-linear hybrid systems. The continuous-linear term is used in the sense of the system theory and, in this sense, continuous-linear hybrid automata will be defined. Thus, some properties and theorems that govern the hybrid automata dynamic behavior to evaluate a limit cycle existence have been established; this content is explained under a theoretical framework.

1. Introduction

Many system structure changes lead to discontinuities in their dynamics. These changes may be caused by discrete events generated by discrete actuators, sensors, or inherent process discontinuities. In general, systems where continuous and discrete components interact are called hybrid systems. This interaction reflects the compositional properties underlying the hybrid system model [1]. Several mathematical models have been proposed for hybrid systems with different modeling capabilities and different purposes [2–6]. In this paper, a continuous dynamic is represented by a differential equation set and the discrete dynamic by a finite state machine. This model is called the hybrid automaton and was introduced by [7] that is probably the most powerful model. In this work some properties are determined for hybrid automata. Using reachability properties we have provided an analytical formulation of the steady state behavior that gives the necessary and sufficient conditions to reach a limit cycle in a continuous-linear hybrid system. Herein, the continuous-linear term is used in the sense of the system theory and in this sense continuous-linear hybrid automata will be defined. The framework used here is introduced in [7]. This paper is organized as follows. Section 1 is an introduction to the topic; Section 2 presents the hybrid automaton model, defines different subclass of hybrid automata, and presents some definitions and properties for discrete transition enabling in the hybrid dynamic evolution. Section 3 briefly presents a reachability analysis of uncoupled case and provides the necessary conditions to obtain a specific limit cycle. In Section 4 a coupled case will be conducted and presents a detailed analysis of a limit cycle existence for the coupled case presented. Finally, Section 5 presents a brief summary and some issues for further research.

2. Theory about Hybrid Automata

In the discrete event systems analysis of hybrid systems, two important approaches are established: the automata approach [8] and the Petri net approach (PN). In this work the automata approach is considered. Hybrid automata take their origin from the finite automaton model [9]. This basic model shows only commutation logic, that is, the discrete event sequences. With this model we can only express, for example, the event sequence generated until a marked state

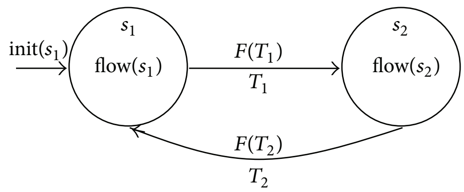


FIGURE 1: Representation of hybrid automaton.

is reached, starting from an initial state. This model is simple but insufficient when time has to be taken into account. Therefore, to enrich the previous model, the timed (finite) automaton was proposed by [10]. Thus, a timed automaton is a finite automaton with a finite set of real-valued clocks. The clocks can be reset to 0 (independently of each other) with the transitions of the automaton and keep track of the time elapsed since the last reset. It provides a simple way to annotate state-transition graphs with timing constraints. On the other hand, the timed automata model is not enough if we wish to model not only the time evolution of the discrete transitions, but also the process variable continuous evolution. Therefore, if we wish to model the continuous evolution of a process model, a hybrid automata model is suggested.

2.1. Hybrid Automata Model. A hybrid automaton is a formal model for a dynamic system with discrete and continuous components (Figure 1). The vertices of a graph, called locations or control modes, model the state of the discrete system, and the arcs, called transitions or control switches, model the discrete dynamics. Thus, the hybrid automaton model (as an extension of the finite automaton) is a bipartite graph, that is, locations (discrete states usually represented by circles) and oriented arcs (transitions represented by arrows). It is compulsory for each oriented arc to have a location at its end. Therefore, locations are connected by arcs. The number of locations is finite and nonzero. The number of arcs is also finite and nonzero. The state of the continuous system is modeled by points in R^r and, in each location, its continuous dynamic is modeled by flow conditions such as differential equations.

Definition 1 (hybrid automata). Based on [8, 11], we consider a hybrid automaton $H = (X, S, \text{flow}, E, F, \sum, \text{init})$ consists of the following:

(i) X: variables, a finite ordered set of real-valued variables. The size r of X is called the automaton dimension.

(ii) S: locations: a finite set of n locations $S = \{s_1, s_2, \ldots, s_n\}$.

(iii) flow: flow conditions: a labeling function flow that assigns a flow condition flow(s_i) to each location $s_i \in S$, where the continuous dynamic is usually represented by differential equations.

(iv) E: transitions, a finite set E of discrete jumps (edges) called transitions and represented by arcs. Each transition $T_i = (s_i, s_j)$ identifies a source location and a target location.

(v) F: jump conditions, a labeling function $F(T_i)$ that assigns a jump condition to each transition $T_i \in E$. The jump conditions relate the values of the variables and their tangents before a (discrete) transition with those after a transition. They also provide the conditions to fire a transition in general as boolean combinations of inequalities.

(vi) $\sum$: events, a finite set of events which made up $\sum$ and a labeling function which is assigned to each *event* in $\sum$ (including empty event) to every transition $T_i \in E$. The events can be used to define and model the parallel composition of hybrid automata.

(vii) init: initial conditions, a labeling function init(s_i) that assigns an initial condition to each location $s_i \in S$. In the automata graph representations, initial conditions appear as labels on incoming arrows without a source location.

From the field of computer science, there are algorithmic techniques for checking certain properties of linear hybrid automata [7, 12]. These automata have linearity restrictions on transitions (linear inequalities between sources and targets of jumps) and continuous flows of variables (differential inequalities $\dot{x} \geq B$). The definition of linearity used there is more restrictive than in system theory. For instance, linear hybrid automata cannot directly model continuous flows of the form $\dot{x} = Ax$.

Definition 2 (linear hybrid automaton). A hybrid automaton H is a linear hybrid automaton H_L if

(1) all variables of H are linear,

(2) for all locations $s_i \in S$, the initial condition init(s_i) and the flow condition flow(s_i) are all convex,

(3) for all transition $T_i \in E$, the jump condition $F(T_i)$ is convex.

For example, in the hybrid automaton H of Figure 2, we obtain a linear hybrid automaton H_L of dimension 2 setting $r = 2$, $\forall i \in \{1, 2\}$:

$$\begin{aligned} \text{init}(s_1) &\Longrightarrow x(0) = [x_1(0) \quad x_2(0)]^T \\ \text{flow}(s_i) &\Longrightarrow \dot{x}(t) = B_i, \\ F(T_i) &\Longrightarrow C_i x(t) \geq K_i \end{aligned} \tag{1}$$

considering the constant values $x(0), B_i, K_i \in R^2$ and $C_i \in R^2 \times R^2$.

It is important to emphasize in the present work that the continuous-linear term is used in the sense of the system theory. Therefore, this paper considers the continuous-linear hybrid automaton to be a general model to represent continuous-linear dynamics in hybrid systems. On the other

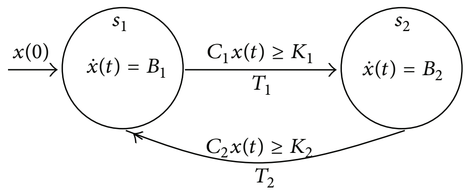


Figure 2: Two-location linear hybrid automaton.

hand, as in the case of linear hybrid automata, continuous-linear hybrid automata have the same linearity restrictions on discrete transitions. Thus in this sense, continuous-linear hybrid automata is defined. As the majority of systems in control theory are modeled by dynamics using coupled differential equations, we have expressed mathematically the continuous coupled flow equations by means of a state space representation. Thus, in this sense, we will consider hybrid systems with continuous flows to be modeled as one r-dimensional first-order differential equation, as defined by [2, 7]

$$\dot{x} = f(t, x, u), \tag{2}$$

where x is the continuous state vector taking values in some subset of the Euclidean space and $f(t, x, u)$ is a controlled vector field. We consider the state vector $x(t) \in X \subseteq R^r$ and the control input vector $u(t) \in U \subseteq R^m$. The size r of x is called the automaton dimension. The size m of u is the control input size. Accordingly we define

$$\begin{aligned} x &= \begin{bmatrix} x_1 & x_2 & \cdots & x_r \end{bmatrix}^T; \\ u &= \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix}^T; \\ f(t, x, u) &= \begin{bmatrix} f_1(t, x, u) & f_2(t, x, u) & \cdots & f_r(t, x, u) \end{bmatrix}^T. \end{aligned} \tag{3}$$

In hybrid systems, continuous-linear flows can be modeled considering in (2) $f(x, u, t) = A_i(t)x(t) + B_i(t)u(t)$ to be the controlled vector field (in all s_i locations), where $A_i(t) \in R^{r\times r}$, $B_i(t) \in R^{r\times m}$, and the automaton dimension r is also called the dimension of $A_i(t)$.

