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Research Article

On a Stochastic Lotka-Volterra Competitive System with Distributed Delay and General Lévy Jumps

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This paper considers a stochastic competitive system with distributed delay and general Lévy jumps. Almost sufficient and necessary conditions for stability in time average and extinction of each population are established under some assumptions. And two facts are revealed: both stability in time average and extinction have closer relationships with the general Lévy jumps, firstly; and secondly, the distributed delay has no effect on the stability in time average and extinction of the stochastic system. Some simulation figures, which are obtained by the split-step θ-method to discretize the stochastic model, are introduced to support the analytical findings.

1. Introduction

In recent years, delay differential equations has been used in the study of population dynamics. A famous competitive system with distributed delay can be expressed by

$$\begin{aligned}\frac{dy_1(t)}{dt} &= y_1(t)\left[b_1 - a_{11}y_1(t) - a_{12}\int_{-\tau_2}^{0} y_2(t+\theta)\,d\mu_2(\theta)\right],\\ \frac{dy_2(t)}{dt} &= y_2(t)\left[b_2 - a_{21}\int_{-\tau_1}^{0} y_1(t+\theta)\,d\mu_1(\theta) - a_{22}y_2(t)\right],\end{aligned} \tag{1}$$

where $y_i(t)$ denotes the size of the ith population, b_i, a_{ij}, and τ_i are all positive constants, and μ_i is a probability measure on $[-\tau_i, 0]$. There is an extensive literature concerned with the dynamics of (1) and we here only mention Kuang and Smith [1], Faria [2], Freedman and Wu [3], Bereketoglu and Győri [4], and Gopalsamy [5] among many others. In particular, Kuang (see [6, p. 231]) claimed that if $\Psi_1 > 0$ and $\Psi_2 > 0$, then model (1) has a positive equilibrium $x^* = (x_1^*, x_2^*) = (\Psi_1/\Psi, \Psi_2/\Psi)$ which is globally asymptotically stable, where $\Psi = a_{11}a_{22} - a_{12}a_{21}$, $\Psi_1 = b_1a_{22} - b_2a_{12}$, and $\Psi_2 = b_2a_{11} - b_1a_{21}$. It is important to point out that if $\Psi_1 > 0$ and $\Psi_2 > 0$, then $\Psi > 0$.

In the real world, the intrinsic growth rates of many species are always disturbed by environmental noises (see, e.g., [7–10]), which was recognized by many scholars in recent years (see, e.g., [11–14]). In particular, May [7] has pointed out that, due to environmental noises, the birth rates, carrying capacity, and other parameters involved in the system should be stochastic. In this paper, we assume that the parameters b_1 and b_2 are stochastic; then by the central limit theorem, we can replace b_1 and b_2 by

$$\begin{aligned} b_1 &\longrightarrow b_1 + \sigma_1\dot{B}_1(t),\\ b_2 &\longrightarrow b_2 + \sigma_2\dot{B}_2(t),\end{aligned} \tag{2}$$

where, for $i = 1, 2$, $B_i(t)$ represents a standard Brownian motion defined on a complete probability space $(\Omega, \mathscr{F}, \mathscr{P})$ and σ_i^2 is the intensity of the noise.

On the other hand, the population systems may suffer sudden environmental perturbations, that is, some jump type stochastic perturbations, for example, earthquakes, hurricanes, and epidemics. Some scholars have concentrated on the population systems with compensator jumps, and some significant and interesting results have been obtained (see,

e.g., [15–19]). Bao et al. [15, 16] did pioneering work in this field. In addition, Zou et al. [20–22] introduce a general Lévy jumps, which is more reasonable and complicated than the compensator jumps from the viewpoint of biomathematics (see [20]), into population models for the first time. However, there are no articles introducing the general Lévy jumps into population models with distributed delay, to the best of our knowledge. Motivated by these, we consider the famous stochastic competitive system with distributed delay and general Lévy jumps:

$$\begin{aligned} dy_1(t) &= y_1(t^-) \\ &\quad \cdot \left[b_1 - a_{11} y_1(t^-) - a_{12} \int_{-\tau_2}^{0} y_2(t^- + \theta)\, d\mu_2(\theta) \right] dt \\ &\quad + \sigma_1 y_1(t^-)\, dB_1(t) + y_1(t^-) \int_{\mathbb{Y}} \gamma_1(u)\, N(dt, du), \\ dy_2(t) &= y_2(t^-) \\ &\quad \cdot \left[b_2 - a_{21} \int_{-\tau_1}^{0} y_1(t^- + \theta)\, d\mu_1(\theta) - a_{22} y_2(t^-) \right] dt \\ &\quad + \sigma_2 y_2(t^-)\, dB_2(t) + y_2(t^-) \int_{\mathbb{Y}} \gamma_2(u)\, N(dt, du), \end{aligned} \tag{3}$$

where $y_i(t^-) = \lim_{s\uparrow t} y_i(s)$, $N(dt, du)$ is a real-valued Poisson counting measure with characteristic measure λ on a measurable subset $\mathbb{Y}$ of $\overline{R}_+$ with $\lambda(\mathbb{Y}) < +\infty$, $\widetilde{N}(dt, du) = N(dt, du) - \lambda(du)dt$, $\gamma(u)$ is bounded function, and $\gamma(u) > -1$, $u \in \mathbb{Y}$; furthermore, we assume that $B_i(t)$ is independent of N. Let the initial data $\xi(t) = (\xi_1(t), \xi_2(t)) \in C([-\tau, 0], R_+^2)$, where $C([-\tau, 0]; R_+^2)$ represents the family of continuous functions from $[-\tau, 0]$ to R_+^2 with the norm $\|\xi_i\| = \sup_{-\tau \le t \le 0} |\xi_i(t)|$, $i = 1, 2$, $\tau = \max\{\tau_1, \tau_2\}$. Other parameters are defined and required as before.

