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Research Article

Bipartite Fuzzy Stochastic Differential Equations with Global Lipschitz Condition

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We introduce and analyze a new type of fuzzy stochastic differential equations. We consider equations with drift and diffusion terms occurring at both sides of equations. Therefore we call them the bipartite fuzzy stochastic differential equations. Under the Lipschitz and boundedness conditions imposed on drifts and diffusions coefficients we prove existence of a unique solution. Then, insensitivity of the solution under small changes of data of equation is examined. Finally, we mention that all results can be repeated for solutions to bipartite set-valued stochastic differential equations.

1. Introduction

Stochastic differential equations are often used in modelling dynamics of uncertain physical systems, where it is assumed that randomness and stochastic noises have an influence on a considered system. The theory of such equations involving stochastic integrals is well established (see, e.g., [1–3]). On the other hand in modelling of many real-world processes there appears uncertainty of different kind than randomness, namely, trying to describe a physical system one encounters, for instance, an imprecision of measurement equipment, imperfect human judgments, and opinions on parameters of such system. These are also symptoms of uncertainty but they do not locate in randomness or stochastic noises. This uncertainty is well treated by fuzzy set theory (c.f. [4–6]). Owing to this theory it is possible to handle mathematically such linguistic opinions, for example, "low pressure," "high temperature," and "about 7%." A usage of fuzzy sets gives ability to study deterministic fuzzy differential equations in modelling various phenomena which include imprecision [7–10]. Moreover, some successful attempts of combining two kinds of uncertainties, that is, randomness and fuzziness, were undertaken for petroleum contamination [11], optimal tracking design of stochastic fuzzy systems [12], random fuzzy differential equations [13–16], stochastic fuzzy neural networks [17, 18], civil engineering and mechanics [19], Markov chains with fuzzy states [20], fuzzy martingales [21], Petri nets [22], optimization [23], ballast water management [24], filtering of fuzzy stochastic systems [25], and fuzzy stochastic differential equations [26–30].

The latter topic on fuzzy stochastic differential equations is quite new and still developed. In papers [26–28] we considered such equations in their natural integral form generalizing one of the crisp stochastic differential equations, that is,

$$x(t) = x_0 + \int_0^t f(s, x(s))\, ds + \int_0^t g(s, x(s))\, dB(s), \quad t \in [0, T], \tag{1}$$

where f is a random fuzzy set-valued drift coefficient, g is a random single-valued diffusion coefficient, and x_0 is a fuzzy random variable. We investigated the problem of existence of a unique solution, since it is almost impossible to find explicit forms of solutions to such equations. This is very similar to the theory of crisp stochastic differential equations. However, unlike crisp equations, fuzzy set-valued equations exhibit new qualitative properties of their solutions. Namely, we mean here nondecreasing (in time) diameter of solution's values, which determines that uncertainty located in fuzziness cannot decrease as time increases. This could be an obstacle in some concrete situations, when an expert knows that fuzziness should be decreasing in his system. Therefore, in the works [29, 30], we proposed to study fuzzy stochastic differential equations in integral form

$$x(t) + (-1)\int_0^t f(s, x(s))\,ds + (-1)\int_0^t g(s, x(s))\,dB(s) = x_0, \quad t \in [0, T]. \tag{2}$$

If one would consider this equation in the crisp setting, then it would be no difference from previous equation. However, these two equations are not equivalent in fuzzy environment. Solutions of the second equation have nonincreasing fuzziness of their values. This property does not refer to solutions of crisp equations. Although some potential applications of fuzzy stochastic differential equations in finance, biology, control systems, and physics were studied, for example, in [26, 28, 30], there is still a need of a further development in this area to know better nature of these equations and properties of their solutions.

In this paper we propose to join two equations mentioned above in a one equation

$$\begin{aligned} x(t) &+ (-1)\int_0^t f(s, x(s))\,ds + (-1)\int_0^t g(s, x(s))\,dB(s) \\ &= x_0 + \int_0^t \widetilde{f}(s, x(s))\,ds + \int_0^t \widetilde{g}(s, x(s))\,d\widetilde{B}(s), \\ &\qquad t \in [0, T]. \end{aligned} \tag{3}$$

This way we introduce a new kind of fuzzy stochastic differential equations which are more general than those studied in our earlier works and mentioned above. Due to the new form of equations with integrals at both sides they will be called the bipartite fuzzy stochastic differential equations. Solutions to such equations may lose property of monotonicity of fuzziness. However, this can be an advantage, since it can allow for future examinations of periodic solutions. In current paper we initiate investigations of the bipartite fuzzy stochastic differential equations. Under the Lipschitz and boundedness conditions imposed on the drift and diffusion coefficients, existence of a unique solution is proved. It is also shown that the solution is stable with respect to small changes of equation's data; that is, the solution does not change much when the changes of drift and diffusion coefficients and initial value are small. This shows that the theory introduced in the paper is well-posed. We also indicate that parallel to bipartite fuzzy stochastic differential equations one can consider bipartite set-valued stochastic differential equations and all the results established for the first equations can be easily repeated for the second equations.

The subsequent part of the paper is organized as follows: in Section 2 we collect a prerequisite knowledge on set-valued random variables, set-valued stochastic processes, fuzzy sets, fuzzy random variables, and fuzzy stochastic Lebesgue-Aumann integral. This is done for convenience of the reader. Section 3 is a main part of the paper. The bipartite fuzzy stochastic differential equations are introduced here. We prove existence and uniqueness of solution to such equations and study properties of solutions.

2. Preliminaries

For a convenience of the reader we set up a framework which we work with.

Let $\mathcal{K}(\mathbb{R}^d)$ be the set of all nonempty, compact, and convex subsets of $\mathbb{R}^d$. This set can be supplied with the Hausdorff metric d_H which is defined by

$$d_H(A, B) := \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}, \tag{4}$$

where $\|\cdot\|$ denotes a norm in $\mathbb{R}^d$. Then the metric space $(\mathcal{K}(\mathbb{R}^d), d_H)$ is complete and separable (see [31]). Also, the addition and scalar multiplication in $\mathcal{K}(\mathbb{R}^d)$ are defined as follows: for $A, B \in \mathcal{K}(\mathbb{R}^d)$, $b \in \mathbb{R}^d$, $\lambda \in \mathbb{R}$

$$\begin{aligned} A + B &:= \{a + b : a \in A,\ b \in B\}, \\ A + \{b\} &:= \{a + b : a \in A\}, \\ \lambda A &:= \{\lambda a : a \in A\}. \end{aligned} \tag{5}$$

Let $(\Omega, \mathscr{A}, P)$ be a complete probability space and $\mathscr{M}(\Omega, \mathscr{A}; \mathcal{K}(\mathbb{R}^d))$ denote the family of $\mathscr{A}$-measurable set-valued mappings $F : \Omega \to \mathcal{K}(\mathbb{R}^d)$ (set-valued random variable) such that

$$\{\omega \in \Omega : F(\omega) \cap O \neq \emptyset\} \in \mathscr{A} \quad \text{for every open set } O \subset \mathbb{R}^d. \tag{6}$$

A set-valued random variable $F \in \mathscr{M}(\Omega, \mathscr{A}; \mathcal{K}(\mathbb{R}^d))$ is called L^p-integrally bounded, $p \geqslant 1$, if there exists $h \in L^p(\Omega, \mathscr{A}, P; \mathbb{R})$ such that $\|a\| \leqslant h(\omega)$ for any a and ω with $a \in F(\omega)$. It is known (see [32]) that F is L^p-integrally bounded iff $\omega \mapsto d_H(F(\omega), \{0\})$ is in $L^p(\Omega, \mathscr{A}, P; \mathbb{R})$, where $L^p(\Omega, \mathscr{A}, P; \mathbb{R})$ is a space of equivalence classes (with respect to the equality P-a.e.) of $\mathscr{A}$-measurable random variables $h : \Omega \to \mathbb{R}$ such that $\mathbb{E}|h|^p = \int_\Omega |h|^p dP < \infty$. Let us denote

$$\begin{aligned} \mathscr{L}^p\left(\Omega, \mathscr{A}, P; \mathcal{K}\left(\mathbb{R}^d\right)\right) := \Big\{&F \in \mathscr{M}\left(\Omega, \mathscr{A}; \mathcal{K}\left(\mathbb{R}^d\right)\right) : \\ &F \text{ is } L^p\text{-integrally bounded}\Big\}, \quad p \geqslant 1. \end{aligned} \tag{7}$$

The set-valued random variables $F, G \in \mathscr{L}^p(\Omega, \mathscr{A}, P; \mathcal{K}(\mathbb{R}^d))$ are considered to be identical, if $F = G$ holds P-a.e.