Definition 3 (continuous-linear hybrid automata). A hybrid automaton H is a linear hybrid automaton H_{CL} if

(1) for all locations $s_i \in S$, the initial condition init(s_i) is convex,

(2) the flow condition flow(s_i) is continuous-linear,

(3) for all transition $T_i \in E$, the jump condition $F(T_i)$ is convex.

Thus, a continuous-linear hybrid automaton H_{CL} is characterized by a set S of locations s_i, by oriented arcs $T_i \in E$ called transitions, by convex relations in the initial and jump conditions, and by a continuous-linear flow equations usually expressed as $\dot{x}(t) = A_i(t)x(t) + B_i(t)u(t)$. From Figure 2, we will obtain a continuous-linear hybrid automaton H_{CL} replacing the flow condition $\dot{x}(t) = B_i$, $\forall i \in \{1, 2\}$, by

$$\dot{x}(t) = A_i(t)x(t) + B_i(t)u(t). \tag{4}$$

Definition 4 (time invariant hybrid automata). A hybrid automaton H is a time invariant hybrid automaton H_I if

(1) for all locations $s_i \in S$, the initial condition init(s_i) and the flow condition flow(s_i) are all time invariant and

(2) for all transitions $T_i \in E$, the jump condition $F(T_i)$ is time invariant.

Remark 5. Further after, we will call continuous-linear time invariant hybrid automaton H_{CLI} to all hybrid automaton that satisfies Definitions 3 and 4. In this case, the vector field in (2) becomes $f(x, u)$.

Definition 6 (uncoupled hybrid automata). A hybrid automaton H is uncoupled hybrid automaton H_D if

(1) for all locations $s_i \in S$, the flow condition flow(s_i) is uncoupled; that is, the variable x is flow independent and the vector field in (2) is given by

$$\begin{aligned} & f(t, x, u) \\ &\quad = \begin{bmatrix} f_1(t, x_1, u) & f_2(t, x_2, u) & \cdots & f_r(t, r, u) \end{bmatrix}^T \end{aligned} \tag{5}$$

and for all transition $T_i \in E$, the jump condition $F(T_i)$ is function of only one state variable.

Remark 7. Hence, we will call uncoupled continuous-linear time invariant hybrid automaton H_{CLID} all hybrid automata that satisfy Definitions 3, 4, and 6. In this case, for all locations $s_i \in SA_i$ matrix in the vector field is diagonal. In consequence, an automaton can be considered a coupled continuous-linear hybrid automaton if it is represented by at least one nondiagonal matrix A_i for any $s_i \in S$ and/or at least two different state variables in any jump condition predicate.

3. Limit Cycle Analysis of Uncoupled Hybrid Systems

In this section, our goal is to establish some definitions, properties, and theorems concerning limit cycle existence and uniqueness of the hybrid automata H_{CLID}. To reach a limit cycle in a dynamic behavior of an automaton, we must consider that at least one closed loop trajectory exists in the automaton model.

3.1. Basic Concepts. In general, the analysis of continuous evolution of a hybrid system is not the main problem. The difficulty stems from the discrete behavior interaction. It is necessary to model the commutation instants as well as considering the continuous state variables evolution. This leads us to an important notion in hybrid system modeling that is called the transition enabling. Consider the automaton partial structure where s_j location has a single input and a single output transition, as in Figure 3.

In each s_j location, differential equation models flow(s_j) for a continuous dynamic evolution for a fixed horizon defined in a closed interval $I_j = [0, \delta_j]$.

Definition 8 (residence time). For all hybrid automata H, a residence time δ_j is defined as the amount of time spent in location s_j before a transition T_j fires.

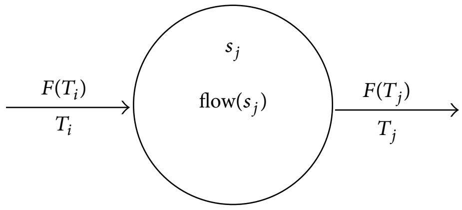


Figure 3: SISO transition location of a hybrid automaton.

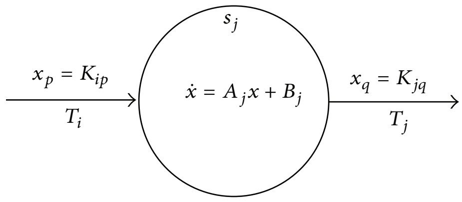


Figure 4: SISO transition location of a H_{CLI} hybrid automaton.

Property 1 (transition enabling). For all hybrid automata being in location s_j, T_j transition is enabled at $t = 0$ input instant, if and only if a finite time $\delta_j \in R_{\geq 0}$ solution exists. This δ_j solution is a function of the initial condition (initial point), the jump condition (final point), and the differential equations (the trajectory) defined for location s_j. A first particular interesting analysis case exists when the dynamic behavior of the continuous-linear time invariant hybrid system is uncoupled; therefore the hybrid automaton model considers diagonal A_j matrices and $u(t)$ a constant control signal, $\forall s_j \in S$.

Hypothesis 1. Let us consider a hybrid automaton H_{CLID} the dynamic representation of uncoupled continuous-linear time invariant hybrid systems. In this representation we suppose $\forall s_j \in S$ a diagonal matrix $A_j = \operatorname{diag}[a_{j1} \ \cdots \ a_{jr}]$ and a real constant vector $B_j u = [b_{j1} \ \cdots \ b_{jr}]^T$. First, for each s_j location (Figure 4), let the jump condition be a simple predicate $x_q = x_q(\delta_j) = K_{jq}$ such as $x_q(t) \mathrel{\hat{=}} x(t)[q]$ is a definite function $R_{\geq 0} \to R$, $\forall q \in [1, r]$, $t \in I_j$, and let K_{jq} be a point in some region $R_{jq} \subseteq R$. Now, let K_{jq} be $x_q(t)$ value at T_j transition instant.

The state vector evolution $x(t)$; $\forall t \in I_j$ is given by

$$x(t) = e^{A_j t} x(0) + \int_0^t e^{A_j (t-\varphi)} B_j u(\varphi)\, d\varphi. \tag{6}$$

For $t = \delta j$, we obtain, where $\forall q \in [1, r]$,

$$K_j = e^{A_j \delta_j} x(0) + \int_0^{\delta_j} e^{A_j(\delta_j - \varphi)} B_j d\varphi, \tag{7}$$

$$\begin{aligned} K_{jq} &= e^{a_{jq}\delta_j} K_{iq} + \int_0^{\delta_j} e^{a_{jq}(\delta_j - \varphi)} b_{jq} d\varphi, \quad \forall a_{jq} \neq 0 \\ K_{jq} &= K_{iq} + b_{jq} \int_0^{\delta_j} d\varphi, \quad a_{jq} = 0. \end{aligned} \tag{8}$$

Herein, K_{ip} and K_{jq} are, respectively, pth and qth constant components of K_i and K_j vectors. Hence solving (8), we can calculate δ_j residence time. Therefore, if δ_j finite real positive solution of (8) exists, then T_j transition is enabled. It must be noted that, in the general case (7), that is, for any A_j matrix, it is difficult to find an explicit solution and in the majority of cases, only δ_j numerical solution is possible.

Property 2. The residence time δ_j in all locations $s_j \in S$ is determined by

$$\begin{aligned} \delta_j &= \frac{1}{a_{jq}} \ln\left(\frac{K_{jq} - P_{jq}}{K_{iq} - P_{jq}}\right), \quad \text{if } a_{jq} \neq 0 \\ \text{or } \delta_j &= \left(\frac{K_{jq} - K_{iq}}{b_{jq}}\right), \quad \text{if } a_{jq} = 0, \end{aligned} \tag{9}$$

where $P_{jq} = -b_{jq}/a_{jq}$.

This last result could be obtained by solving (8) at δ_j residence time. For $a_{jq} \in R_{\neq 0}$ it must be noted that the corresponding equilibrium point $x^*{}_q \in R$ to x_q is given by relation $x^*{}_q = P_{jq} = -b_{jq}/a_{jq}$. In general $\forall s_j \in S$ and for this uncoupled case, the equilibrium point of $x(t)$ in the Euclidean space R^r is given by $x^* = [P_{j1} \ \ P_{j2} \ \cdots \ P_{jr}]^T$.