For convenience, we introduce the following notations:

$$\begin{aligned} R_+^2 &= \left\{ g = (g_1, g_2) \in R^2 \mid g_i > 0,\ i = 1, 2 \right\}, \\ \langle f(t) \rangle &= t^{-1} \int_0^t f(s)\, ds, \\ \eta_i &= \int_{\mathbb{Y}} \ln\left(\frac{1}{1 + \gamma_i(u)} \right) \lambda(du), \quad i = 1, 2, \\ \widetilde{\Psi}_1 &= 0.5 a_{22} \sigma_1^2 - 0.5 a_{12} \sigma_2^2 + a_{22} \eta_1 - a_{12} \eta_2, \\ \widetilde{\Psi}_2 &= 0.5 a_{11} \sigma_2^2 - 0.5 a_{21} \sigma_1^2 + a_{11} \eta_2 - a_{21} \eta_1. \end{aligned} \tag{4}$$

Moreover, we impose the following assumptions in this paper.

Assumption 1. There exists a positive constant c such that $|\ln(1 + \gamma(u))| \le c$ for $\gamma(u) > -1$.

In this paper, we consider a stochastic competitive system with distributed delay and general Lévy jumps. Unlike the deterministic system, the stochastic system does not have an interior equilibrium. Therefore, we cannot investigate the stability of the stochastic system. In Section 2, we show that the solution to system (3) will tend to a point in time average. Furthermore, we establish almost sufficient and necessary conditions for stability in time average and extinction of each population. In Section 3, we present an example to illustrate our mathematical findings. Section 4 gives the conclusions and future directions of the research.

2. Main Content

Lemma 2 (see Liu et al. [23]). *Suppose that $z(t) \in C(\Omega \times [0, +\infty), R_+)$.*

(i) *If there exist two positive constants T and ρ_0 such that $\ln z(t) \le \rho t - \rho_0 \int_0^t z(s) ds + \sum_{i=1}^2 \alpha_i B_i(t)$ for all $t \ge T$, where α_i, $i = 1, 2$, are constants, then*

$$\begin{aligned} \limsup_{t \to +\infty} \langle z(t) \rangle &\le \frac{\rho}{\rho_0} \quad a.s.,\ if\ \rho \ge 0; \\ \lim_{t \to +\infty} z(t) &= 0 \quad a.s.,\ if\ \rho < 0. \end{aligned} \tag{5}$$

(ii) *If there exist three positive constants T, ρ, and ρ_0 such that $\ln z(t) \ge \lambda t - \rho_0 \int_0^t z(s) ds + \sum_{i=1}^2 \alpha_i B_i(t)$ for all $t \ge T$, then $\liminf_{t \to +\infty} \langle z(t) \rangle \ge \rho / \rho_0$ a.s.*

In order for the model to be significant, we shall show that the solution is global and nonnegative. However, theorem of existence and uniqueness ([24–28]) is not satisfied in system (3). By using method established by Mao et al. [8], we will show existence and uniqueness of the global positive solution of system (3).

Lemma 3. *Let Assumption 1 hold. For any given initial value $\xi(t) = (\xi_1(t), \xi_2(t)) \in C([-\tau, 0], R_+^2)$; then system (3) has a unique positive solution $x(t) = (y_1(t), y_2(t))$ on $t \ge -\tau$ a.s. and the solution satisfies*

$$\limsup_{t \to +\infty} \frac{\ln y_i(t)}{\ln t} \le 1 \quad a.s.,\ i = 1, 2. \tag{6}$$

Proof. The proof is similar to Han et al. [29] by defining

$$V(x) = V_1(x) + V_2(x), \tag{7}$$

where

$$\begin{aligned} V_1(x) &= \sqrt{x_1} - 1 - 0.5 \ln x_1, \\ V_2(x) &= \sqrt{x_2} - 1 - 0.5 \ln x_2. \end{aligned} \tag{8}$$

In addition, applying the inequality, for $i = 1, 2$,

$$\begin{aligned} &\int_0^t \int_{-\tau}^0 x_i^2 \left((s + \theta)^- \right) d\mu(\theta)\, ds \\ &\quad = \int_{-\tau}^0 d\mu(\theta) \int_\theta^t x_i^2(s^-)\, ds \\ &\quad \le \int_{-\tau}^0 d\mu(\theta) \int_{-\tau}^t x_i^2(s^-)\, ds \\ &\quad \le \int_{-\tau}^0 \xi_i^2(s^-)\, ds + \int_0^t x_i^2(s^-)\, ds. \end{aligned} \tag{9}$$

So we omit it here. Now let us prove inequality (6).

Case 1 ($i = 1$). For any $t \geq 0$, applying the generalized Itô's formula [30] to (3) results in

$$\begin{aligned} d\left(e^t \ln y_1\right) &= e^t \ln y_1\, dt + e^t d \ln y_1 = e^t \Big[\ln y_1 + b_1 \\ &\quad - 0.5\sigma_1^2 + \int_{\mathbb{Y}} \ln\left(1+\gamma_1(u)\right)\lambda(du) - a_{11} y_1 \\ &\quad - a_{12}\int_{-\tau_2}^{0} y_2\left(t^- + \theta\right) d\mu_2(\theta)\Big] dt + \sigma_1 dB_1(t) \\ &\quad + \int_{\mathbb{Y}} \ln\left(1+\gamma_1(u)\right)\widetilde{N}(dt,du) \leq e^t \Big[\ln y_1 + b_1 \\ &\quad - 0.5\sigma_1^2 + \int_{\mathbb{Y}} \ln\left(1+\gamma_1(u)\right)\lambda(du) - a_{11} y_1\Big] dt \\ &\quad + e^t \sigma_1 dB_1(t) + e^t \int_{\mathbb{Y}} \ln\left(1+\gamma_1(u)\right)\widetilde{N}(dt,du). \end{aligned} \tag{10}$$