Let $T > 0$, and denote $I := [0, T]$. Let the system $(\Omega, \mathscr{A}, \{\mathscr{A}_t\}_{t\in I}, P)$ be a complete, filtered probability space with a filtration $\{\mathscr{A}_t\}_{t\in I}$ satisfying the usual hypotheses; that is, $\{\mathscr{A}_t\}_{t\in I}$ is an increasing and right continuous family of sub-σ-algebras of $\mathscr{A}$, and $\mathscr{A}_0$ contains all P-null sets. We call $X : I \times \Omega \to \mathcal{K}(\mathbb{R}^d)$ a set-valued stochastic process, if for every $t \in I$ a mapping $X(t) : \Omega \to \mathcal{K}(\mathbb{R}^d)$ is a set-valued random variable. We say that a set-valued stochastic process X is d_H-continuous, if almost all (with respect to the probability measure P) its paths, that is, the mappings $X(\cdot,\omega) : I \to \mathscr{K}(\mathbb{R}^d)$, are the d_H-continuous functions. A set-valued stochastic process X is said to be $\{\mathscr{A}_t\}_{t\in I}$-adapted, if for every $t \in I$ the set-valued random variable $X(t) : \Omega \to \mathscr{K}(\mathbb{R}^d)$ is $\mathscr{A}_t$-measurable. It is called measurable, if $X : I\times\Omega \to \mathscr{K}(\mathbb{R}^d)$ is a $\mathscr{B}(I)\otimes\mathscr{A}$-measurable set-valued random variable, where $\mathscr{B}(I)$ denotes the Borel σ-algebra of subsets of I. If $X : I\times\Omega \to \mathscr{K}(\mathbb{R}^d)$ is $\{\mathscr{A}_t\}_{t\in I}$-adapted and measurable, then it will be called nonanticipating. Equivalently, X is nonanticipating iff X is measurable with respect to the σ-algebra $\mathscr{N}$ which is defined as follows:

$$\mathscr{N} := \left\{A \in \mathscr{B}(I) \otimes \mathscr{A} : A^t \in \mathscr{A}_t \text{ for every } t \in I\right\}, \tag{8}$$

where $A^t = \{\omega : (t,\omega) \in A\}$. A set-valued nonanticipating stochastic process $X : I \times \Omega \to \mathscr{K}(\mathbb{R}^d)$ is called $\mathscr{L}^p$-integrally bounded, if there exists a measurable stochastic process $h : I \times \Omega \to \mathbb{R}$ such that $\mathbb{E}(\int_I |h(s)|^p ds) < \infty$ and $|||X(t,\omega)||| \leqslant h(t,\omega)$ for a.a. $(t,\omega) \in I \times \Omega$. By $\mathscr{L}^p(I\times\Omega,\mathscr{N};\mathscr{K}(\mathbb{R}^d))$ we denote the set of all equivalence classes (with respect to the equality $\gamma\times P$-a.e., γ denotes the Lebesgue measure) of nonanticipating and $\mathscr{L}^p$-integrally bounded set-valued stochastic processes.

A fuzzy set u in $\mathbb{R}^d$ (see [4]) is characterized by its membership function (denoted by u again) $u : \mathbb{R}^d \to [0,1]$ and $u(x)$ (for each $x \in \mathbb{R}^d$) is interpreted as the degree of membership of x in the fuzzy set u. As the value $u(x)$ expresses "degree of membership of x in" or a "degree of satisfying by x a property," one can work with imprecise information. Obviously, every ordinary set u in $\mathbb{R}^d$ is a fuzzy set, since then $u(x) = 1$ if $x \in u$ and $u(x) = 0$ if $x \notin u$.

Let $\mathscr{F}(\mathbb{R}^d)$ denote the fuzzy sets $u : \mathbb{R}^d \to [0,1]$ such that $[u]^\alpha \in \mathscr{K}(\mathbb{R}^d)$ for every $\alpha \in [0,1]$, where $[u]^\alpha := \{a \in \mathbb{R}^d : u(a) \geqslant \alpha\}$ for $\alpha \in (0,1]$ and $[u]^0 := \mathrm{cl}\{a \in \mathbb{R}^d : u(a) > 0\}$. Note that the set $\mathbb{R}^d$ can be embedded into $\mathscr{F}(\mathbb{R}^d)$ by the embedding $\langle\cdot\rangle : \mathbb{R}^d \to \mathscr{F}(\mathbb{R}^d)$ defined as follows: for $r \in \mathbb{R}^d$ we have $\langle r\rangle(x) = 1$ if $x = r$, and $\langle r\rangle(x) = 0$ if $x \neq r$.

Addition $u + v$ and scalar multiplication βu in fuzzy set space $\mathscr{F}(\mathbb{R}^d)$ can be defined levelwise (see [33]):

$$\begin{aligned} [u+v]^\alpha &= [u]^\alpha + [v]^\alpha, \\ [\beta u]^\alpha &= \beta [u]^\alpha, \end{aligned} \tag{9}$$

where $u, v \in \mathscr{F}(\mathbb{R}^d)$, $\beta \in \mathbb{R}$, and $\alpha \in [0,1]$.

Let $u, v \in \mathscr{F}(\mathbb{R}^d)$. If there exists $w \in \mathscr{F}(\mathbb{R}^d)$ such that $u = v + w$ then we call w the Hukuhara difference of u and v and we denote it by $u\ominus v$. Note that $u\ominus v \neq u+(-1)v$. Also $u\ominus v$ may not exist, but if it exists it is unique. For u, $v \in \mathscr{F}(\mathbb{R}^d)$ and $r_1, r_2 \in \mathbb{R}^d$ we have the following:

(P1) $(u + \langle r_1\rangle) \ominus \langle r_2\rangle = u + \langle r_1 - r_2\rangle$,

(P2) the Hukuhara difference $(u + \langle r_1\rangle) \ominus v$ exists iff $u \ominus v$ exists. Moreover, $(u + \langle r_1\rangle) \ominus v = (u \ominus v) + \langle r_1\rangle$.

Define $d_\infty : \mathscr{F}(\mathbb{R}^d)\times\mathscr{F}(\mathbb{R}^d) \to [0,\infty)$ by the expression

$$d_\infty(u,v) := \sup_{\alpha\in[0,1]} d_H\left([u]^\alpha, [v]^\alpha\right). \tag{10}$$

The mapping d_∞ is a metric in $\mathscr{F}(\mathbb{R}^d)$. It is known that $(\mathscr{F}(\mathbb{R}^d), d_\infty)$ is a complete metric space, but it is not separable and it is not locally compact. For every $u, v, w, z \in \mathscr{F}(\mathbb{R}^d)$, $\beta \in \mathbb{R}$, one has (see, e.g., [14, 34])

(P3) $d_\infty(u + w, v + w) = d_\infty(u, v)$,

(P4) $d_\infty(u + v, w + z) \leqslant d_\infty(u, w) + d_\infty(v, z)$,

(P5) $d_\infty(\beta u, \beta v) = |\beta| d_\infty(u, v)$,

(P6) $d_\infty(u \ominus v, \langle 0\rangle) = d_\infty(u, v)$,

(P7) $d_\infty(u \ominus v, u \ominus w) = d_\infty(v, w)$,

(P8) $d_\infty(u \ominus v, w \ominus z) \leqslant d_\infty(u, w) + d_\infty(v, z)$.

A mapping $x : \Omega \to \mathscr{F}(\mathbb{R}^d)$ is said to be a fuzzy random variable (see [34]), if $[x]^\alpha : \Omega \to \mathscr{K}(\mathbb{R}^d)$ is an $\mathscr{A}$-measurable set-valued random variable for all $\alpha \in [0,1]$. It is known from [35] that $x : \Omega \to \mathscr{F}(\mathbb{R}^d)$ is the fuzzy random variable iff $x : (\Omega,\mathscr{A}) \to (\mathscr{F}(\mathbb{R}^d), \mathscr{B}_{d_S})$ is $\mathscr{A} \mid \mathscr{B}_{d_S}$-measurable, where d_S denotes the Skorohod metric in $\mathscr{F}(\mathbb{R}^d)$ and $\mathscr{B}_{d_S}$ denotes the σ-algebra generated by the topology induced by d_S. A fuzzy random variable $x : \Omega \to \mathscr{F}(\mathbb{R}^d)$ is said to be L^p-integrally bounded, $p \geqslant 1$, if $[x]^0$ belongs to $\mathscr{L}^p(\Omega,\mathscr{A},P;\mathscr{K}(\mathbb{R}^d))$. By $\mathscr{L}^p(\Omega,\mathscr{A},P;\mathscr{F}(\mathbb{R}^d))$ we denote the set of all L^p-integrally bounded fuzzy random variables, where we consider $x, y \in \mathscr{L}^p(\Omega,\mathscr{A},P;\mathscr{F}(\mathbb{R}^d))$ as identical if $x = y$ holds P-a.e.