Theorem 9. *Given a hybrid automaton H_{CLID} with $x_q = K_{iq}$ input (initial) condition and $x_q = K_{jq}$ jump condition predicate of T_j (output) transition, the necessary and sufficient condition to enable T_j is given by the following logical equations:*

$$\begin{aligned} &[P_{jq} < K_{jq} \leq K_{iq}] \vee [P_{jq} > K_{jq} \geq K_{iq}] = 1; \\ &\qquad\qquad if \ a_{jq} < 0, \\ &[P_{jq} < K_{iq} \leq K_{jq}] \vee [P_{jq} > K_{iq} \geq K_{jq}] = 1; \\ &\qquad\qquad if \ a_{jq} > 0, \\ &[K_{jq} \leq K_{iq}][b_{jq} < 0] \vee [K_{jq} \geq K_{iq}][b_{jq} > 0] = 1; \\ &\qquad\qquad if \ a_{jq} = 0. \end{aligned} \tag{10}$$

The proof, from the above relations, is a simple one and it is based on the fact that for enabling a transition T_j it is enough that $\exists \delta_j \mid \delta_j \in R_{\geq 0}$. It must be noted in (9) that the equality $K_{jq} = K_{iq}$ gives the lower bound of residence time $(\delta_j = 0)$ in location s_j. On the other hand, a value $K_{jq} = P_{jq}$ where $K_{jq} \neq K_{iq}$, will produce the upper limit of residence time $(\delta_j = \inf)$. In this last case, T_j transition will not be enabled; therefore s_j will be a sink location. For reachability analysis, we consider a region in the state space R_i as initial condition and a region R_j as jump condition of T_j transition. K_{iq} and K_{jq} are two points in R_i and R_j regions, respectively. The final region R_j is reachable from the initial region R_i, $R_i \to R_j$, if for each point $K_{jq} \in R_j$ in the final region, there is at least one point $K_{iq} \in R_i$ that satisfy the logical equations (10).

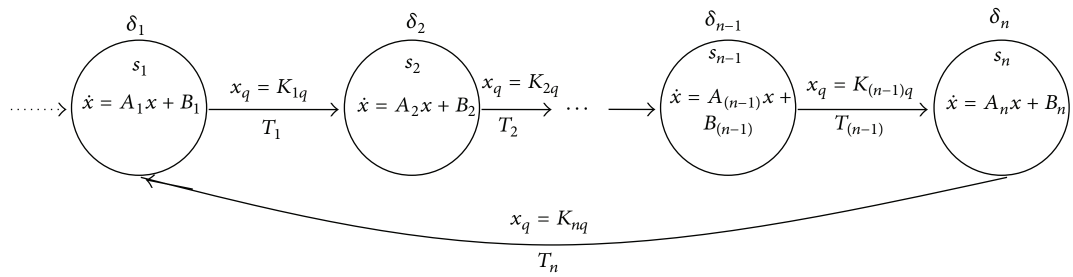


Figure 5: H_{CLID} hybrid automaton with a closed loop structure.

Hypothesis 2. Based on the Hypothesis 1, let us consider β closed loop trajectory obtained from automaton locations (Figure 5).

Theorem 10. *For each β closed loop trajectory in H_{CLID} hybrid automaton, the necessary condition to reach a limit cycle is given by the logical expression:*

$$\bigcup_{j=1}^{n} \left[\overline{C_{jq}} \wedge C_{(j-1)q} \wedge C^{*}_{(j+1)q} \right] = 1. \tag{11}$$

In this case

$$C^{*}_{(j+1)q} = \begin{cases} 1, & \text{if } n = 2, \\ \bigcap_{i=1}^{n-2} \left[C_{(j+i)q} \vee \overline{C_{(j+i)q}} \right], & \text{if } n > 2, \end{cases} \tag{12}$$

where

$$\begin{aligned} C_{jq} = {} & \left[\left(a_{jq} < 0 \right) \wedge \left(P_{jq} < K_{jq} \le K_{(j-1)q} \right) \right] \\ & \vee \left[\left(a_{jq} > 0 \right) \wedge \left(P_{jq} < K_{(j-1)q} \le K_{jq} \right) \right] \\ & \vee \left[\left(a_{jq} = 0 \wedge b_{jq} < 0 \right) \wedge \left(K_{jq} \le K_{(j-1)q} \right) \right], \\ \overline{C_{jq}} = {} & \left[\left(a_{jq} < 0 \right) \wedge \left(P_{jq} > K_{jq} \ge K_{(j-1)q} \right) \right] \\ & \vee \left[\left(a_{jq} > 0 \right) \wedge \left(P_{jq} > K_{(j-1)q} \ge K_{jq} \right) \right] \\ & \vee \left[\left(a_{jq} = 0 \wedge b_{jq} > 0 \right) \wedge \left(K_{jq} \ge K_{(j-1)q} \right) \right]. \end{aligned} \tag{13}$$

Considering all subindices $(n+i) = i$; $\forall i \ge 0$, due the closed loop configuration.

Theorem 11. *For β closed loop structure H_{CLID} hybrid automaton (Figure 5), the initial condition point to reach a limit cycle must belong to the initial region defined by*

$$R_{\text{init}} = C_{\text{init}} \vee \overline{C_{\text{init}}}, \tag{14}$$

$$\begin{aligned} C_{\text{init}} = {} & \left[\left(a_{1q} < 0 \right) \wedge \left(P_{1q} < K_{1q} \le x_0 (q) \right) \right] \\ & \vee \left[\left(a_{1q} > 0 \right) \wedge \left(P_{1q} < x_0 (q) \le K_{1q} \right) \right] \\ & \vee \left[\left(a_{1q} = 0 \wedge b_{1q} < 0 \right) \wedge \left(K_{1q} \le x_0 (q) \right) \right], \end{aligned}$$

$$\begin{aligned} \overline{C_{\text{init}}} = {} & \left[\left(a_{1q} < 0 \right) \wedge \left(P_{1q} > K_{1q} \ge x_0 (q) \right) \right] \\ & \vee \left[\left(a_{1q} > 0 \right) \wedge \left(P_{1q} > x_0 (q) \ge K_{1q} \right) \right] \\ & \vee \left[\left(a_{1q} = 0 \wedge b_{1q} > 0 \right) \wedge \left(K_{1q} \ge x_0 (q) \right) \right]. \end{aligned} \tag{15}$$

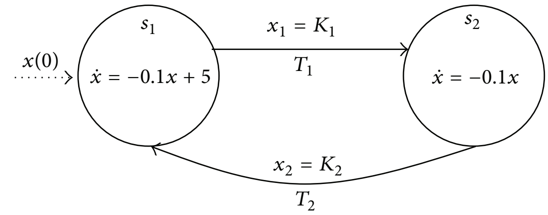


Figure 6: Thermostat hybrid automaton model.

As we can see in Example 12, the limit cycle reachability is function of the automaton jump conditions as well as its initial condition region. Therefore, (11) and (14) enable us to evaluate in a rapid, easy, and direct way if $x(t)$ hybrid dynamics will reach a limit cycle.

Example 12. H_{CLID} hybrid automaton of Figure 6 models a thermostat that controls the temperature, $x(t)$, of a heating system. The task is to evaluate its reachability of a limit cycle for $x(t)$ and find the region of initial conditions that make it possible.

Using (11) for $a_{jq} < 0$, $j \in \{1, 2\}$, $n = 2$, $q = 0$ (simplifying index notations because there is only one continuous variable $x_q(t) = x(t)$) and again considering index zero equal to n, we get the necessary condition to reach the limit cycle (Theorem 10):

$$\begin{gathered} \bigcup_{j=1}^{2} \left[\overline{C_j} \wedge C_{(j-1)} \right] = 1, \\ \left(\overline{C_1} \wedge C_2 \right) \vee \left(\overline{C_2} \wedge C_1 \right) = 1, \\ (P_1 > K_1 \ge K_2 \wedge P_2 < K_2 \le K_1) \\ \vee (P_1 < K_1 \le K_2 \wedge P_2 > K_2 \ge K_1) = 1, \\ (P_2 < K_2 \le K_1 < P_1) \vee (P_2 > K_2 \ge K_1 > P_1) = 1 \end{gathered} \tag{16}$$