Thus

$$\begin{aligned} e^t \ln y_1(t) &\leq \ln y_1(0) + \int_0^t e^s \Big[\ln y_1(s) + b_1 - 0.5\sigma_1^2 \\ &\quad + \int_{\mathbb{Y}} \ln\left(1+\gamma_1(u)\right)\lambda(du) - a_{11} y_1(s)\Big] ds \\ &\quad + \int_0^t e^s \sigma_1 dB_1(s) \\ &\quad + \int_0^t e^s \int_{\mathbb{Y}} \ln\left(1+\gamma_1(u)\right)\widetilde{N}(ds,du). \end{aligned} \tag{11}$$

The rest of proof is analogous with Lemma 4.4 in [15]; we omitted it here.

Case 2 ($i = 2$). The proof is similar to Case 1; we left out it here. The proof is complete. □

Theorem 4. *For system (3), we suppose that Assumption 1, $\Psi_1 > 0$ and $\Psi_2 > 0$, holds.*

(I) *If $b_1 < 0.5\sigma_1^2 + \eta_1$ and $b_2 < 0.5\sigma_2^2 + \eta_2$, then both y_1 and y_2 are extinctive almost surely (a.s.); that is, $\lim_{t\to+\infty} y_i(t) = 0$ a.s., $i = 1, 2$.*

(II) *If $b_1 > 0.5\sigma_1^2 + \eta_1$ and $b_2 < 0.5\sigma_2^2 + \eta_2$, then y_2 is extinctive a.s. and y_1 is stable in time average a.s.; that is,*

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_1(s)\,ds = \frac{\left[b_1 - 0.5\sigma_1^2 - \eta_1\right]}{a_{11}}, \quad a.s.. \tag{12}$$

(III) *If $b_1 < 0.5\sigma_1^2 + \eta_1$ and $b_2 > 0.5\sigma_2^2 + \eta_2$, then y_1 is extinctive a.s. and y_2 is stable in time average a.s.; that is,*

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_2(s)\,ds = \frac{\left[b_2 - 0.5\sigma_2^2 - \eta_2\right]}{a_{22}}, \quad a.s.. \tag{13}$$

(IV) *If $b_1 > 0.5\sigma_1^2 + \eta_1$, $b_2 > 0.5\sigma_2^2 + \eta_2$,*

(A) *If $\Psi_1 > \widetilde{\Psi}_1$ and $\Psi_2 < \widetilde{\Psi}_2$, then y_2 is extinctive a.s. and y_1 is stable in time average a.s.:*

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_1(s)\,ds = \frac{\left[b_1 - 0.5\sigma_1^2 - \eta_1\right]}{a_{11}}, \quad a.s.; \tag{14}$$

(B) *If $\Psi_1 < \widetilde{\Psi}_1$ and $\Psi_2 > \widetilde{\Psi}_2$, then y_1 is extinctive a.s. and y_2 is stable in time average a.s.:*

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_2(s)\,ds = \frac{\left[b_2 - 0.5\sigma_2^2 - \eta_2\right]}{a_{22}}, \quad a.s.; \tag{15}$$

(C) *If $\Psi_1 > \widetilde{\Psi}_1$ and $\Psi_2 > \widetilde{\Psi}_2$, then both y_1 and y_2 are stable in time average a.s.:*

$$\begin{aligned} \lim_{t\to+\infty} t^{-1}\int_0^t y_1(s)\,ds &= \frac{\Psi_1 - \widetilde{\Psi}_1}{\Psi}, \\ \lim_{t\to+\infty} t^{-1}\int_0^t y_2(s)\,ds &= \frac{\Psi_2 - \widetilde{\Psi}_2}{\Psi}, \quad a.s. \end{aligned} \tag{16}$$

Proof. Applying Itô's formula [30] to the first equality in (3), we get

$$\begin{aligned} &\ln y_1(t) - \ln y_1(0) \\ &\quad = \left[b_1 - 0.5\sigma_1^2 - \eta_1\right] t - a_{11}\int_0^t y_1(s)\,ds \\ &\qquad - a_{12}\int_0^t\int_{-\tau_2}^{0} y_2(s+\theta)\,d\mu_2(\theta)\,ds + \sigma_1 B_1(t) \\ &\qquad + \int_0^t\int_{\mathbb{Y}} \ln\left(1+\gamma_1(u)\right)\widetilde{N}(ds,du). \end{aligned} \tag{17}$$

Making use of the Fubini theorem and a substitution technique, we have

$$\begin{aligned} &\int_0^t\int_{-\tau_2}^{0} y_2(s+\theta)\,d\mu_2(\theta)\,ds = \int_{-\tau_2}^{0} d\mu_2(\theta) \\ &\quad \cdot \int_0^t y_2(s+\theta)\,ds = \int_{-\tau_2}^{0} d\mu_2(\theta)\int_\theta^{t+\theta} y_2(s)\,ds \\ &\quad = \int_{-\tau_2}^{0} d\mu_2(\theta) \\ &\quad \cdot \left[\int_\theta^0 y_2(s)\,ds + \int_0^t y_2(s)\,ds + \int_t^{t+\theta} y_2(s)\,ds\right] \\ &\quad = \int_{-\tau_2}^{0} d\mu_2(\theta)\int_\theta^0 y_2(s)\,ds + \int_0^t y_2(s)\,ds \\ &\quad + \int_{-\tau_2}^{0} d\mu_2(\theta)\int_t^{t+\theta} y_2(s)\,ds. \end{aligned} \tag{18}$$