We call $x : I\times\Omega \to \mathscr{F}(\mathbb{R}^d)$ a fuzzy stochastic process, if for every $t \in I$ the mapping $x(t,\cdot) : \Omega \to \mathscr{F}(\mathbb{R}^d)$ is a fuzzy random variable. We say that a fuzzy stochastic process x is d_∞-continuous, if almost all (with respect to the probability measure P) its trajectories, that is, the mappings $x(\cdot,\omega) : I \to \mathscr{F}(\mathbb{R}^d)$, are the d_∞-continuous functions. A fuzzy stochastic process x is called $\{\mathscr{A}_t\}_{t\in I}$-adapted, if for every $\alpha \in [0,1]$ the multifunction $[x(t)]^\alpha : \Omega \to \mathscr{K}(\mathbb{R}^d)$ is $\mathscr{A}_t$-measurable for all $t \in I$. It is called measurable, if $[x]^\alpha : I\times\Omega \to \mathscr{K}(\mathbb{R}^d)$ is a $\mathscr{B}(I)\otimes\mathscr{A}$-measurable multifunction for all $\alpha \in [0,1]$, where $\mathscr{B}(I)$ denotes the Borel σ-algebra of subsets of I. If $x : I\times\Omega \to \mathscr{F}(\mathbb{R}^d)$ is $\{\mathscr{A}_t\}_{t\in I}$-adapted and measurable, then it is called nonanticipating. Equivalently, x is nonanticipating iff for every $\alpha \in [0,1]$ the set-valued random variable $[x]^\alpha$ is measurable with respect to the σ-algebra $\mathscr{N}$. A fuzzy stochastic process x is called L^p-integrally bounded ($p \geqslant 1$), if there exists a measurable stochastic process $h : I\times\Omega \to \mathbb{R}$ such that $\mathbb{E}\int_I |h(t)|^p dt < \infty$ and $d_\infty(x(t,\omega), \langle 0\rangle) \leqslant h(t,\omega)$ for a.a. $(t,\omega) \in I\times\Omega$. By $\mathscr{L}^p(I\times\Omega,\mathscr{N};\mathscr{F}(\mathbb{R}^d))$ we denote the set of nonanticipating and L^p-integrally bounded fuzzy stochastic processes.

In the whole paper, notation $x \overset{P.1}{=} y$ stands for abbreviation of $P(x = y) = 1$, where x, y are some random elements. Also we will write $x(t) \overset{I\ P.1}{=} y(t)$ instead of $P(x(t) = y(t)\ \forall t \in I) = 1$, where x, y are some stochastic processes. Similar notations will be used for inequalities.

Let $x \in \mathscr{L}^p(I \times \Omega, \mathscr{N}; \mathscr{F}(\mathbb{R}^d))$, $p \geqslant 1$. For such process x we can define (see, e.g., [26]) the fuzzy stochastic Lebesgue-Aumann integral which is a fuzzy random variable

$$\Omega \ni \omega \longmapsto \int_I x(s,\omega)\,ds \in \mathscr{F}(\mathbb{R}^d). \tag{11}$$

Then $\int_0^t x(s)ds$ (from now on we do not write the argument ω) is understood as $\int_I \mathbf{1}_{[0,t]}(s)x(s)ds$. For the fuzzy stochastic Lebesgue-Aumann integral we have the following properties (see [26]).

Proposition 1. *Let $p \geqslant 1$. If $x, y \in \mathscr{L}^p(I \times \Omega, \mathscr{N}; \mathscr{F}(\mathbb{R}^d))$ then*

(i) $I \times \Omega \ni (t,\omega) \mapsto \int_0^t x(s,\omega)ds \in \mathscr{F}(\mathbb{R}^d)$ *belongs to* $\mathscr{L}^p(I \times \Omega, \mathscr{N}; \mathscr{F}(\mathbb{R}^d))$,

(ii) *the fuzzy stochastic process* $(t,\omega) \mapsto \int_0^t x(s,\omega)ds$ *is* d_∞-*continuous,*

(iii) $\sup_{u\in[0,t]} d_\infty^p(\int_0^u x(s)ds, \int_0^u y(s)ds) \overset{I\ P.1}{\leqslant} t^{p-1}\int_0^t d_\infty^p(x(s), y(s))ds$,

(iv) *for every* $t \in I$ *it holds* $\mathbb{E}\sup_{u\in[0,t]} d_\infty^p(\int_0^u x(s)ds, \int_0^u y(s)ds) \leqslant t^{p-1}\mathbb{E}\int_0^t d_\infty^p(x(s), y(s))ds$.

3. Main Results

Let $B = \{B(t)\}_{t\in I}$ denote an m-dimensional $\{\mathscr{A}_t\}$-Brownian motion defined on $(\Omega, \mathscr{A}, \{\mathscr{A}_t\}_{t\in I}, P)$, $m \in \mathbb{N}$. The process B is defined as follows: $B = (B^1, B^2, \ldots, B^m)'$, where $\{B^1(t)\}_{t\in I}, \{B^2(t)\}_{t\in I}, \ldots, \{B^m(t)\}_{t\in I}$ are the independent, one-dimensional $\{\mathscr{A}_t\}_{t\in I}$-Brownian motions, and the symbol $'$ denotes transposition. Similarly $\widetilde{B} = \{\widetilde{B}\}_{t\in I}$ stays for an n-dimensional Brownian motion which is assumed to be independent of B.

In the paper we make an examination of initial value problem for fuzzy stochastic differential equations of a new form

$$\begin{aligned} &dx(t) + (-1)f(t,x(t))\,dt + \langle (-1)g(t,x(t))\,dB(t)\rangle \\ &\quad \overset{I\ P.1}{=} \widetilde{f}(t,x(t))\,dt + \langle \widetilde{g}(t,x(t))\,d\widetilde{B}(t)\rangle, \\ &x(0) \overset{P.1}{=} x_0, \end{aligned} \tag{12}$$

with $f, \widetilde{f} : I \times \Omega \times \mathscr{F}(\mathbb{R}^d) \to \mathscr{F}(\mathbb{R}^d)$, $g : I \times \Omega \times \mathscr{F}(\mathbb{R}^d) \to \mathbb{R}^d \times \mathbb{R}^m$, $\widetilde{g} : I \times \Omega \times \mathscr{F}(\mathbb{R}^d) \to \mathbb{R}^d \times \mathbb{R}^n$, and $x_0 : \Omega \to \mathscr{F}(\mathbb{R}^d)$ being a fuzzy random variable. Note that problem (12) can be reformulated as

$$\begin{aligned} &dx(t) + (-1)f(t,x(t))\,dt \\ &\quad + \left\langle \sum_{i=1}^m (-1)g^i(t,x(t))\,dB^i(t) \right\rangle \overset{I\ P.1}{=} \widetilde{f}(t,x(t))\,dt \\ &\quad + \left\langle \sum_{j=1}^n \widetilde{g}^j(t,x(t))\,d\widetilde{B}^j(t) \right\rangle, \\ &x(0) \overset{P.1}{=} x_0, \end{aligned} \tag{13}$$

since we consider m-dimensional Brownian motion B, n-dimensional Brownian motion $\widetilde{B}$, $g = (g^1, g^2, \ldots, g^m)$, $\widetilde{g} = (\widetilde{g}^1, \widetilde{g}^2, \ldots, \widetilde{g}^n)$, where $g^i, \widetilde{g}^j : I \times \Omega \times \mathscr{F}(\mathbb{R}^d) \to \mathbb{R}^d$ for each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

The way of writing fuzzy stochastic differential equations in differential forms (12) and (13) is symbolic only, because these equations are always considered as integral equations:

$$\begin{aligned} &x(t) + \int_0^t (-1)f(s,x(s))\,ds \\ &\quad + \left\langle \sum_{i=1}^m \int_0^t (-1)g^i(s,x(s))\,dB^i(s) \right\rangle \\ &\overset{I\ P.1}{=} x_0 + \int_0^t \widetilde{f}(s,x(s))\,ds \\ &\quad + \left\langle \sum_{j=1}^n \int_0^t \widetilde{g}^j(s,x(s))\,d\widetilde{B}^j(s) \right\rangle, \end{aligned} \tag{14}$$

where the first integrals on both sides are the fuzzy stochastic Lebesgue-Aumann integrals and the remaining integrals are the crisp stochastic Itô integrals.

One can observe that if $f \equiv \langle 0 \rangle$ and $g^i \equiv 0$ for $i = 1, 2, \ldots, m$, then (14) takes form

$$\begin{aligned} x(t) &\overset{I\ P.1}{=} x_0 + \int_0^t \widetilde{f}(s,x(s))\,ds \\ &\quad + \left\langle \sum_{j=1}^n \int_0^t \widetilde{g}^j(s,x(s))\,d\widetilde{B}^j(s) \right\rangle. \end{aligned} \tag{15}$$

Such equations were studied in [26–28]. On the other hand if $\widetilde{f} \equiv \langle 0 \rangle$ and $\widetilde{g}^j \equiv 0$ for $j = 1, 2, \ldots, n$, then (14) reduces to

$$\begin{aligned} &x(t) + \int_0^t (-1)f(s,x(s))\,ds \\ &\quad + \left\langle \sum_{i=1}^m \int_0^t (-1)g^i(s,x(s))\,dB^i(s) \right\rangle \overset{I\ P.1}{=} x_0 \end{aligned} \tag{16}$$

and only investigations concerning these equations are presented in [29, 30]. If the data f, g, x_0 are the single-valued and singleton-defined mappings in (15) and (16), then we arrived at the same type of crisp stochastic differential equation. However, in fuzzy case, (15) and (16) are of different type, because fuzzy solutions to fuzzy equations (15) and (16) exhibit different geometric properties. The solutions x to (15) have nondecreasing fuzziness in their values as t increases; that is, with $P.1$ for every $\alpha \in [0,1]$ the mappings $t \mapsto \operatorname{diam}([x(t,\omega)]^\alpha)$ are nondecreasing (Theorem 3.8 [26]), but for the solutions $\widetilde{x}$ to (16) the mappings $t \mapsto \operatorname{diam}([x(t,\omega)]^\alpha)$ are nonincreasing (Theorem 3.3 [30]).