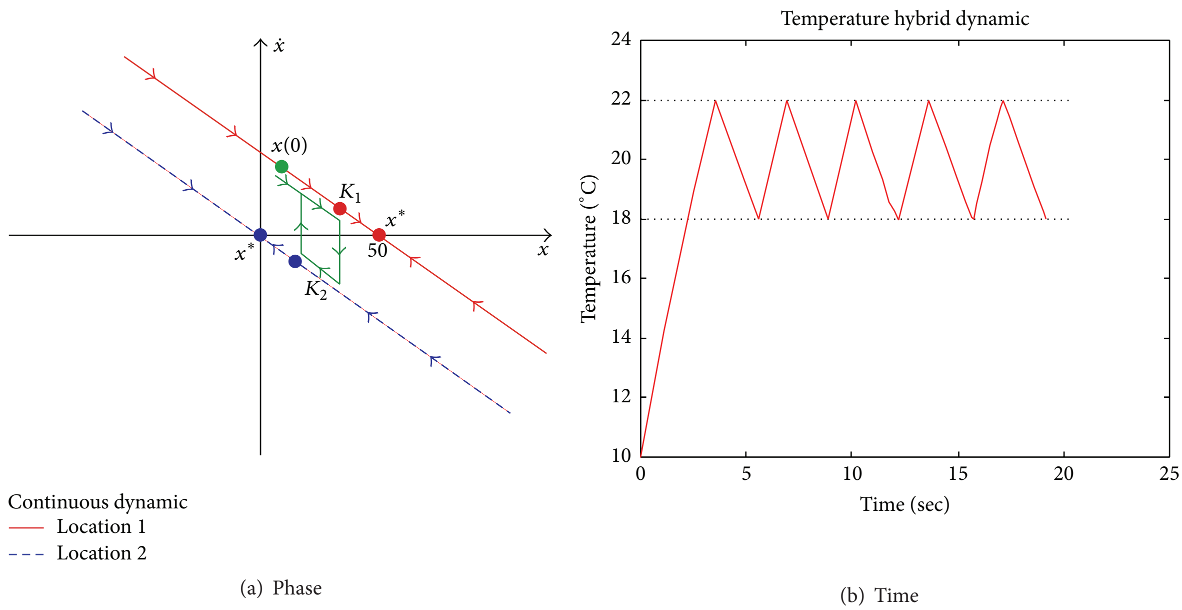


FIGURE 7: Cyclic behavior of temperature.

with $P_1 = 50$, $P_2 = 0$; we obtain $(0 < K_2 \leq K_1 < 50) = 1$. Therefore, it is possible to reach a limit cycle if the previous inequality holds true. Then, the region that defines the initial condition is given by

$$\begin{aligned} R_{\text{init}} &= C_{\text{init}} \vee \overline{C_{\text{init}}} \\ &= [P_1 < K_1 \leq x(0)] \vee [P_1 > K_1 \geq x(0)] \\ &= [50 > K_1 \geq x(0)]. \end{aligned} \tag{17}$$

Thus, if we consider $x(0) = 10$, $K_1 = 22$, and $K_2 = 18$, all the previous relations are satisfied, and, therefore, the temperature response $x(t)$ will be cyclic (Figure 7).

On the other hand, if we consider $x(0) = -15$, $K_1 = -10$, and $K_2 = -5$, $x(0)$ is in the initial condition region, but Theorem 10 is not satisfied. Therefore, in steady state, the temperature $x(t)$ will not reach a limit cycle (Figure 8).

4. Limit Cycle Analysis for a Coupled Case

The fact of having the automaton jump condition predicates related to two different state variables gives a certain coupling between them and consequently a more difficult analytical solution. Therefore for this case, x_q variables in the jump condition predicates will not be solvable in the solvability sense defined in [7] and as a result a more difficult reachability analysis. Our goal in this section is then to provide some theorems and properties to reach a limit cycle for this coupled case.

Hypothesis 3. Let Figure 9 be s_j location of two dimensions for H_{CLI} hybrid automaton. In this coupled case $A_j = \text{diag}[a_{j1} \ \ a_{j2}]$, vectors $B_j = [b_{j1} \ \ b_{j2}]^T$, K_{j1}, and K_{j2} are real constants, and the jump condition predicates are related to one of two different state variables $x_q(t) = K_{jq} \mid q \in \{1, 2\}$.

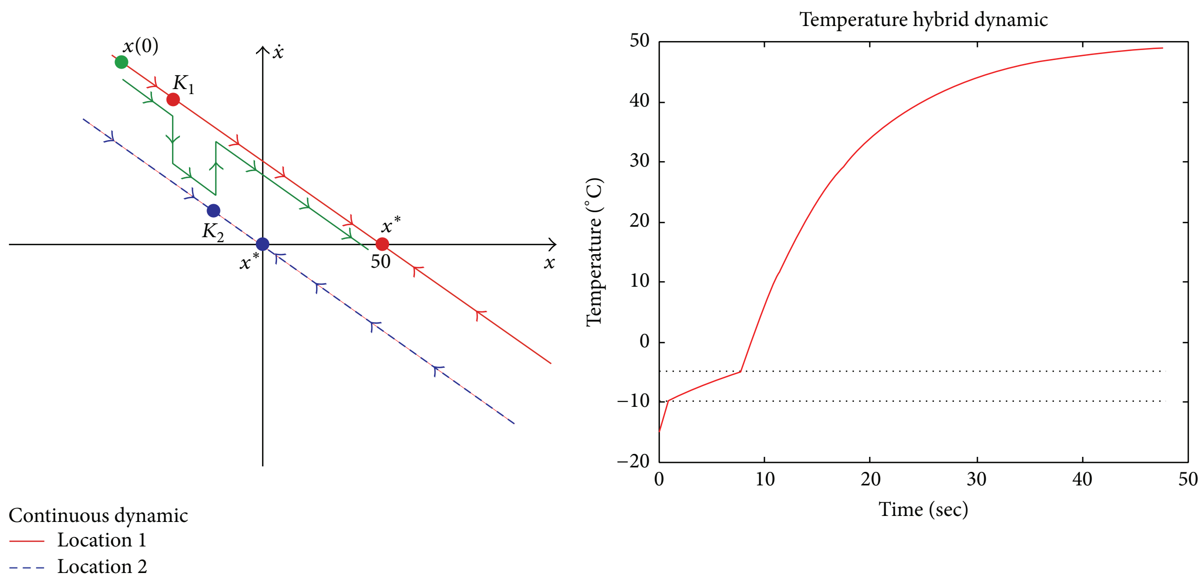
The state vector at transition instants is defined as $x(\delta_j) = [K_{j1} \ \ K_{j2}]^T$. The discrete values K_{j1} and K_{j2} are obtained at each T_j transition instant. The trajectory equation $x_2 = f(x_1)$, $\forall t \in [0, \delta_j]$ in the state space (for each s_j location) is given by

$$\begin{aligned} x_2 &= x_2(0) + \frac{b_{j2}}{b_{j1}}[x_1 - x_1(0)] \quad \text{for } a_{j1} = a_{j2} = 0, \\ x_2 &= P_{j2} + [x_2(0) - P_{j2}]\, e^{(a_{j2}/b_{j1})[x_1 - x_1(0)]}, \\ &\qquad \text{for } a_{j1} = 0,\ a_{j2} \neq 0, \\ x_2 &= P_{j2} + [x_2(0) - P_{j2}]\left[\frac{x_1 - P_{j1}}{x_1(0) - P_{j1}}\right]^{a_{j2}/a_{j1}}, \\ &\qquad \text{for } a_{j1} \neq 0,\ a_{j2} \neq 0, \end{aligned} \tag{18}$$

where $[x_1(0) \ \ x_2(0)]^T = [K_{i1} \ \ K_{i2}]^T$ is s_j initial condition.

Property 3. The discrete dynamic relation between K_{j2} and K_{j1} at each T_j transition instant is given by

$$\begin{aligned} K_{j2} &= K_{i2} + \frac{b_{j2}}{b_{j1}}[K_{j1} - K_{i1}], \quad \text{for } a_{j1} = a_{j2} = 0, \\ K_{j2} &= P_{j2} + [K_{i2} - P_{j2}]\, e^{(a_{j2}/b_{j1})[K_{j1} - K_{i1}]}, \\ &\qquad \text{for } a_{j1} = 0,\ a_{j2} \neq 0, \\ K_{j2} &= P_{j2} + [K_{i2} - P_{j2}]\left[\frac{K_{j1} - P_{j1}}{K_{i1} - P_{j1}}\right]^{a_{j2}/a_{j1}}, \\ &\qquad \text{for } a_{j1} \neq 0,\ a_{j2} \neq 0. \end{aligned} \tag{19}$$



(a) Phase (b) Time

FIGURE 8: Temperature $x(t)$ with noncyclic behavior.

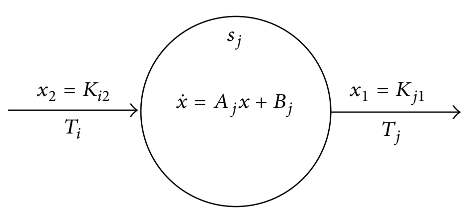


FIGURE 9: An automaton location for a coupled case.

This is an analytical result obtained by finding the commutation point in the state space trajectory equation (18), at T_j transition firing instant. Now, for limit cycle analysis we must consider that there exists a closed loop β obtained from H_{CLI} hybrid automaton with jump condition predicates function of x_1 or x_2 state variables.