Therefore, we derive

$$\begin{aligned}
&\ln y_1(t) - \ln y_1(0) \\
&\quad = \left[b_1 - 0.5\sigma_1^2 - \eta_1\right] t - a_{11} \int_0^t y_1(s)\, ds \\
&\qquad - a_{12} \int_0^t y_2(s)\, ds - a_{12} \int_{-\tau_2}^0 d\mu_2(\theta) \int_\theta^0 y_2(s)\, ds \\
&\qquad - a_{12} \int_{-\tau_2}^0 d\mu_2(\theta) \int_t^{t+\theta} y_2(s)\, ds + \sigma_1 B_1(t) \\
&\qquad + \int_0^t \int_{\mathbb{Y}} \ln\left(1 + \gamma_1(u)\right) \widetilde{N}(ds, du).
\end{aligned} \tag{19}$$

Similarly,

$$\begin{aligned}
&\ln y_2(t) - \ln y_2(0) \\
&\quad = \left(b_2 - 0.5\sigma_2^2 - \eta_2\right) t - a_{22} \int_0^t y_2(s)\, ds \\
&\qquad - a_{21} \int_0^t y_1(s)\, ds - a_{21} \int_{-\tau_1}^0 d\mu_1(\theta) \int_\theta^0 y_1(s)\, ds \\
&\qquad - a_{21} \int_{-\tau_1}^0 d\mu_1(\theta) \int_t^{t+\theta} y_1(s)\, ds + \sigma_2 B_2(t) \\
&\qquad + \int_0^t \int_{\mathbb{Y}} \ln\left(1 + \gamma_2(u)\right) \widetilde{N}(ds, du).
\end{aligned} \tag{20}$$

(I) Assume that $b_1 < 0.5\sigma_1^2 + \eta_1$ and $b_2 < 0.5\sigma_2^2 + \eta_2$. From (9),

$$\begin{aligned}
t^{-1} \ln \frac{y_1(t)}{y_1(0)} &\le b_1 - 0.5\sigma_1^2 - \eta_1 + t^{-1}\sigma_1 B_1(t) \\
&\quad + t^{-1} \int_0^t \int_{\mathbb{Y}} \ln\left(1 + \gamma_1(u)\right) \widetilde{N}(ds, du).
\end{aligned} \tag{21}$$

By the strong law of large numbers for martingales, we therefore have $\lim_{t\to+\infty} B_i(t)/t = 0$ a.s., $i = 1, 2$; meanwhile, we define, for $t \ge 0$, $M_i(t) = \int_0^t \int_{\mathbb{Y}} \ln(1 + \gamma_i(u)) \widetilde{N}(ds, du)$, $i = 1, 2$. Under Assumption 1, $\langle M_i \rangle(t) = \int_0^t \int_{\mathbb{Y}} (\ln(1 + \gamma_i(u))^2 \lambda(du) ds \le c^2 t \lambda(\mathbb{Y})$, making use of the strong law of large numbers for local martingales (see, e.g., [31]), we then derive

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t \int_{\mathbb{Y}} \ln\left(1 + \gamma_i(u)\right) \widetilde{N}(ds, du) = 0 \quad \text{a.s.}, \tag{22}$$

and $b_1 < 0.5\sigma_1^2 + \eta_1$ that

$$\limsup_{t\to+\infty} t^{-1} \ln y_1(t) \le b_1 - 0.5\sigma_1^2 - \eta_1 < 0. \tag{23}$$

Consequently, $\lim_{t\to+\infty} y_1(t) = 0$, a.s. Similarly, by (19), we can show that if $b_2 < 0.5\sigma_2^2 + \eta_2$, then $\lim_{t\to+\infty} y_2(t) = 0$, a.s.

(II) Suppose that $b_1 > 0.5\sigma_1^2 + \eta_1$ and $b_2 < 0.5\sigma_2^2 + \eta_2$. Since $b_2 < 0.5\sigma_2^2 + \eta_2$, then, by (I), $\lim_{t\to+\infty} y_2(t) = 0$, a.s. Hence, making use of (9), for arbitrary $\varepsilon > 0$, there is $T > 0$ such that, for $t \ge T$,

$$\begin{aligned}
&\frac{-\varepsilon}{2} \le a_{12} t^{-1} \int_0^t \int_{-\tau_2}^0 y_2(s + \theta)\, d\mu_2(\theta)\, ds \le \frac{\varepsilon}{2}, \\
&\frac{-\varepsilon}{2} \le t^{-1} \ln y_1(0) \le \frac{\varepsilon}{2}.
\end{aligned} \tag{24}$$

Substituting the above inequalities into (19), we can see that, for $t \ge T$,

$$\begin{aligned}
\ln y_1(t) &\le \left(b_1 - 0.5\sigma_1^2 - \eta_1 + \varepsilon\right) t - a_{11} \int_0^t y_1(s)\, ds \\
&\quad + \sigma_1 B_1(t) \\
&\quad + \int_0^t \int_{\mathbb{Y}} \ln\left(1 + \gamma_1(u)\right) \widetilde{N}(ds, du),
\end{aligned} \tag{25}$$

$$\begin{aligned}
\ln y_1(t) &\ge \left(b_1 - 0.5\sigma_1^2 - \eta_1 - \varepsilon\right) t - a_{11} \int_0^t y_1(s)\, ds \\
&\quad + \sigma_1 B_1(t) \\
&\quad + \int_0^t \int_{\mathbb{Y}} \ln\left(1 + \gamma_1(u)\right) \widetilde{N}(ds, du).
\end{aligned} \tag{26}$$

Since $b_1 > 0.5\sigma_1^2 + \eta_1$, we can choose ε sufficiently small such that $b_1 - 0.5\sigma_1^2 - \eta_1 - \varepsilon > 0$. Applying (i) and (ii) in Lemma 2 to (25) and (26), respectively, we derive

$$\begin{aligned}
\frac{b_1 - 0.5\sigma_1^2 - \eta_1 - \varepsilon}{a_{11}} &\le \liminf_{t\to+\infty} \langle y_1(t) \rangle \\
&\le \limsup_{t\to+\infty} \langle y_1(t) \rangle \\
&\le \frac{b_1 - 0.5\sigma_1^2 - \eta_1 + \varepsilon}{a_{11}}, \quad \text{a.s.}
\end{aligned} \tag{27}$$

Let $\varepsilon \to 0$. Then, we have $\lim_{t\to+\infty} \langle y_1(t) \rangle = [b_1 - 0.5\sigma_1^2 - \eta_1]/a_{11}$, a.s.