In this paper we establish a new kind of fuzzy stochastic differential equations (14) by joining (15) and (16). Therefore (14) is called the *bipartite* fuzzy stochastic differential equation. The solutions x to (14) can lose property that the mappings $t \mapsto \operatorname{diam}([x(t,\omega)]^\alpha)$ are monotone. Indeed, the fuzzy stochastic Lebesgue-Aumann integral on the left-hand side of (14) is an item which affects monotonicity of functions $t \mapsto \operatorname{diam}([x(t,\omega)]^\alpha)$. It makes them nonincreasing ones, but simultaneously the fuzzy stochastic Lebesgue-Aumann integral on the right-hand side of (14) forces that the functions $t \mapsto \operatorname{diam}([x(t,\omega)]^\alpha)$ do not decrease. However, the loss of monotonicity could be an advantage in the future, since it could open a gate for future studies of periodic solutions to fuzzy stochastic differential equations.

Note that, using (P1) and (P2), (14) can be viewed as

$$\begin{aligned} x(t) &\overset{I\ P.1}{=} \left[\left(x_0 + \int_0^t \widetilde{f}(s,x(s))\,ds\right)\right. \\ &\quad \left.\ominus\left((-1)\int_0^t f(s,x(s))\,ds\right)\right] \\ &\quad + \left\langle \sum_{j=1}^{n} \int_0^t \widetilde{g}^j(s,x(s))\,dB^j(s)\right\rangle \\ &\quad + \left\langle \sum_{i=1}^{m} \int_0^t g^i(s,x(s))\,dB^i(s)\right\rangle. \end{aligned} \tag{17}$$

Hence, without loss of generality, we can consider bipartite fuzzy stochastic differential equations of the following integral form:

$$\begin{aligned} x(t) &\overset{I\ P.1}{=} \left[\left(x_0 + \int_0^t \widetilde{f}(s,x(s))\,ds\right)\right. \\ &\quad \left.\ominus\left((-1)\int_0^t f(s,x(s))\,ds\right)\right] \\ &\quad + \left\langle \sum_{k=1}^{\ell} \int_0^t h^k(s,x(s))\,dW^k(s)\right\rangle, \end{aligned} \tag{18}$$

where $\ell \in \mathbb{N}$, $h^1,h^2,\ldots,h^\ell : I\times\Omega\times\mathscr{F}(\mathbb{R}^d) \to \mathbb{R}^d$, and $W^1,W^2,\ldots,W^\ell$ are the independent one-dimensional $\{\mathscr{A}_t\}_{t\in I}$-Brownian motions, and x_0 is a fuzzy random variable.

Below we write what we mean by a solution to bipartite fuzzy stochastic differential equation. Let $\widetilde{T} \in (0,T]$, $\widetilde{I} = [0,\widetilde{T}]$.

Definition 2. Let a fuzzy stochastic process $x : \widetilde{I}\times\Omega \to \mathscr{F}(\mathbb{R}^d)$ satisfy the following: (i) $x \in \mathscr{L}^2(\widetilde{I}\times\Omega,\mathscr{N};\mathscr{F}(\mathbb{R}^d))$, (ii) x is d_∞-continuous, and (iii) it holds (18). If $\widetilde{T} < T$, then x is said to be the local solution to bipartite fuzzy stochastic differential equation (18), and if $\widetilde{T} = T$, then x is called the global solution to (18). A local solution $x : \widetilde{I}\times\Omega \to \mathscr{F}(\mathbb{R}^d)$ to (18) is said to be unique, if $x(t) \overset{\widetilde{I}\ P.1}{=} y(t)$, where $y : \widetilde{I}\times\Omega \to \mathscr{F}(\mathbb{R}^d)$ is any other local solution to (18). The uniqueness of the global solution to (18) is defined similarly.

Since existence of Hukuhara differences in (18) depends on x_0, existence of solution to (18) cannot be independent of x_0. This fact differs bipartite fuzzy differential equations from crisp stochastic differential equations.

In what follows we begin our study with a first and most important issue of existence and uniqueness of solutions to (18). In the paper we require that $x_0 : \Omega \to \mathscr{F}(\mathbb{R}^d)$, $f,\widetilde{f} : I\times\Omega\times\mathscr{F}(\mathbb{R}^d) \to \mathscr{F}(\mathbb{R}^d)$, $h^k : I\times\Omega\times\mathscr{F}(\mathbb{R}^d) \to \mathbb{R}^d$ ($k = 1,2,\ldots,\ell$) satisfy the following:

(A0) $x_0 \in \mathscr{L}^2(\Omega,\mathscr{A}_0,P;\mathscr{F}(\mathbb{R}^d))$,

(A1) the mappings $f,\widetilde{f} : (I\times\Omega)\times\mathscr{F}(\mathbb{R}^d) \to \mathscr{F}(\mathbb{R}^d)$ are $\mathscr{N}\otimes\mathscr{B}_{d_S} \mid \mathscr{B}_{d_S}$-measurable and $h^1,h^2,\ldots,h^\ell : (I\times\Omega)\times\mathscr{F}(\mathbb{R}^d) \to \mathbb{R}^d$ are $\mathscr{N}\otimes\mathscr{B}_{d_S} \mid \mathscr{B}(\mathbb{R}^d)$-measurable,

(A2) there exists a constant $L > 0$ such that for $\gamma\times P$-a.a. (t,ω) and for every $u,v \in \mathscr{F}(\mathbb{R}^d)$ it holds

$$\begin{aligned} \max&\left\{d_\infty^2\left(f(t,\omega,u),f(t,\omega,v)\right),\right. \\ &\quad d_\infty^2\left(\widetilde{f}(t,\omega,u),\widetilde{f}(t,\omega,v)\right), \\ &\quad \left.\left\|h^k(t,\omega,u)-h^k(t,\omega,v)\right\|^2\right\} \leqslant L d_\infty^2(u,v), \\ &\qquad k = 1,2,\ldots,\ell, \end{aligned} \tag{19}$$

(A3) there exists $K \in L^1(I\times\Omega,\mathscr{B}(I)\otimes\mathscr{A},\gamma\times P;\mathbb{R})$ such that for $\gamma\times P$-a.a. (t,ω) it holds:

$$\begin{aligned} \max&\left\{d_\infty^2\left(f(t,\omega,\langle 0\rangle),\langle 0\rangle\right), d_\infty^2\left(\widetilde{f}(t,\omega,\langle 0\rangle),\langle 0\rangle\right),\right. \\ &\quad \left.\left\|h^k(t,\omega,\langle 0\rangle)\right\|^2\right\} \leqslant K(t,\omega), \quad k = 1,2,\ldots,\ell, \end{aligned} \tag{20}$$

(A4) there exists a constant $\widetilde{T} \in (0,T]$ such that the sequence $\{x_n\}_{n=0}^\infty$ of the fuzzy mappings $x_n : \widetilde{I}\times\Omega \to \mathscr{F}(\mathbb{R}^d)$ described as

$$x_0(t) \overset{\widetilde{I}\ P.1}{:=} x_0, \tag{21}$$

and for $n = 1,2,\ldots$

$$\begin{aligned} x_n(t) &\overset{\widetilde{I}\ P.1}{=} \left[\left(x_0 + \int_0^t \widetilde{f}(s,x_{n-1}(s))\,ds\right)\right. \\ &\quad \left.\ominus\left((-1)\int_0^t f(s,x_{n-1}(s))\,ds\right)\right] \\ &\quad + \left\langle \sum_{k=1}^{\ell} \int_0^t h^k(s,x_{n-1}(s))\,dW^k(s)\right\rangle, \end{aligned} \tag{22}$$

is well defined; that is, in particular, the Hukuhara differences appearing above do exist.

Remark 3. Assume that (A0)–(A4) are satisfied for x_0, f, $\widetilde{f}$, h^k. Then the mappings x_n described in (A4) are the fuzzy stochastic processes, and they are d_∞-continuous and belong to $\mathscr{L}^2(\widetilde{I}\times\Omega,\mathscr{N};\mathscr{F}(\mathbb{R}^d))$.

A preliminary result is on the sequence $\{x_n\}$ and it shows that $\{x_n\}$ is uniformly bounded. As we intend to use $\{x_n\}$ as the sequence of approximate solutions, we will be able to infer later on the fact that the exact solution is bounded as well.