Hypothesis 4. Let β be a closed loop obtained by n single output locations $s_j \in \overline{S} \mid \overline{S} \subseteq S$ of H_{CLI} hybrid automaton. In the jump condition predicates, the enabled T_j transitions are function of $x_q(t)$, for $q \in \{1, 2\}$. It is important to note that in each location s_j there are two unknown discrete variables K_{i1} and K_{j2} at input and output commutation instants, respectively. In general, for each location s_j we have one equation obtained from (19), with two unknowns (K_{i1} and K_{j2}) that will determine (by mean of its convergence analysis) the existence of a limit cycle.

Property 4. For all closed loops $\beta \in H_{CLI}$, dynamic relations among state variables in discrete commutation instants are governed by a set of equations obtained from (19), $\forall s_j \in \overline{S}$ and $j \in \{1, 2, \dots, n\}$. Thus, in general, in β closed loop structure we will have the same number of equations (and unknowns) as locations in the loop.

Remark 13. The existence and uniqueness of a limit cycle are concluded from the convergence analysis of K_{jq} values at commutation instants $\forall s \in \overline{S}$ and $q \in \{1, 2\}$ according to the fixed point theorem.

Theorem 14 (about the existence of a limit cycle). *For n locations closed loop $\beta \in H_{CLI}$ with jump conditions predicate $x_q(t) = K_{jq} \mid q \in \{1, 2, \dots, r\}$, the necessary condition to obtain a limit cycle is given by*

$$\bigcap_{q=1}^{r} \left[\bigcup_{j=1}^{n} \overline{C_{jq}} \wedge C_{(j-1)q} \wedge C^{*}_{(j+1)q} \right] = 1. \qquad (20)$$

This result is a natural extension of (11) for the case of n locations and r commutation variables. Therefore, the existence of a limit defined, by (11), must be true for each commutation variable that gives the logical product of (20).

Property 5. K_{jq} convergence is assured by $\forall j \in \{1, 2, \dots, n\}$ and $\forall q \in \{1, 2, \dots, r\}$ if in the hybrid system behavior, the state vector trajectory enters and remains inside the state space region defined by (20). In this case, the limit cycle will be uniquely defined for a system region of initial conditions. For each particular case, this region of initial conditions could be found by means of a system phase portrait (isoclines) analysis considering the transitions enabling conditions imposed by (20). A geometric signification of (20) and Property 5 will be conducted in Section 5.

Property 6. If Theorem 14 has been satisfied, we can find an initial condition region to reach a limit cycle replacing $[K_{n1} \;\; K_{n2} \;\; \cdots \;\; K_{nr}]^T$ by $[x_1(0) \;\; x_2(0) \;\; \cdots \;\; x_r(0)]^T$ in (20).

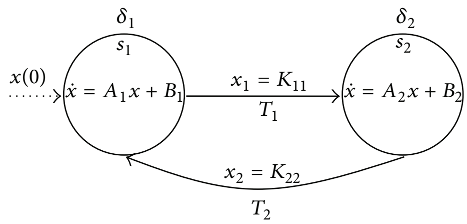


FIGURE 10: H_{CL}, two-location coupling case.

Remark 15. Thus, we must note that a sufficient condition to reach a limit cycle is given by Theorem 14 plus the initial condition region found from Property 6.

Property 7. For the case where the diagonal matrix A_j is stable $\forall s_j \in S$ and jump condition predicates $x_q(t) = K_{jq} \mid q \in \{1, 2, \ldots, r\}$, the $x(t)$ values are bounded. For the limit cycle case, the bounds are defined by (20) and for another case by the sink location equilibrium point.

The previous analysis procedure can be applied to automata for n-location case. In Section 5, we present an analysis of the two-location case. A geometric interpretation of results is then conducted.

5. Results Interpretation

Consider two-location $s_j \mid j \in \{1, 2\}$ of a continuous-linear hybrid automaton, as in Figure 10.

One has $x(t) = [x_1(t)\ \ x_2(t)]^T$, $A_1 = [a_{11}\ \ a_{12}]$, $A_2 = [a_{21}\ \ a_{22}]$, $B_1 = [b_{11}\ \ b_{12}]^T$, $B_2 = [b_{21}\ \ b_{22}]^T$, $F(T_1) \Rightarrow x_1(t) = K_{11}$, and $F(T_2) \Rightarrow x_2(t) = K_{22}$. K_{11} and K_{22} are real constants of known magnitude. So without loss of generality, because the analysis procedure is the same, we will consider stable continuous equations ($a_{jq} < 0\ \forall s_j \mid j \in \{1, 2\}$ and $\forall q \in \{1, 2\}$). According to Property 5, to reach a limit cycle from a given initial condition region, the state vector dynamic must always enter and remain in a state space region (transition enabling conditions) defined by

$$\bigcap_{q=1}^{2} \left[\left(\overline{C_{1q}} \vee C_{2q} \right) \wedge \left(\overline{C_{2q}} \vee C_{1q} \right) \right] = 1, \tag{21}$$

where

$$\begin{aligned} \overline{C_{1q}} &= \left(P_{21} < K_{21} < K_{11} < P_{11} \right), \\ C_{2q} &= \left(P_{21} > K_{21} > K_{11} > P_{11} \right), \\ \overline{C_{2q}} &= \left(P_{22} < K_{22} < K_{12} < P_{12} \right), \\ C_{1q} &= \left(P_{22} > K_{22} > K_{12} > P_{12} \right), \end{aligned} \tag{22}$$

and it gives 4 bounded regions of the state space, where a cyclic dynamic will take place, as in Figure 11.

In Figure 11, the trajectory attraction points $\dot{x}^* = P_1$ and $x^* = P_2$ (equilibrium points) for locations s_1 and s_2, respectively, are shown. Thus, a unique limit cycle is assured if the system dynamic starts from an initial state space region that makes that $x(t)$ enters the shadow zone and remains inside. Hence, it can be seen in this case that the shadow zone respects the invariant principle and consequently all initial point from this zone will produce a limit cycle. According to Property 6, a sufficient initial condition in the state space region to reach a unique limit cycle is obtained replacing $[K_{21}\ \ K_{22}]^T$ by $[x_1(0)\ \ x_2(0)]^T$ in (21). Because K_{22} is a known constant, we obtain

$$\begin{aligned} &\left[\left(P_{21} < x_1(0) < K_{11} < P_{11} \right) \right. \\ &\quad \left. \vee \left(P_{21} > x_1(0) > K_{11} > P_{11} \right) \right] \wedge \left[x_2(0) = K_{22} \right] \\ &\quad = 1. \end{aligned} \tag{23}$$

The necessary and sufficient initial condition region could be found using the automaton trajectory equations, that is, from its phase portrait analysis. In this particular case, the following relations give the initial condition region:

$$\begin{aligned} &x_1(0) \rhd_1 K_{11}, \\ &x_2(0) \rhd_2 P_{12} + \left[K_{22} - P_{12} \right] \left[\frac{x_1(0) - P_{11}}{K_{11} - P_{11}} \right]^{a_{12}/a_{11}}, \end{aligned} \tag{24}$$

where relations $\rhd = (<, >)$ and are defined according to the (known) automaton relations $K_{11} \rhd_1 P_{11}$ and $K_{22} \rhd_2 P_{22}$. This result will be obtained by the automaton phase portrait analysis using trajectories equations (18), still under the restrictions imposed by (21). Now, a limit cycle existence can be determined the K_{12} and K_{21} convergence analysis (Remark 13). Its dynamic behaviors are governed by (Property 3)

$$\begin{aligned} K_{12} &= P_{12} + \left[K_{22} - P_{12} \right] \left[\frac{K_{11} - P_{11}}{K_{21} - P_{11}} \right]^{a_{12}/a_{11}}, \\ K_{21} &= P_{21} + \left[K_{11} - P_{21} \right] \left[\frac{K_{22} - P_{22}}{K_{12} - P_{22}} \right]^{a_{21}/a_{22}}. \end{aligned} \tag{25}$$

Considering $K_{12} = g_1(K_{21})$ and $K_{21} = g_2(K_{12})$ from (25), Figure 12 shows $g_1(K_{21})$ and the curves generated under restrictions imposed by (21). Herein, we can observe the 4 possible cases to reach a limit cycle. Additionally, in this particular case we will note that the automaton dynamic behavior will reach one of two possible fixed points in $K_{12} - K_{21}$ space. The limit cycle existence and uniqueness demonstration can be done starting from K_{12} and K_{21} fixed point convergence analysis. This is the formal method that proves the cyclic steady state convergence.