The proof of (III) is homogeneous with (II) by symmetry; hence it is omitted.

Now let us prove (IV). For $i = 1, 2$, consider the following equation:

$$\begin{aligned}
dz_i(t) &= z_i(t) \left[b_i - a_{ii} z_i(t)\right] dt + \sigma_i z_i(t)\, dB_i(t) \\
&\quad + z_1(t^-) \int_{\mathbb{Y}} \gamma_i(u)\, N(dt, du), \\
z_i(\theta) &= y_i(\theta), \quad \theta \in [-\tau, 0].
\end{aligned} \tag{28}$$

In virtue of the classic stochastic comparison theorem [32], we can find that

$$\begin{aligned}
y_1(t) &\le z_1(t), \\
y_2(t) &\le z_2(t).
\end{aligned} \tag{29}$$

Since $b_i > 0.5\sigma_i^2 + \eta_i$, $i = 1, 2$, similar to the proof of (II), we can show that

$$\begin{aligned}\lim_{t\to+\infty}\langle z_i(t)\rangle &= \lim_{t\to+\infty} t^{-1}\int_0^t z_i(s)\,ds \\ &= \frac{\left(b_i - 0.5\sigma_i^2 - \eta_2\right)}{a_{ii}}, \quad \text{a.s.,}\ i = 1, 2, \\ \lim_{t\to+\infty}\langle z_i(t)\rangle &= \lim_{t\to+\infty} t^{-1}\int_0^{t+\theta} z_i(s)\,ds \\ &= \frac{\left(b_i - 0.5\sigma_i^2 - \eta_2\right)}{a_{ii}}, \\ &\qquad \text{a.s.,}\ -\tau \le \theta \le 0,\ i = 1, 2.\end{aligned} \tag{30}$$

Thus

$$\begin{aligned}&\lim_{t\to+\infty} t^{-1}\int_{t+\theta}^t z_1(s)\,ds \\ &\quad = \lim_{t\to+\infty}\left(t^{-1}\int_0^t z_1(s)\,ds - t^{-1}\int_0^{t+\theta} z_1(s)\,ds\right) = 0, \\ &\lim_{t\to+\infty} t^{-1}\int_{t+\theta}^t z_2(s)\,ds = 0, \quad \text{a.s.,}\end{aligned} \tag{31}$$

which, together with (29), implies that

$$\begin{aligned}&\lim_{t\to+\infty} t^{-1}\int_{t+\theta}^t y_1(s)\,ds = 0, \\ &\lim_{t\to+\infty} t^{-1}\int_{t+\theta}^t y_2(s)\,ds = 0, \quad \text{a.s.}\end{aligned} \tag{32}$$

On the other hand, calculating $(2.5)\times a_{11} - (2.4)\times a_{21}$ deduces

$$\begin{aligned}a_{11}\ln\frac{y_2(t)}{y_2(0)} &= -a_{11}a_{21}\int_{-\tau_1}^0 d\mu_1(\theta)\int_\theta^0 y_1(s)\,ds \\ &\quad - a_{11}a_{21}\int_{-\tau_1}^0 d\mu_1(\theta)\int_t^{t+\theta} y_1(s)\,ds \\ &\quad + a_{21}a_{12}\int_{-\tau_2}^0 d\mu_2(\theta)\int_\theta^0 y_2(s)\,ds \\ &\quad + a_{21}a_{12}\int_{-\tau_2}^0 d\mu_2(\theta)\int_t^{t+\theta} y_2(s)\,ds \\ &\quad + a_{21}\ln\frac{y_1(t)}{y_1(0)} + \left(\Psi_2 - \widetilde{\Psi}_2\right)t \\ &\quad - \Psi\int_0^t y_2(s)\,ds - a_{21}\sigma_1 B_1(t) \\ &\quad - a_{21}\int_0^t\int_{\mathbb{Y}}\ln\left(1+\gamma_1(u)\right)\widetilde{N}(ds,du) \\ &\quad + a_{11}\sigma_2 B_2(t) \\ &\quad + a_{11}\int_0^t\int_{\mathbb{Y}}\ln\left(1+\gamma_2(u)\right)\widetilde{N}(ds,du).\end{aligned} \tag{33}$$

By virtue of (6) and (32), for arbitrary $\varepsilon > 0$, there is $T > 0$ such that, for $t \ge T$,

$$\begin{aligned}&t^{-1}a_{21}\ln\frac{y_1(t)}{y_1(0)} < \frac{\varepsilon}{3}, \\ &t^{-1}a_{11}\ln y_2(0) < \frac{\varepsilon}{3}, \\ &\quad - a_{11}a_{21}t^{-1}\int_{-\tau_1}^0 d\mu_1(\theta)\int_\theta^0 y_1(s)\,ds \\ &\quad - a_{11}a_{21}t^{-1}\int_{-\tau_1}^0 d\mu_1(\theta)\int_t^{t+\theta} y_1(s)\,ds \\ &\quad + a_{21}a_{12}t^{-1}\int_{-\tau_2}^0 d\mu_2(\theta)\int_\theta^0 y_2(s)\,ds \\ &\quad + a_{21}a_{12}t^{-1}\int_{-\tau_2}^0 d\mu_2(\theta)\int_t^{t+\theta} y_2(s)\,ds \le \frac{\varepsilon}{3}.\end{aligned} \tag{34}$$