Proposition 4. *Assume that (A0)–(A4) are satisfied for x_0, f, $\widetilde{f}$, h^k. Then, for the sequence $\{x_n\}_{n=1}^{\infty}$ defined like in (22) we have*

$$\mathbb{E}\sup_{t\in\widetilde{I}} d_\infty^2\left(x_n(t),\langle 0\rangle\right) \leqslant \left(C_1 + C_2\widetilde{T}\mathbb{E}d_\infty^2\left(x_0,\langle 0\rangle\right)\right)e^{C_2\widetilde{T}}, \quad n\in\mathbb{N}, \tag{23}$$

where $C_1 = (\ell+3)[\mathbb{E}d_\infty^2(x_0,\langle 0\rangle) + 4(\widetilde{T}+2\ell)\mathbb{E}\int_{\widetilde{I}}K(s)ds]$ and $C_2 = 4L(\ell+3)(\widetilde{T}+2\ell)$.

Proof. Denote $z_n(t) = \mathbb{E}\sup_{u\in[0,t]}d_\infty^2(x_n(u),\langle 0\rangle)$ for $n\in\mathbb{N}$ and $t\in\widetilde{I}$. Applying properties (P4), (P6), and (P5) we arrive at

$$\begin{aligned} z_n(t) &= \mathbb{E}\sup_{u\in[0,t]} d_\infty^2\Bigg(\Bigg[\left(x_0 + \int_0^u \widetilde{f}\left(s,x_{n-1}(s)\right)ds\right) \\ &\ominus\left((-1)\int_0^u f\left(s,x_{n-1}(s)\right)ds\right)\Bigg] \\ &+\left\langle\sum_{k=1}^{\ell}\int_0^u h^k\left(s,x_{n-1}(s)\right)dW^k(s)\right\rangle,\langle 0\rangle\Bigg) \leqslant (\ell \\ &+3)\Bigg[\mathbb{E}d_\infty^2\left(x_0,\langle 0\rangle\right) \\ &+\mathbb{E}\sup_{u\in[0,t]}d_\infty^2\left(\int_0^u\widetilde{f}\left(s,x_{n-1}(s)\right)ds,\langle 0\rangle\right) \\ &+\mathbb{E}\sup_{u\in[0,t]}d_\infty^2\left(\int_0^u f\left(s,x_{n-1}(s)\right)ds,\langle 0\rangle\right) \\ &+\sum_{k=1}^{\ell}\mathbb{E}\sup_{u\in[0,t]}\left\|\int_0^u h^k\left(s,x_{n-1}(s)\right)dW^k(s)\right\|^2\Bigg]. \end{aligned} \tag{24}$$

Further,

$$\begin{aligned} z_n(t) &\leqslant (\ell+3)\Bigg[\mathbb{E}d_\infty^2\left(x_0,\langle 0\rangle\right) \\ &+2\mathbb{E}\sup_{u\in[0,t]}d_\infty^2\left(\int_0^u\widetilde{f}\left(s,x_{n-1}(s)\right)ds,\int_0^u\widetilde{f}\left(s,\langle 0\rangle\right)ds\right) \\ &+2\mathbb{E}\sup_{u\in[0,t]}d_\infty^2\left(\int_0^u\widetilde{f}\left(s,\langle 0\rangle\right)ds,\langle 0\rangle\right) \\ &+2\mathbb{E}\sup_{u\in[0,t]}d_\infty^2\left(\int_0^u f\left(s,x_{n-1}(s)\right)ds,\int_0^u f\left(s,\langle 0\rangle\right)ds\right) \\ &+2\mathbb{E}\sup_{u\in[0,t]}d_\infty^2\left(\int_0^u f\left(s,\langle 0\rangle\right)ds,\langle 0\rangle\right) \\ &+2\sum_{k=1}^{\ell}\mathbb{E}\sup_{u\in[0,t]}\left\|\int_0^u\left(h^k\left(s,x_{n-1}(s)\right)-h^k\left(s,\langle 0\rangle\right)\right)dW^k(s)\right\|^2 \\ &+2\sum_{k=1}^{\ell}\mathbb{E}\sup_{u\in[0,t]}\left\|\int_0^u h^k\left(s,\langle 0\rangle\right)dW^k(s)\right\|^2\Bigg]. \end{aligned} \tag{25}$$

Invoking Proposition 1 and the Doob inequality we get

$$\begin{aligned} z_n(t) &\leqslant (\ell+3)\Bigg[\mathbb{E}d_\infty^2\left(x_0,\langle 0\rangle\right) \\ &+2t\mathbb{E}\int_0^t d_\infty^2\left(\widetilde{f}\left(s,x_{n-1}(s)\right),\widetilde{f}\left(s,\langle 0\rangle\right)\right)ds \\ &+2t\mathbb{E}\int_0^t d_\infty^2\left(\widetilde{f}\left(s,\langle 0\rangle\right),\langle 0\rangle\right)ds \\ &+2t\mathbb{E}\int_0^t d_\infty^2\left(f\left(s,x_{n-1}(s)\right),f\left(s,\langle 0\rangle\right)\right)ds \\ &+2t\mathbb{E}\int_0^t d_\infty^2\left(f\left(s,\langle 0\rangle\right),\langle 0\rangle\right)ds \\ &+8\sum_{k=1}^{\ell}\mathbb{E}\int_0^t\left\|h^k\left(s,x_{n-1}(s)\right)-h^k\left(s,\langle 0\rangle\right)\right\|^2ds \\ &+8\sum_{k=1}^{\ell}\mathbb{E}\int_0^t\left\|h^k\left(s,\langle 0\rangle\right)\right\|^2ds\Bigg]. \end{aligned} \tag{26}$$

By assumptions (A2) and (A3) we obtain

$$\begin{aligned} z_n(t) &\leqslant (\ell+3)\Bigg[\mathbb{E}d_\infty^2\left(x_0,\langle 0\rangle\right) \\ &+4(t+2\ell)\mathbb{E}\int_0^t K(s)ds \\ &+4(t+2\ell)L\mathbb{E}\int_0^t d_\infty^2\left(x_{n-1}(s),\langle 0\rangle\right)ds\Bigg] \leqslant C_1 \\ &+C_2\int_0^t z_{n-1}(s)ds. \end{aligned} \tag{27}$$

Due to the last inequality we can infer that $\max_{1\leqslant n\leqslant k}z_n(t) \leqslant M_1 + M_2\int_0^t\max_{1\leqslant n\leqslant k}z_{n-1}(s)ds$ for $k\in\mathbb{N}$. Hence for $k\in\mathbb{N}$

$$\max_{1\leqslant n\leqslant k}z_n(t) \leqslant C_1 + C_2\widetilde{T}\mathbb{E}d_\infty^2\left(x_0,\langle 0\rangle\right) + C_2\int_0^t\max_{1\leqslant n\leqslant k}z_n(s)ds, \quad t\in\widetilde{I}. \tag{28}$$

Applying Gronwall's inequality we arrive at $\max_{1\leqslant n\leqslant k}z_n(t) \leqslant [C_1 + C_2\widetilde{T}\mathbb{E}d_\infty^2(x_0,\langle 0\rangle)]e^{C_2t}$ for $t\in\widetilde{I}$. This allows us to infer that $z_n(\widetilde{T}) \leqslant (C_1 + C_2\widetilde{T}\mathbb{E}d_\infty^2(x_0,\langle 0\rangle))e^{C_2\widetilde{T}}$. □

In what follows we formulate an existence and uniqueness theorem for solutions to the bipartite fuzzy stochastic differential equations.

Theorem 5. *Assume that conditions (A0)–(A4) are satisfied. Then the bipartite fuzzy stochastic differential equation (18) has a unique local solution.*

Proof. Denote $z_n(t) = \mathbb{E}\sup_{u\in[0,t]} d_\infty^2(x_n(u), x_{n-1}(u))$ for $n \in \mathbb{N}$ and $t \in \widetilde{I}$. Then for $t \in \widetilde{I}$ applying properties (P4), (P8), and (P5) we can write

$$\begin{aligned}
z_1(t) &= \mathbb{E}\sup_{u\in[0,t]} d_\infty^2\Bigg(\Bigg[\left(x_0 + \int_0^u \widetilde{f}(s,x_0)\,ds\right) \\
&\quad \ominus \left((-1)\int_0^u f(s,x_0)\,ds\right)\Bigg] \\
&\quad + \left\langle \sum_{k=1}^{\ell}\int_0^u h^k(s,x_0)\,dW^k(s)\right\rangle, x_0\Bigg) \\
&\leqslant 2\mathbb{E}\sup_{u\in[0,t]} d_\infty^2\Bigg(\left(x_0 + \int_0^u \widetilde{f}(s,x_0)\,ds\right) \\
&\quad \ominus \left((-1)\int_0^u f(s,x_0)\,ds\right), x_0\Bigg) \\
&\quad + 2\mathbb{E}\sup_{u\in[0,t]}\left\|\sum_{k=1}^{\ell}\int_0^u h^k(s,x_0)\,dW^k(s)\right\|^2 \\
&\leqslant 4\mathbb{E}\sup_{u\in[0,t]} d_\infty^2\left(\int_0^u \widetilde{f}(s,x_0)\,ds, \langle 0\rangle\right) \\
&\quad + 4\mathbb{E}\sup_{u\in[0,t]} d_\infty^2\left(\int_0^u f(s,x_0)\,ds, \langle 0\rangle\right) \\
&\quad + 2\ell\sum_{k=1}^{\ell}\mathbb{E}\sup_{u\in[0,t]}\left\|\int_0^u h^k(s,x_0)\,dW^k(s)\right\|^2.
\end{aligned} \tag{29}$$