Therefore, from (25), by means of K_{12} and K_{21} and a fixed point analysis, we find two convergence possibilities. possibilities: first, the possibility of having only one fixed point (Figure 12) and second the possibility of having two. In case of one stable fixed point the residence time in each location is $0 < \delta_j < \inf$, $\forall s_j \in S$, and for an unstable point, a boundary case, where $K_{12} = K_{22}$ and $K_{21} = K_{11}$, the residence times are zero; that is, $\delta_j = 0$, $\forall s_j \in S$.

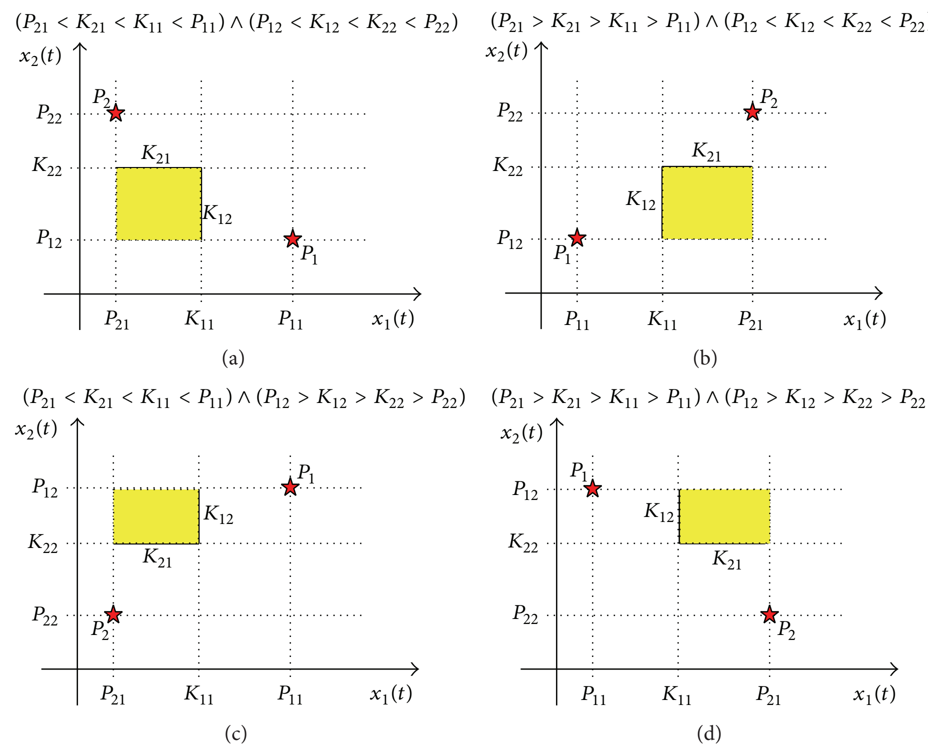


FIGURE 11: State space regions where cyclic dynamics must take place.

Thus, according to the fixed point theory, the fixed point uniqueness is assured if

$$\left|\frac{dK_{12(i+1)}}{dK_{12}(i)}\right| < 1, \qquad \left|\frac{dK_{21(i+1)}}{dK_{21}(i)}\right| < 1, \tag{26}$$

where

$$\frac{dK_{12}(i+1)}{dK_{12}(i)} = c\left[f_1(K_{12})\right]\left[f_2(K_{12})\right], \qquad \frac{dK_{21}(i+1)}{dK_{21}(i)} = c\left[f_3(K_{21})\right]\left[f_4(K_{21})\right] \tag{27}$$

with

$$c = \left[\frac{K_{11}-P_{21}}{K_{11}-P_{11}}\right]\left[\frac{K_{22}-P_{12}}{K_{22}-P_{22}}\right]\left[\frac{a_{12}}{a_{11}}\right]\left[\frac{a_{21}}{a_{22}}\right],$$

$$f_1(K_{12}) = \left\{\left[\frac{P_{21}-P_{11}}{K_{11}-P_{11}}\right] + \left[\frac{K_{11}-P_{21}}{K_{11}-P_{11}}\right]\left[\frac{K_{22}-P_{22}}{K_{12}-P_{22}}\right]^{(a_{21}/a_{22})}\right\}^{-(1+a_{12}/a_{11})},$$

$$f_2(K_{12}) = \left[\frac{K_{22}-P_{22}}{K_{12}-P_{22}}\right]^{(1+a_{21}/a_{22})},$$

$$f_3(K_{21}) = \left\{\left[\frac{P_{12}-P_{22}}{K_{22}-P_{22}}\right] + \left[\frac{K_{22}-P_{12}}{K_{22}-P_{22}}\right]\left[\frac{K_{11}-P_{11}}{K_{21}-P_{11}}\right]^{(a_{12}/a_{11})}\right\}^{-(1+a_{21}/a_{22})},$$

$$f_4(K_{21}) = \left[\frac{K_{11}-P_{11}}{K_{21}-P_{11}}\right]^{(1+a_{12}/a_{11})}. \tag{28}$$

By mean of preceding terms analysis (restricted of limit cycle condition (20)), we find always $|f_i| < 1$, $\forall i \in \{1,2,3,4\}$. Thus, we can guarantee the fixed point existence and uniqueness assuring the constant value $|c| < 1$, but this fixed point produces a zero period limit cycle (Figures 12(a), 12(b), 12(c), and 12(d)). Therefore, in general, the desirable solution to avoid the zero period limit cycle is one which gives two fixed points (Figures 12(a), 12(b), 12(c), and 12(d)). So we must assure the relation $|c| > 1$; that is $c > 1$ (because in the four regions defined in Figure 11, c is positive). Then, choosing adequately K_{11} and K_{22} values to achieve $c > 1$, we can assure the discrete values convergence toward the stable fixed point for a defined initial condition region.

Example 16. Consider the hybrid automaton from Figure 10. Therein, a continuous dynamic behavior is modeled by

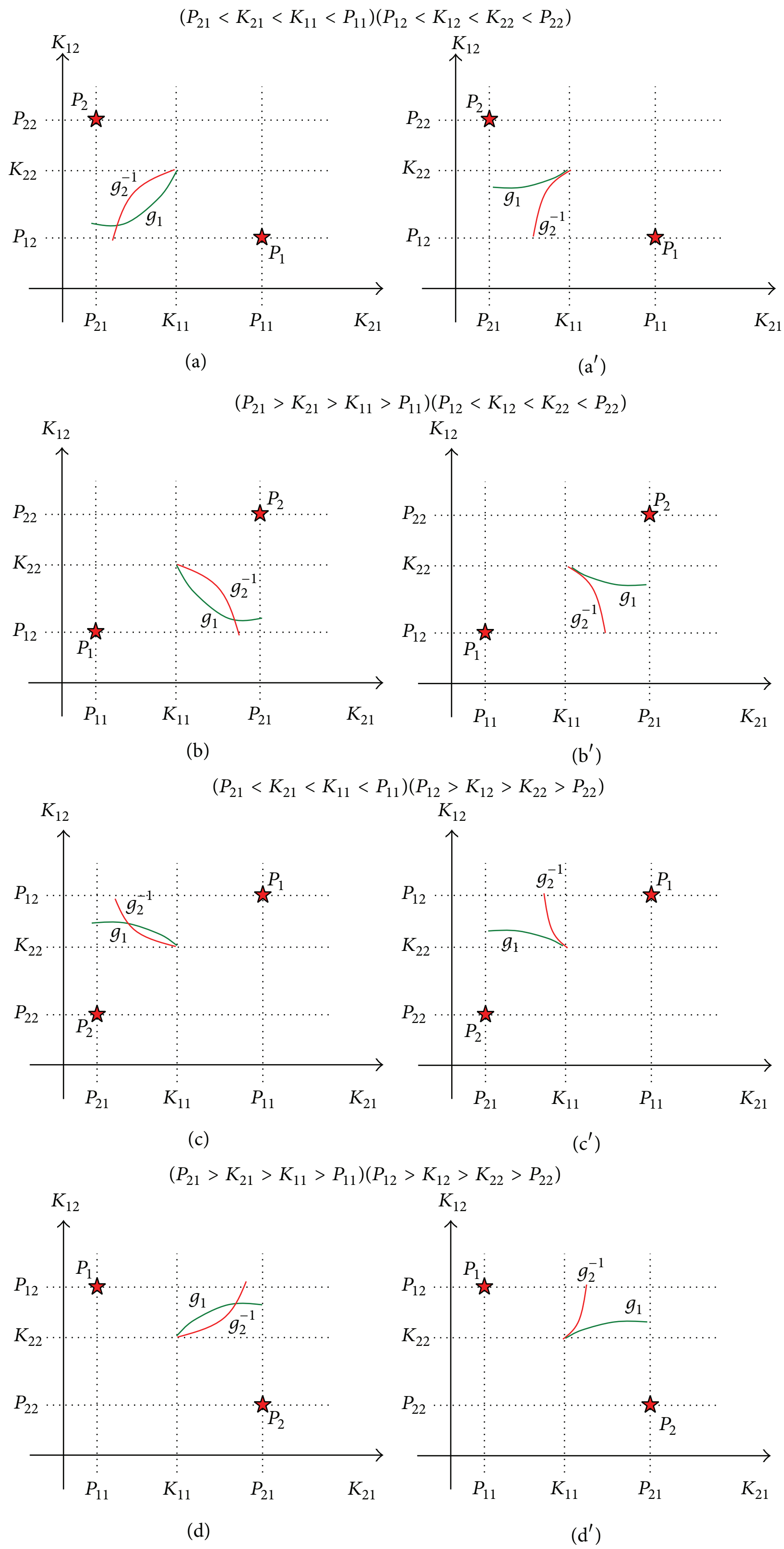


FIGURE 12: $g_1(K_{21})$ and $g_2^{-1}(K_{21})$ curves: (a), (b), (c), and (d) with two fixed points' convergence; (a′), (b′), (c′), and (d′) with only one fixed point.