Substituting the above inequalities into (33) results in

$$\begin{aligned}a_{11}\ln y_2(t) &\le \left(\Psi_2 - \widetilde{\Psi}_2 + \varepsilon\right)t - \Psi\int_0^t y_2(s)\,ds \\ &\quad - a_{21}\sigma_1 B_1(t) + a_{11}\sigma_2 B_2 \\ &\quad + a_{11}\int_0^t\int_{\mathbb{Y}}\ln\left(1+\gamma_2(u)\right)\widetilde{N}(ds,du),\end{aligned} \tag{35}$$

for $t > T$. Meanwhile, calculating $(2.4)\times a_{22} - (2.5)\times a_{12}$ yields

$$\begin{aligned}a_{22}\ln\frac{y_1(t)}{y_1(0)} &= -a_{11}a_{21}\int_{-\tau_1}^0 d\mu_1(\theta)\int_\theta^0 y_1(s)\,ds \\ &\quad - a_{11}a_{21}\int_{-\tau_1}^0 d\mu_1(\theta)\int_t^{t+\theta} y_1(s)\,ds \\ &\quad + a_{12}a_{21}\int_{-\tau_1}^0 d\mu_1(\theta)\int_\theta^0 y_1(s)\,ds \\ &\quad - a_{12}a_{21}\int_{-\tau_1}^0 d\mu_1(\theta)\int_t^{t+\theta} y_1(s)\,ds \\ &\quad + a_{12}\ln\frac{y_2(t)}{y_2(0)} + \left(\Psi_1 - \widetilde{\Psi}_1\right)t \\ &\quad - \Psi\int_0^t y_1(s)\,ds + a_{22}\sigma_1 B_1(t) \\ &\quad + a_{22}\int_0^t\int_{\mathbb{Y}}\ln\left(1+\gamma_1(u)\right)\widetilde{N}(ds,du) \\ &\quad - a_{12}\sigma_2 B_2(t) \\ &\quad - a_{12}\int_0^t\int_{\mathbb{Y}}\ln\left(1+\gamma_2(u)\right)\widetilde{N}(ds,du).\end{aligned} \tag{36}$$

Using the same way, by (36) we can have that, for $t > T$,

$$\begin{aligned} a_{22}\ln y_1(t) \le{}& \left(\Psi_1 - \widetilde{\Psi}_1 + \varepsilon\right)t - \Psi\int_0^t y_1(s)\,ds \\ &+ a_{22}\sigma_1 B_1(t) \\ &+ a_{22}\int_0^t\int_{\mathbb{Y}} \ln\left(1+\gamma_1(u)\right)\widetilde{N}(ds,du) \\ &- a_{12}\sigma_2 B_2(t) \\ &- a_{12}\int_0^t\int_{\mathbb{Y}} \ln\left(1+\gamma_2(u)\right)\widetilde{N}(ds,du). \end{aligned} \tag{37}$$

(A) Suppose $\Psi_1 > \widetilde{\Psi}_1$ and $\Psi_2 < \widetilde{\Psi}_2$. Note that $\Psi_2 < \widetilde{\Psi}_2$, and then let ε be sufficiently small such that $\Psi_2 - \widetilde{\Psi}_2 + \varepsilon < 0$. Applying (i) in Lemma 2 to (35) gives $\lim_{t\to+\infty} y_2(t) = 0$, a.s. The proof of $\lim_{t\to+\infty}\langle y_1(t)\rangle = (b_1 - 0.5\sigma_1^2 - \eta_1)/a_{11}$, a.s., is similar to (II) and hence is omitted.

The proof of (B) is similar to (A) by symmetry and hence is left out.

(C) Suppose that $\Psi_1 > \widetilde{\Psi}_1$ and $\Psi_2 > \widetilde{\Psi}_2$. Since $\Psi_2 > \widetilde{\Psi}_2$, it then follows from (33) and Lemma 2 that

$$\limsup_{t\to+\infty}\langle y_2(t)\rangle \le \frac{\Psi_2 - \widetilde{\Psi}_2 + \varepsilon}{\Psi}, \quad \text{a.s.} \tag{38}$$

Making use of the arbitrariness of ε, we can see that

$$\limsup_{t\to+\infty}\langle y_2(t)\rangle \le \frac{\Psi_2 - \widetilde{\Psi}_2}{\Psi}, \quad \text{a.s.} \tag{39}$$

It follows from (37), Lemma 2, and the arbitrariness of ε that

$$\limsup_{t\to+\infty}\langle y_1(t)\rangle \le \frac{\Psi_1 - \widetilde{\Psi}_1}{\Psi}, \quad \text{a.s.} \tag{40}$$

likewise. Let ε be sufficiently small such that $a_{11}((\Psi_2 - \widetilde{\Psi}_2)/\Psi) - \varepsilon > 0$. When (32) and (39) are used in (19), we get