By Proposition 1, Doob's inequality, and assumptions (A2) and (A3) we obtain

$$\begin{aligned}
z_1(t) &\leqslant 16L\left(t+\ell^2\right)t\mathbb{E}d_\infty^2(x_0,\langle 0\rangle) \\
&\quad + 16\left(t+\ell^2\right)\mathbb{E}\int_0^t K(s)\,ds \leqslant C_3,
\end{aligned} \tag{30}$$

where $C_3 = 16(\widetilde{T}+\ell^2)[L\widetilde{T}\mathbb{E}d_\infty^2(x_0,\langle 0\rangle) + \mathbb{E}\int_{\widetilde{I}} K(s)ds] < \infty$. Moreover for $n \in \mathbb{N}$ we obtain $z_{n+1}(t) \leqslant 8L(\widetilde{T}+\ell^2)\int_0^t z_n(s)ds$. Thus one can infer that for $n \in \mathbb{N}$

$$z_n(t) \leqslant C_3\frac{\left[8L\left(\widetilde{T}+\ell^2\right)t\right]^{n-1}}{(n-1)!}, \quad t \in \widetilde{I}. \tag{31}$$

Using Chebyshev's inequality and (31) we arrive at

$$\begin{aligned}
P\left(\sup_{u\in\widetilde{I}} d_\infty^2(x_n(u), x_{n-1}(u)) > \frac{1}{4^n}\right) &\leqslant 4^n z_n\left(\widetilde{T}\right) \\
&\leqslant 4C_3\frac{\left[32L\left(\widetilde{T}+\ell^2\right)\widetilde{T}\right]^{n-1}}{(n-1)!}.
\end{aligned} \tag{32}$$

Since the series $\sum_{n=1}^{\infty}[32L(\widetilde{T}+\ell^2)\widetilde{T}]^{n-1}/(n-1)!$ is convergent, due to the Borel-Cantelli lemma we obtain

$$\begin{aligned}
&P\left(\sup_{u\in\widetilde{I}} d_\infty(x_n(u), x_{n-1}(u)) > \frac{1}{2^n} \text{ infinitely often}\right) \\
&\quad = 0.
\end{aligned} \tag{33}$$

Now, similarly like in [26], we infer that there exists a d_∞-continuous fuzzy stochastic process $x \in \mathcal{L}^2(\widetilde{I}\times\Omega,\mathcal{N};\mathcal{F}(\mathbb{R}^d))$ such that $\sup_{t\in\widetilde{I}} d_\infty(x_n(t,\omega), x(t,\omega)) \xrightarrow{P.1} 0$, as $n \to \infty$. It can also be verified that $\mathbb{E}d_\infty^2(x_n(t), x(t)) \to 0$, $t \in \widetilde{I}$.

We shall show that x is a solution to (18). Indeed, let us notice that

$$\begin{aligned}
&\mathbb{E}d_\infty^2\Bigg(x_n, \\
&\quad \left[\left(x_0 + \int_0^t \widetilde{f}(s,x(s))\,ds\right) \ominus \left((-1)\int_0^t f(s,x(s))\,ds\right)\right] \\
&\quad + \left\langle \sum_{k=1}^{\ell}\int_0^t h^k(s,x(s))\,dW^k(s)\right\rangle\Bigg) \\
&\quad \leqslant 2\mathbb{E}d_\infty^2\Bigg(\left(x_0 + \int_0^t \widetilde{f}(s,x_{n-1}(s))\,ds\right) \\
&\quad \ominus \left((-1)\int_0^t f(s,x_{n-1}(s))\,ds\right), \left(x_0 + \int_0^t \widetilde{f}(s,x(s))\,ds\right) \\
&\quad \ominus \left((-1)\int_0^t f(s,x(s))\,ds\right)\Bigg) \\
&\quad + 2\mathbb{E}\left\|\sum_{k=1}^{\ell}\int_0^t \left(h^k(s,x_{n-1}(s)) - h^k(s,x(s))\right)dW^k(s)\right\|^2 \\
&\quad \leqslant 4\mathbb{E}d_\infty^2\left(\int_0^t \widetilde{f}(s,x_{n-1}(s))\,ds, \int_0^t \widetilde{f}(s,x(s))\,ds\right) \\
&\quad + 4\mathbb{E}d_\infty^2\left(\int_0^t f(s,x_{n-1}(s))\,ds, \int_0^t f(s,x(s))\,ds\right) \\
&\quad + 2\ell\sum_{k=1}^{\ell}\mathbb{E}\left\|\int_0^t \left(h^k(s,x_{n-1}(s)) - h^k(s,x(s))\right)dW^k(s)\right\|^2.
\end{aligned} \tag{34}$$

By Proposition 1 and Itô's isometry, assumption (A2), and Lebesgue's Dominated Convergence Theorem we get

$$\begin{aligned}
&\mathbb{E}d_\infty^2\Bigg(x_n, \Bigg[\left(x_0 + \int_0^t \widetilde{f}(s,x(s))\,ds\right) \\
&\quad \ominus \left((-1)\int_0^t f(s,x(s))\,ds\right)\Bigg] \\
&\quad + \left\langle \sum_{k=1}^{\ell}\int_0^t h^k(s,x(s))\,dW^k(s)\right\rangle\Bigg) \leqslant 2L\Big(4\widetilde{T} \\
&\quad + \ell^2\Big)\int_{\widetilde{I}} \mathbb{E}d_\infty^2(x_{n-1}(s), x(s))\,ds \longrightarrow 0, \\
&\qquad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{35}$$

Hence for every $t \in \widetilde{I}$

$$\begin{aligned} &\mathbb{E} d_\infty^2 \Bigg(x(t), \Bigg[\Bigg(x_0 + \int_0^t \widetilde{f}(s, x(s))\, ds \Bigg) \\ &\ominus \Bigg((-1) \int_0^t f(s, x(s))\, ds \Bigg) \Bigg] \\ &+ \Bigg\langle \sum_{k=1}^{\ell} \int_0^t h^k(s, x(s))\, dW^k(s) \Bigg\rangle \Bigg) = 0. \end{aligned} \tag{36}$$

Thus we can infer that for every $t \in \widetilde{I}$

$$\begin{aligned} &d_\infty \Bigg(x(t), \Bigg[\Bigg(x_0 + \int_0^t \widetilde{f}(s, x(s))\, ds \Bigg) \\ &\ominus \Bigg((-1) \int_0^t f(s, x(s))\, ds \Bigg) \Bigg] \\ &+ \Bigg\langle \sum_{k=1}^{\ell} \int_0^t h^k(s, x(s))\, dW^k(s) \Bigg\rangle \Bigg) \overset{P.1}{=} 0. \end{aligned} \tag{37}$$

Now, since the processes are d_∞-continuous, we get

$$\begin{aligned} &d_\infty \Bigg(x(t), \Bigg[\Bigg(x_0 + \int_0^t \widetilde{f}(s, x(s))\, ds \Bigg) \\ &\ominus \Bigg((-1) \int_0^t f(s, x(s))\, ds \Bigg) \Bigg] \\ &+ \Bigg\langle \sum_{k=1}^{\ell} \int_0^t h^k(s, x(s))\, dW^k(s) \Bigg\rangle \Bigg) \overset{\widetilde{I}\ P.1}{=} 0. \end{aligned} \tag{38}$$

This shows that $x : \widetilde{I} \times \Omega \to \mathscr{F}(\mathbb{R}^d)$ is a solution (possibly a local solution) to (18).

What is left is to prove that the solution x is unique. Let us assume that $x, y : \widetilde{I} \times \Omega \to \mathscr{F}(\mathbb{R}^d)$ are two solutions to the bipartite fuzzy stochastic differential equation (18). Denote $z(t) = \mathbb{E} \sup_{u \in [0,t]} d_\infty^2(x(u), y(u))$ for $t \in \widetilde{I}$. Let us notice that for every $t \in \widetilde{I}$ we have

$$\begin{aligned} z(t) &\leqslant 4\mathbb{E} d_\infty^2 \Bigg(\int_0^t \widetilde{f}(s, x(s))\, ds, \int_0^t \widetilde{f}(s, y(s))\, ds \Bigg) \\ &+ 4\mathbb{E} d_\infty^2 \Bigg(\int_0^t f(s, x(s))\, ds, \int_0^t f(s, y(s))\, ds \Bigg) \\ &+ 2\ell \sum_{k=1}^{\ell} \mathbb{E} \Bigg\| \int_0^t \left(h^k(s, x(s)) - h^k(s, y(s)) \right) dW^k(s) \Bigg\|^2 \\ &\leqslant 8L \left(\widetilde{T} + \ell^2 \right) \int_0^t j(s)\, ds. \end{aligned} \tag{39}$$

Invoking Gronwall's inequality we get $z(t) \leqslant 0$ for $t \in \widetilde{I}$, which leads to the conclusion that $\sup_{t \in \widetilde{I}} d_\infty(x(t), y(t)) \overset{P.1}{=} 0$. This ends the proof. □

As we mentioned earlier the sequence $\{x_n\}$ can be treated as a sequence of approximate solutions. The next result presents an upper bound for the error of nth approximation x_n.