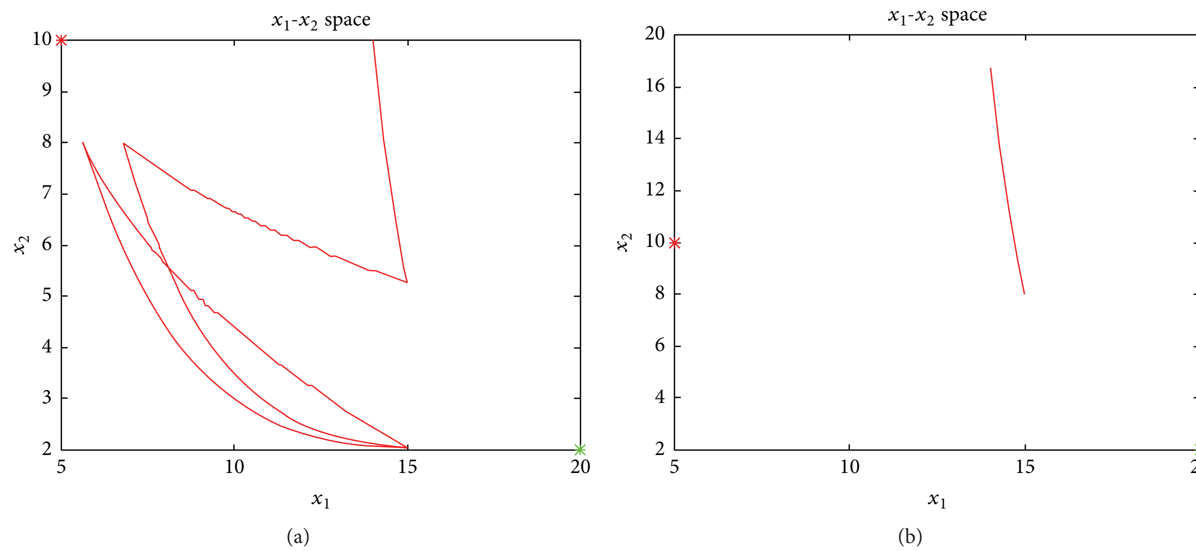


FIGURE 13: Limit cycle convergence in $x_1 - x_2$ space with $[x_1(0) \;\; x_2(0)]^T$ (a) $[14 \;\; 10]^T$, and for the boundary case (b) $[14 \;\; 16.9]^T$.

$A_1 = \text{diag}[-0.1 \;\; -0.5]$, $A_2 = \text{diag}[-0.2 \;\; -0.1]$, $B_1 = [2 \;\; 1]^T$, $B_2 = [1 \;\; 1]^T$, and $x(0) = [10 \;\; 0]^T$ considering the following:

(a) If jump conditions are $F(T_1) \Rightarrow x_1(t) = K_{11} = 15$ and $F(T_2) \Rightarrow x_2(t) = K_{22} = 8$, determine the limit cycle existence and uniqueness.

(b) For the same jump conditions, find the necessary and sufficient initial region in s_1 that will give a limit cycle system behavior.

(c) Design a 10-second limit cycle period by considering steady state residence times of $\delta_1 = 6$ sec. and $\delta_2 = 4$ sec. Find K_{11} and K_{22} values as well as the initial condition region that guarantees it.

(d) For $K_{22} = 8$, determine the state space region of K_{11} values to allow obtaining a nonzero period limit cycle.

For this example, the trajectory attraction (equilibrium) points corresponding to each location are given by

$$P_1 = [P_{11} \;\; P_{12}]^T = [20 \;\; 2]^T, \qquad P_2 = [P_{21} \;\; P_{22}]^T = [5 \;\; 10]^T. \tag{29}$$

Thus according to Property 5, to guarantee the limit cycle existence and uniqueness, the hybrid state vector evolution must enter and remain in the state space region defined by

$$(5 < K_{21} < 15 < 20) \wedge (2 < K_{12} < 8 < 10). \tag{30}$$

So considering $x(0) = [10 \;\; 0]^T$ as an initial point, we will verify if this point is in the initial condition region. Therefore, using (24) and considering $K_{11} < P_{11}$ and $K_{22} < P_{22}$,

$$x_1(0) < K_{11} \Longrightarrow 10 < 15,$$
$$x_2(0) < P_{12} + [K_{22} - P_{22}] \left[\frac{x_1(0) - P_{11}}{K_{11} - P_{11}} \right]^{a_{12}/a_{11}} \Longrightarrow 0 < 194. \tag{31}$$

Because the necessary and sufficient conditions are satisfied, we can assure the limit cycle existence and uniqueness.

As in part (b), the necessary and sufficient initial condition region to guarantee the limit cycle existence (24) are given by

$$x_1(0) < 15, \qquad x_2(0) < 2 + 6[4 - 0.2x_1(0)]^5. \tag{32}$$

System simulations (using MATLABÝ) for different initial conditions are shown in Figure 13.

In Figure 13(a) the system dynamic starts from a point in the resulting initial condition region and the system reaches a limit cycle. Otherwise, for the boundary case, Figure 13(b), where $x_1(0) = 14$ and $x_2(0) = 2 + 6[4 - 0.2x_1(0)]^5 = 16.9$, the automaton dynamic behavior will reach its second (unstable) fixed point. In this case, the residence time for the found locations will be equal to zero and $K_{12} = K_{22}$ and $K_{21} = K_{11}$.

For (c), the residence time in each location is deduced by

$$\delta_1 = \frac{1}{a_{1q}} \ln\left(\frac{K_{1q} - P_{1q}}{K_{2q} - P_{1q}}\right), \qquad \delta_2 = \frac{1}{a_{2q}} \ln\left(\frac{K_{2q} - P_{2q}}{K_{1q} - P_{2q}}\right) \tag{33}$$

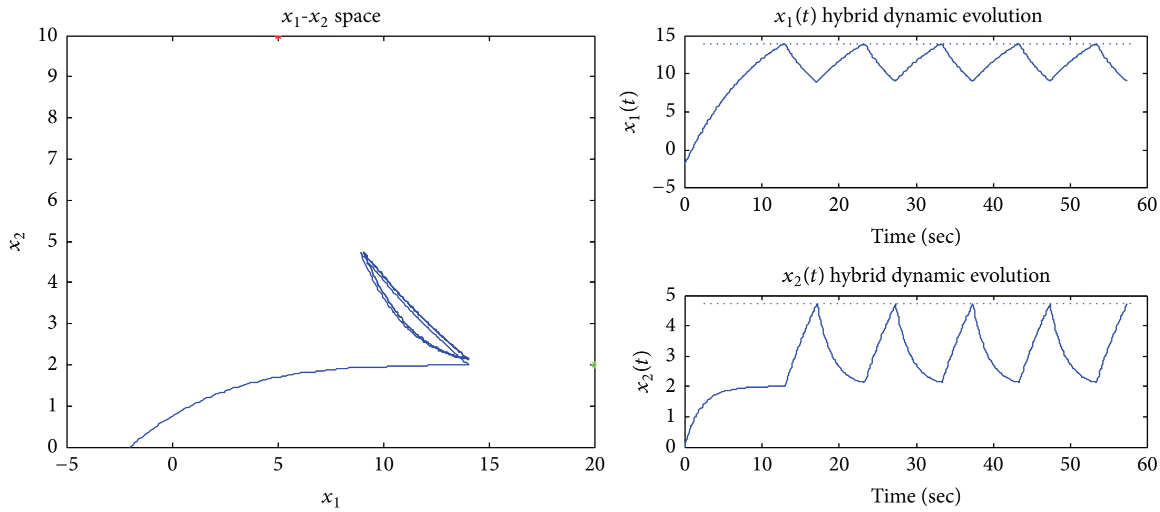


FIGURE 14: Dynamic simulation of a limit cycle design, $x_1 - x_2$ space, and time response for $[x_1(0) \;\; x_2(0)]^T = [8 \;\; 8]^T$.