$$\begin{aligned} t^{-1}\ln y_1(t) ={}& t^{-1}\ln y_1(0) + b_1 - 0.5\sigma_1^2 - \eta_1 \\ &- a_{11}\langle y_1(t)\rangle - a_{12}\langle y_2(t)\rangle + \frac{\sigma_1 B_1(t)}{t} \\ &+ t^{-1}\int_0^t\int_{\mathbb{Y}} \ln\left(1+\gamma_1(u)\right)\widetilde{N}(ds,du) \\ &- a_{12}t^{-1}\left[\int_{-\tau_2}^0 d\mu_2(\theta)\int_\theta^0 y_2(s)\,ds \right. \\ &\left. + \int_{-\tau_2}^0 d\mu_2(\theta)\int_t^{t+\theta} y_2(s)\,ds\right] \ge b_1 - 0.5\sigma_1^2 - \eta_1 - \varepsilon \\ &- a_{11}\langle y_1(t)\rangle - a_{12}\limsup_{t\to+\infty}\langle y_2(t)\rangle + \frac{\sigma_1 B_1(t)}{t} \\ &+ t^{-1}\int_0^t\int_{\mathbb{Y}} \ln\left(1+\gamma_1(u)\right)\widetilde{N}(ds,du) \ge b_1 - 0.5\sigma_1^2 \\ &- \eta_1 - \varepsilon - a_{11}\langle y_1(t)\rangle - a_{12}\frac{\Psi_2 - \widetilde{\Psi}_2}{\Psi} + \frac{\sigma_1 B_1(t)}{t} \\ &+ t^{-1}\int_0^t\int_{\mathbb{Y}} \ln\left(1+\gamma_1(u)\right)\widetilde{N}(ds,du) = a_{11} \\ &\cdot \frac{\Psi_1 - \widetilde{\Psi}_1}{\Psi} - \varepsilon - a_{11}\langle y_1(t)\rangle + \frac{\sigma_1 B_1(t)}{t} \\ &+ t^{-1}\int_0^t\int_{\mathbb{Y}} \ln\left(1+\gamma_1(u)\right)\widetilde{N}(ds,du), \end{aligned} \tag{41}$$

for sufficiently large t. In virtue of (ii) in Lemma 2 and the arbitrariness of ε, we get

$$\liminf_{t\to+\infty}\langle y_1(t)\rangle \ge \frac{\Psi_1 - \widetilde{\Psi}_1}{\Psi}, \quad \text{a.s.} \tag{42}$$

Similarly, substituting (32) and (40) into (20) brings about $\liminf_{t\to+\infty}\langle y_2(t)\rangle \ge (\Psi_2 - \widetilde{\Psi}_2)/\Psi$, a.s. This, together with (39), (40), and (42), means $\lim_{t\to+\infty}\langle y_1(t)\rangle = (\Psi_1 - \widetilde{\Psi}_1)/\Psi$ and $\lim_{t\to+\infty}\langle y_2(t)\rangle = (\Psi_2 - \widetilde{\Psi}_2)/\Psi$, a.s. □

Remark 5. It is important to designate that if $b_1 > 0.5\sigma_1^2 + \eta_1$, $b_2 > 0.5\sigma_2^2 + \eta_2$, and $\Psi > 0$, then $\Psi_1 < \widetilde{\Psi}_1$ and $\Psi_2 < \widetilde{\Psi}_2$ cannot hold simultaneously.

Remark 6. Theorem 4 implies an important fact that when $-1 < \gamma_i(u) < 0$, $i = 1, 2$, the jump process can result in extinction of the population $y_i(t)$, for example, earthquakes and hurricanes, and when $\gamma_i(u) > 0$, $i = 1, 2$, the jump process is always advantage for the population $y_i(t)$, for example, ocean red tide.

Remark 7. From the perspective of the condition in Theorem 4, the distributed delay does not influence some the properties including extinction and stability in time average.

3. Numerical Simulations

In this section, we employ the split-step θ-method, whose approximate solution is mean-square convergent with order $p = 0.5$ (see [32, 33]), to discretize (3). Here, we choose the initial data $\xi(t) = (0.3e^t, 0.4e^t)$, $b_1 = 0.59$, $b_2 = 0.51$, $a_{11} = 0.8$, $a_{12} = 0.39$, $a_{21} = 0.51$, $a_{22} = 0.69$, $\tau_1 = \tau_2 = 0.3$, $\mathbb{Y} = (0, +\infty)$, and $\lambda(\mathbb{Y}) = 1$. Then $\Psi = 0.36$. The main difference between the conditions of the following Case 1–Case 4 is that the values of $\gamma_1(u)$ and $\gamma_2(u)$ are different. In Case 1, we let $\gamma_1(u) = \gamma_2(u) = 0$ and $\sigma_1^2 = \sigma_2^2 = 0$. Then by virtue of Kuangs work [6], we have that the positive equilibrium $x^* = (\Psi_1/\Psi, \Psi_2/\Psi) = (0.612, 0.277)$ is globally asymptotically stable. Figure 1(a) verifies this. In Case 2, we set $\gamma_1(u) = -0.1$, $\gamma_2(u) = -0.2$, and $\sigma_1^2 = 0.2, \sigma_2^2 = 0.4$. Then