Proposition 6. *Assume that (A0)–(A4) hold for x_0, f, $\widetilde{f}$, and h^k's. Then, for the approximations x_n defined in (22) and the exact solution x to the bipartite fuzzy stochastic differential equation (18) we have*

$$\begin{aligned} &\mathbb{E} \sup_{t \in \widetilde{I}} d_\infty^2 (x_n(t), x(t)) \\ &\leqslant \frac{2C_3 \left[8L \left(\widetilde{T} + \ell^2 \right) L\widetilde{T} \right]^n}{n!} e^{16(\widetilde{T} + \ell^2) L \widetilde{T}}, \end{aligned} \tag{40}$$

where C_3 is like in (31).

Proof. Denote $z(t) = \mathbb{E} \sup_{u \in [0,t]} d_\infty^2(x_n(u), x(u))$ for $t \in \widetilde{I}$. Notice that

$$\begin{aligned} z(t) &\leqslant 2\mathbb{E} \sup_{u \in [0,t]} d_\infty^2 \Bigg(\Bigg(x_0 + \int_0^u \widetilde{f}(s, x_{n-1}(s))\, ds \Bigg) \\ &\ominus \Bigg((-1) \int_0^u f(s, x_{n-1}(s))\, ds \Bigg), \Bigg(x_0 + \int_0^u \widetilde{f}(s, x(s))\, ds \Bigg) \\ &\ominus \Bigg((-1) \int_0^u f(s, x(s))\, ds \Bigg) \Bigg) \\ &+ 2\mathbb{E} \sup_{u \in [0,t]} \Bigg\| \sum_{k=1}^{\ell} \int_0^u \left(h^k(s, x_{n-1}(s)) - h^k(s, x(s)) \right) dW^k(s) \Bigg\|^2 \\ &\leqslant 4t\mathbb{E} \int_0^t d_\infty^2 \left(\widetilde{f}(s, x_{n-1}(s)), \widetilde{f}(s, x(s)) \right) ds \\ &+ 4t\mathbb{E} \int_0^t d_\infty^2 \left(f(s, x_{n-1}(s)), f(s, x(s)) \right) ds \\ &+ 8\ell \sum_{k=1}^{\ell} \mathbb{E} \int_0^t \left\| h^k(s, x_{n-1}(s)) - h^k(s, x(s)) \right\|^2 ds \leqslant 8L \Big(t \\ &+ \ell^2 \Big) \int_0^t \mathbb{E} \sup_{u \in [0,s]} d_\infty^2 (x_{n-1}(u), x(u))\, ds \leqslant 16L \left(t + \ell^2 \right) \\ &\cdot \int_0^t \mathbb{E} \sup_{u \in [0,s]} d_\infty^2 (x_{n-1}(u), x_n(u))\, ds + 16L \left(t + \ell^2 \right) \\ &\cdot \int_0^t \mathbb{E} \sup_{u \in [0,s]} d_\infty^2 (x_n(u), x(u))\, ds. \end{aligned} \tag{41}$$

Applying (31) we arrive at

$$\begin{aligned} z(t) &\leqslant \frac{2C_3 \left[8L \left(\widetilde{T} + \ell^2 \right) \widetilde{T} \right]^n}{n!} \\ &+ 16L \left(\widetilde{T} + \ell^2 \right) \int_0^t z(s)\, ds. \end{aligned} \tag{42}$$

Now invoking Gronwall's inequality we can write

$$z(t) \leqslant \frac{2C_3 \left[8L \left(\widetilde{T} + \ell^2 \right) \widetilde{T} \right]^n}{n!} e^{16L(\widetilde{T} + \ell^2) t} \quad \text{for every } t \in \widetilde{I}. \tag{43}$$

This leads us to the inequality $z(\widetilde{T}) \leqslant (2C_3[8L(\widetilde{T} + \ell^2)\widetilde{T}]^n / n!) e^{16L(\widetilde{T} + \ell^2)\widetilde{T}}$. □

As an immediate consequence of the assertion presented above, we have the following property $\mathbb{E}\sup_{t\in\widetilde{I}} d_\infty^2(x_n(t), x(t)) \to 0$ as $n\to\infty$, which, together with Proposition 4, allows us to find a bound for the expression $\mathbb{E}\sup_{t\in\widetilde{I}} d_\infty^2(x(t), \langle 0\rangle)$. Another estimation is contained in the next claim.

Proposition 7. *Assume that conditions (A0)–(A4) are satisfied for x_0, f, $\widetilde{f}$, and h^k's. Let $x : \widetilde{I}\times\Omega \to \mathscr{F}(\mathbb{R}^d)$ denote unique (possibly local) solution to (18). Then it holds*

$$\begin{aligned}
&\mathbb{E}\sup_{t\in\widetilde{I}} d_\infty^2(x(t),\langle 0\rangle)\\
&\quad\leqslant 8\left[\mathbb{E}d_\infty^2(x_0,\langle 0\rangle) + \left(3\widetilde{T}+2\ell^2\right)\mathbb{E}\int_{\widetilde{I}} K(s)\,ds\right]\\
&\quad\cdot e^{8L(3\widetilde{T}+2\ell^2)\widetilde{T}}.
\end{aligned} \tag{44}$$

Proof. Denote $z(t) = \mathbb{E}\sup_{u\in[0,t]} d_\infty^2(x(u),\langle 0\rangle)$ for $t\in\widetilde{I}$. Observe that

$$\begin{aligned}
z(t) &= \mathbb{E}\sup_{u\in[0,t]} d_\infty^2\Bigg(\Bigg[\left(x_0+\int_0^u \widetilde{f}(s,x(s))\,ds\right)\\
&\quad\ominus\left((-1)\int_0^u f(s,x(s))\,ds\right)\Bigg]\\
&\quad+\left\langle\sum_{k=1}^{\ell}\int_0^u h^k(s,x(s))\,dW^k(s)\right\rangle,\langle 0\rangle\Bigg)\\
&\leqslant 2\mathbb{E}\sup_{u\in[0,t]} d_\infty^2\Bigg(\Bigg[\left(x_0+\int_0^u \widetilde{f}(s,x(s))\,ds\right)\\
&\quad\ominus\left((-1)\int_0^u f(s,x(s))\,ds\right)\Bigg],\langle 0\rangle\Bigg)\\
&\quad+2\mathbb{E}\sup_{u\in[0,t]}\left\|\sum_{k=1}^{\ell}\int_0^u h^k(s,x(s))\,dW^k(s)\right\|^2\\
&\leqslant 8\mathbb{E}d_\infty^2(x_0,\langle 0\rangle)\\
&\quad+8t\mathbb{E}\int_0^t d_\infty^2\left(\widetilde{f}(s,x(s)),\langle 0\rangle\right)ds\\
&\quad+4t\mathbb{E}\int_0^t d_\infty^2\left(f(s,x(s)),\langle 0\rangle\right)ds\\
&\quad+8\ell\sum_{k=1}^{\ell}\mathbb{E}\int_0^t\left\|h^k(s,x(s))\right\|^2 ds.
\end{aligned} \tag{45}$$

Further it can be verified that

$$\begin{aligned}
z(t) &\leqslant 8\mathbb{E}d_\infty^2(x_0,\langle 0\rangle) + \left(24\widetilde{T}+16\ell^2\right)\mathbb{E}\int_{\widetilde{I}} K(s)\,ds\\
&\quad+\left(24\widetilde{T}+16\ell^2\right)L\int_0^t z(s)\,ds.
\end{aligned} \tag{46}$$

Hence, by Gronwall's inequality we can infer that

$$\begin{aligned}
&z(\widetilde{T})\\
&\quad\leqslant\left[8\mathbb{E}d_\infty^2(x_0,\langle 0\rangle)+\left(24\widetilde{T}+16\ell^2\right)\mathbb{E}\int_{\widetilde{I}} K(s)\,ds\right]\\
&\quad\cdot e^{(24\widetilde{T}+16\ell^2)L\widetilde{T}}.
\end{aligned} \tag{47}$$

□

The next part of this section is focused on well-posedness of the theory of bipartite fuzzy stochastic differential equations. We shall prove that the solutions are insensitive with respect to small changes of the equation's data. We start with insensitivity with respect to initial value x_0.

Let $x, y : \widetilde{I}\times\Omega \to \mathscr{F}(\mathbb{R}^d)$ denote solutions to the bipartite fuzzy stochastic differential equation (18) and

$$\begin{aligned}
y(t) &\overset{I\ P.1}{=} \left[\left(y_0+\int_0^t \widetilde{f}(s,y(s))\,ds\right)\right.\\
&\quad\left.\ominus\left((-1)\int_0^t f(s,y(s))\,ds\right)\right]\\
&\quad+\left\langle\sum_{k=1}^{\ell}\int_0^t h^k(s,y(s))\,dW^k(s)\right\rangle,
\end{aligned} \tag{48}$$

respectively. The initial values x_0, y_0 can differ on a set of positive probability and the remaining data are the same.