for $q \in \{1, 2\}$. Thus the discrete evolution of K_{1q} and K_{2q} may be obtained as

$$\begin{aligned} K_{1q} &= \frac{1}{1 - e^{(a_{1q}\delta_1 + a_{2q}\delta_2)}} \left[P_{1q} + \left(P_{2q} - P_{1q}\right) e^{a_{1q}\delta_1} \right. \\ &\quad \left. - P_{2q} e^{(a_{1q}\delta_1 + a_{2q}\delta_2)}\right], \\ K_{2q} &= \frac{1}{1 - e^{(a_{2q}\delta_2 + a_{1q}\delta_1)}} \left[P_{2q} + \left(P_{1q} - P_{2q}\right) e^{a_{2q}\delta_2} \right. \\ &\quad \left. - P_{1q} e^{(a_{2q}\delta_2 + a_{1q}\delta_1)}\right]. \end{aligned} \tag{34}$$

In this way, to obtain a 10-second period limit cycle by considering $\delta_1 = 6$ sec. and $\delta_2 = 4$ seconds, it is necessary that

$$\begin{aligned} K_{11} &= \frac{1}{1 - e^{(a_{11}\delta_1 + a_{21}\delta_2)}} \left[P_{11} + \left(P_{21} - P_{11}\right) e^{a_{11}\delta_1} \right. \\ &\quad \left. - P_{21} e^{(a_{11}\delta_1 + a_{21}\delta_2)}\right] = 13.983, \\ K_{22} &= \frac{1}{1 - e^{(a_{22}\delta_2 + a_{12}\delta_1)}} \left[P_{22} + \left(P_{12} - P_{22}\right) e^{a_{22}\delta_2} \right. \\ &\quad \left. - P_{12} e^{(a_{22}\delta_2 + a_{12}\delta_1)}\right] = 4.7285. \end{aligned} \tag{35}$$

The initial condition region is given by

$$\begin{aligned} &x_1(0) < 13.983, \\ &x_2(0) < 2 + 2.73\left[3.32 - 0.16 x_1(0)\right]^5. \end{aligned} \tag{36}$$

Figure 14 shows system simulations using the above results. Therein, how the period specifications are obtained should be observed.

Finally, using the results obtained from recursive equations K_{12} and K_{21} fixed point analysis, to assure the existence of two fixed points, it is necessary that $c > 1$ (sufficient condition). Therefore

$$\begin{aligned} &\left[\frac{K_{11} - P_{21}}{K_{11} - P_{11}}\right] \left[\frac{K_{22} - P_{12}}{K_{22} - P_{22}}\right] \left[\frac{a_{12}}{a_{11}}\right] \left[\frac{a_{21}}{a_{22}}\right] > 1 \\ &\qquad \Longrightarrow K_{11} > 5.48 \end{aligned} \tag{37}$$

and also, to assure the limit cycle existence and uniqueness, initial state variables values must be an element of the following initial condition region:

$$\begin{aligned} &x_1(0) < 15, \\ &x_2(0) < 2 + 6\left[\frac{x_1(0) - 20}{K_{11} - 20}\right]^5. \end{aligned} \tag{38}$$

Figure 15 shows the simulation results considering $[x_1(0) \;\; x_2(0)]^T = [5 \;\; 6]^T$ and $K_{11} = 5.6$ that satisfy the previous inequality and $K_{11} = 5.2$ that does not.

6. Conclusion

The main theoretical results in this paper are concerned to establish limit cycle properties for uncoupled and a class of coupled hybrid systems. In this way, we have provided an analytical formulation of the steady state behavior that gives the necessary and sufficient conditions to reach a limit cycle. We have established that when the hybrid automaton has a cyclic structure and when the steady state exists, there will be a transient behavior followed by a steady state. This transient behavior depends on the automaton initial condition region as well as its jump conditions. The limit cycle in the steady state behavior is obtained from an initial region and is independent of the initial point within the region. In this paper we also have shown a way to design a specific limit cycle for a coupled hybrid system case. The sufficient and necessary conditions have been formulated. One goal of current research is to extend the analysis to a more general

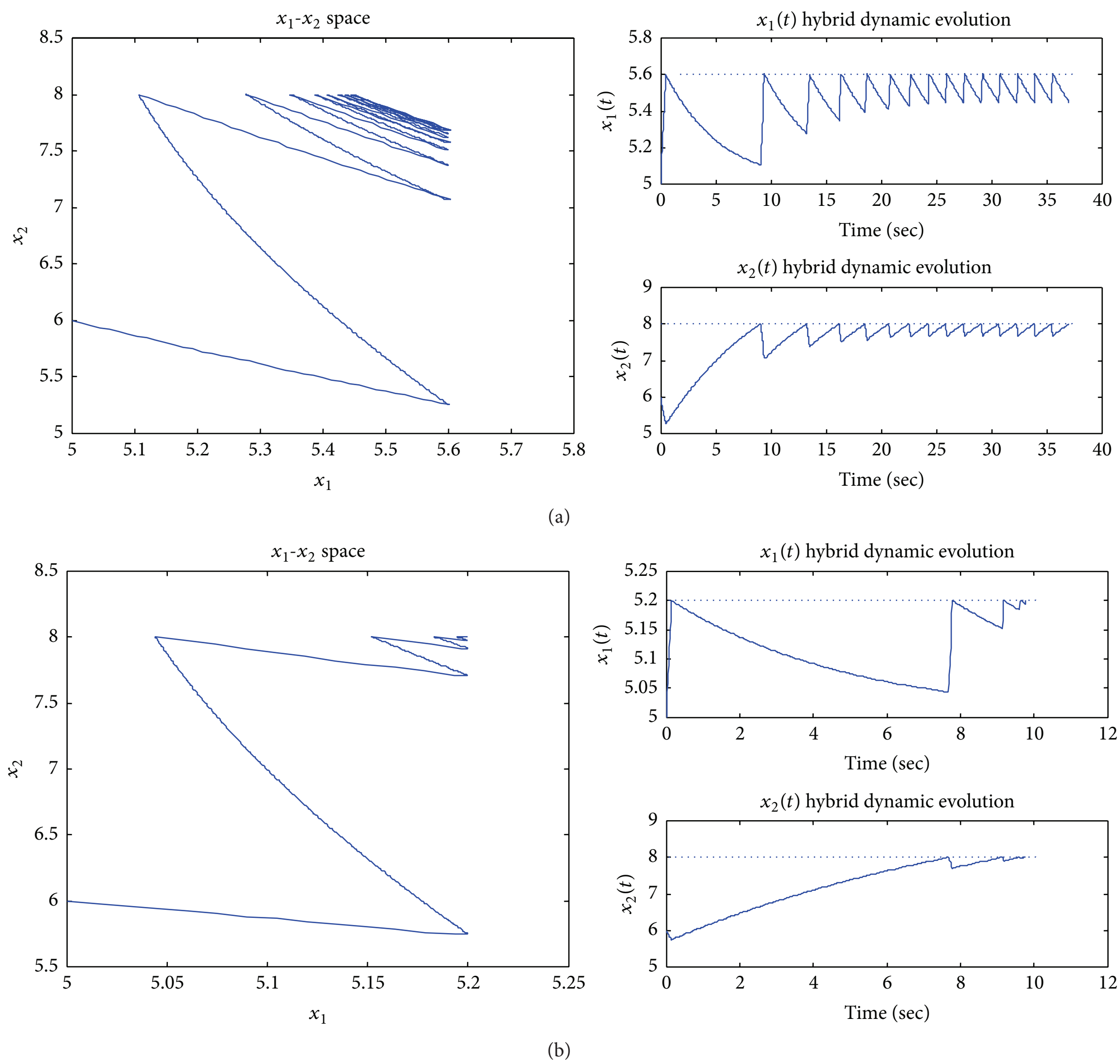


Figure 15: Automaton hybrid dynamics designed to have a limit cycle period: (a) nonzero ($K_{11} = 5.6$), (b) zero ($K_{11} = 5.2$).

class of coupled linear hybrid automata with more general jump conditions. This will extend the model applicability to a wide process class of dynamic hybrid systems.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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