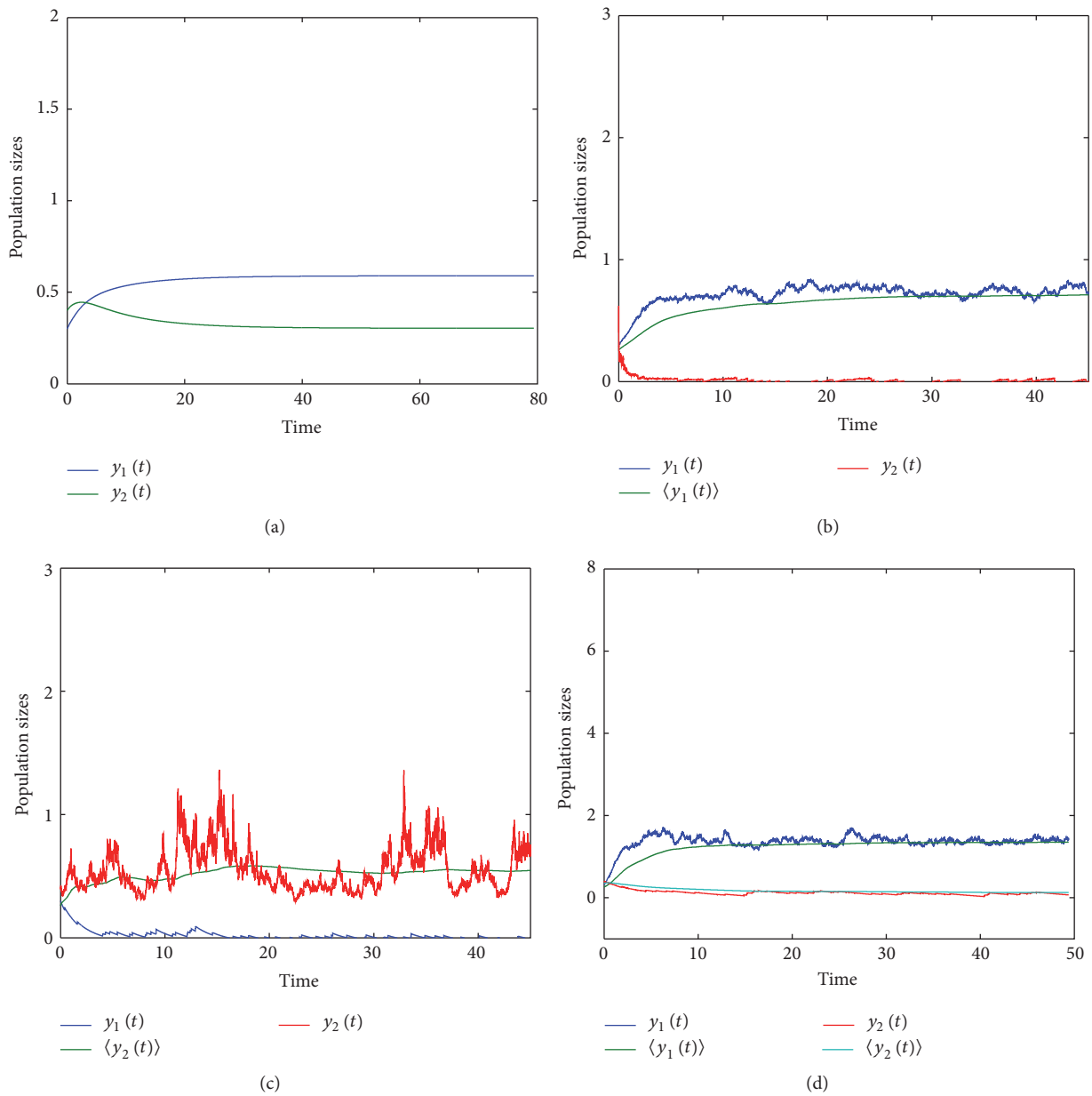

Figure 1: The horizontal axis and the vertical axis in this and following figures represent the time t and the populations size (step size $\Delta t = 0.001$).

$\Psi_1 > \widetilde{\Psi}_1 = -0.03216$ and $\Psi_2 = 0.1 < \widetilde{\Psi}_2 = 0.2386$. In view of (A) in Theorem 4, y_2 goes to extinction and

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_1(s)\,ds = \frac{b_1 - 0.5\sigma_1^2 - \eta_1}{a_{11}} = \frac{0.595}{0.8} = 0.74. \tag{43}$$

Figure 1(b) confirms this. In Case 3, we choose $\gamma_1(u) = -0.8$, $\gamma_2(u) = 0.01$, $\sigma_1^2 = 0.2$, and $\sigma_2^2 = 0.4$. Then $\Psi_1 = 0.22 < \widetilde{\Psi}_1 = 1.11$ and $\Psi_2 = 0.1 > \widetilde{\Psi}_2 = 0.03$. It follows from (B) in Theorem 4 that y_1 goes to extinction and

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_2(s)\,ds = \frac{b_2 - 0.5\sigma_2^2 - \eta_2}{a_{22}} = \frac{0.41}{0.7} = 0.59. \tag{44}$$

See Figure 1(c). In Case 4, we let $\gamma_1(u) = -0.03$, $\gamma_2(u) = 0.01$, $\sigma_1^2 = 0.2$, and $\sigma_2^2 = 0.4$, that is, $\Psi_1 = 0.22 > \widetilde{\Psi}_1 = -0.22$ and

$\Psi_2 = 0.1 > \widetilde{\Psi}_2 = 0.095$. According to (C) in Theorem 4, we obtain

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_1(s)\,ds = \frac{\Psi_1 - \widetilde{\Psi}_1}{\Psi} = 1.22,$$
$$\lim_{t\to+\infty} t^{-1}\int_0^t y_2(s)\,ds = \frac{\Psi_2 - \widetilde{\Psi}_2}{\Psi} = 0.0139. \tag{45}$$

Figure 1(d) validates this.

4. Conclusions and Remarks

This paper investigates a stochastic competitive system with distributed delay and general Lévy jumps. Under the assumption $\Psi > 0$, the almost complete parameter analysis is fulfilled in detail. Our results imply that the general Lévy jumps can significantly change the properties of population models.

Some interesting and significant topics deserve our further engagement. One may put forward a more realistic and sophisticated model to integrate the colored noise into the model [10, 11, 34]. Another significant problem is devoted to stochastic model with infinite delays and general Lévy jumps. We will leave these for future investigation.

It should also be mentioned that "stability in time average" is not a good definition of persistence for stochastic population models. Some papers have introduced more appropriate definitions of permanence for stochastic population models, that is, stochastically persistent in probability or stochastic permanence (see, e.g., [35–37]). We will research these kinds of permanence of model (3) in detail in our following study.

Competing Interests

The authors declare that they have no competing interests.

Acknowledgments

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