Theorem 8. *Assume that x_0, y_0 satisfy condition (A0) and f, $\widetilde{f}$, h^k, $k=1,2,\ldots,\ell$, satisfy (A1)–(A3). Suppose that x_0, f, $\widetilde{f}$, h^k's satisfy (A4) and y_0, f, $\widetilde{f}$, h^k's fulfill (A4) and this happens for a common $\widetilde{T}\in(0,T]$. Then, for the unique local (or global) solutions $x, y : \widetilde{I}\times\Omega \to \mathscr{F}(\mathbb{R}^d)$ to the bipartite fuzzy stochastic differential equations (18) and (48) we have*

$$\mathbb{E}\sup_{t\in\widetilde{I}} d_\infty^2(x(t),y(t)) \leqslant 8\mathbb{E}d_\infty^2(x_0,y_0)\,e^{4L(3\widetilde{T}+2\ell^2)L\widetilde{T}}. \tag{49}$$

Proof. The unique solutions x, y to (18) and (48) exist owing to Theorem 5. For $t\in\widetilde{I}$ one gets

$$\begin{aligned}
&\mathbb{E}\sup_{u\in[0,t]} d_\infty^2(x(u),y(u)) \leqslant 2\mathbb{E}\sup_{u\in[0,t]} d_\infty^2\Bigg(\Bigg(x_0\\
&\quad+\int_0^u \widetilde{f}(s,x(s))\,ds\Bigg)\ominus\left((-1)\int_0^u f(s,x(s))\,ds\right),\Bigg(y_0\\
&\quad+\int_0^u \widetilde{f}(s,y(s))\,ds\Bigg)\ominus\left((-1)\int_0^u f(s,y(s))\,ds\right)\Bigg)\\
&\quad+2\mathbb{E}\sup_{u\in[0,t]}\left\|\sum_{k=1}^{\ell}\int_0^u\left(h^k(s,x(s))-h^k(s,y(s))\right)dW^k(s)\right\|^2\\
&\quad\leqslant 8\mathbb{E}d_\infty^2(x_0,y_0)\\
&\quad+8t\mathbb{E}\int_0^t d_\infty^2\left(\widetilde{f}(s,x(s)),\widetilde{f}(s,y(s))\right)ds\\
&\quad+4t\mathbb{E}\int_0^t d_\infty^2\left(f(s,x(s)),f(s,y(s))\right)ds\\
&\quad+8\ell\sum_{k=1}^{\ell}\mathbb{E}\int_0^t\left\|h^k(s,x(s))-h^k(s,y(s))\right\|^2 ds\\
&\quad\leqslant 8\mathbb{E}d_\infty^2(x_0,y_0)+4L\left(3\widetilde{T}+2\ell^2\right)\\
&\quad\cdot\int_0^t\mathbb{E}\sup_{u\in[0,s]} d_\infty^2(x(u),y(u))\,ds.
\end{aligned} \tag{50}$$

Invoking Gronwall's inequality we infer that

$$\mathbb{E}\sup_{u\in[0,t]} d_{\infty}^{2}\left(x\left(u\right),y\left(u\right)\right)\leqslant 8\mathbb{E}d_{\infty}^{2}\left(x_{0},y_{0}\right)e^{4L(3\widetilde{T}+2\ell^{2})t} \tag{51}$$

for every $t\in[0,\widetilde{T}]$. □

By this theorem one can infer that the solutions to (18) and (48) are close to each other provided that $\mathbb{E}d_{\infty}^{2}(x_{0},y_{0})$ is small. As an immediate consequence of that, one can state that solution to (18) depends continuously on x_0. Indeed, consider (18) and

$$\begin{aligned} y_{n}\left(t\right) &\overset{I\ P.1}{=} \left[\left(y_{0}^{(n)}+\int_{0}^{t}\widetilde{f}\left(s,y_{n}\left(s\right)\right)ds\right)\right.\\ &\quad\left.\ominus\left(\left(-1\right)\int_{0}^{t}f\left(s,y_{n}\left(s\right)\right)ds\right)\right]\\ &\quad+\left\langle\sum_{k=1}^{\ell}\int_{0}^{t}h^{k}\left(s,y_{n}\left(s\right)\right)dW^{k}\left(s\right)\right\rangle,\quad n\in\mathbb{N}. \end{aligned} \tag{52}$$

Let x, y_n denote solutions to (18) and (52), respectively.

Corollary 9. *Assume that x_0, $y_0^{(n)}$ ($n\in\mathbb{N}$) satisfy (A0), and f, $\widetilde{f}$, h^k, $k=1,2,\ldots,\ell$, satisfy (A1)–(A3). Suppose that x_0, f, $\widetilde{f}$, h^k's satisfy (A4) and $y_0^{(n)}$, f, $\widetilde{f}$, h^k's fulfill (A4) for each n and this happens for a common $\widetilde{T}\in(0,T]$. Suppose that*

$$\mathbb{E}d_{\infty}^{2}\left(y_{0}^{(n)},x_{0}\right)\longrightarrow 0\quad \text{as } n\longrightarrow\infty. \tag{53}$$

Then for the unique solution $x:\widetilde{I}\times\Omega\to\mathcal{F}(\mathbb{R}^{d})$ to (18) and the unique solutions $y_n:\widetilde{I}\times\Omega\to\mathcal{F}(\mathbb{R}^{d})$ to the bipartite fuzzy stochastic differential equations we have

$$\mathbb{E}\sup_{t\in\widetilde{I}} d_{\infty}^{2}\left(y_{n}\left(t\right),x\left(t\right)\right)\longrightarrow 0\quad \text{as } n\longrightarrow\infty. \tag{54}$$

Now, we consider (18) and

$$\begin{aligned} y_{n}\left(t\right) &\overset{I\ P.1}{=} \left[\left(x_{0}+\int_{0}^{t}\widetilde{f}_{n}\left(s,y_{n}\left(s\right)\right)ds\right)\right.\\ &\quad\left.\ominus\left(\left(-1\right)\int_{0}^{t}f_{n}\left(s,y_{n}\left(s\right)\right)ds\right)\right]\\ &\quad+\left\langle\sum_{k=1}^{\ell}\int_{0}^{t}h_{n}^{k}\left(s,y_{n}\left(s\right)\right)dW^{k}\left(s\right)\right\rangle,\quad n\in\mathbb{N} \end{aligned} \tag{55}$$

to investigate continuous dependence of solution to (18) with respect to coefficients f, $\widetilde{f}$, and h^k's. Let x, y_n denote solutions to (18) and (55), respectively.

Theorem 10. *Assume that x_0 satisfy (A0) and f, f_n, $\widetilde{f}$, $\widetilde{f}_n$, h^k, h_n^k ($n\in\mathbb{N}$, $k=1,2,\ldots,\ell$) satisfy (A1)–(A3) with the same constant L and function K. Assume that condition (A4) is satisfied with the same constant $\widetilde{T}\in(0,T]$ for x_0, f, $\widetilde{f}$, h^k's and for x_0, f_n, $\widetilde{f}_n$, h_n^k's ($n\in\mathbb{N}$). Suppose that for every $u\in\mathcal{F}(\mathbb{R}^{d})$ it holds*

$$\begin{aligned} &\mathbb{E}\int_{\widetilde{I}} d_{\infty}^{2}\left(f_{n}\left(t,u\right),f\left(t,u\right)\right)dt\longrightarrow 0\quad \text{as } n\longrightarrow\infty,\\ &\mathbb{E}\int_{\widetilde{I}} d_{\infty}^{2}\left(\widetilde{f}_{n}\left(t,u\right),\widetilde{f}\left(t,u\right)\right)dt\longrightarrow 0\quad \text{as } n\longrightarrow\infty,\\ &\mathbb{E}\int_{\widetilde{I}}\left\|h_{n}^{k}\left(t,u\right)-h^{k}\left(t,u\right)\right\|^{2}dt\longrightarrow 0\\ &\qquad\qquad \text{as } n\longrightarrow\infty,\ k=1,2,\ldots,\ell. \end{aligned} \tag{56}$$

Then, for the unique solution $x:\widetilde{I}\times\Omega\to\mathcal{F}(\mathbb{R}^{d})$ to (18) and the unique solutions y_n to (55) we have

$$\mathbb{E}\sup_{t\in\widetilde{I}} d_{\infty}^{2}\left(y_{n}\left(t\right),x\left(t\right)\right)\longrightarrow 0\quad \text{as } n\longrightarrow\infty. \tag{57}$$

Proof. Owing to Theorem 5 the solutions x to (18) and y_n to (55) exist on $\widetilde{I}$ and are unique. For $t\in\widetilde{I}$ we have

$$\begin{aligned} &\mathbb{E}\sup_{u\in[0,t]} d_{\infty}^{2}\left(y_{n}\left(u\right),x\left(u\right)\right)\\ &\leqslant 4t\mathbb{E}\int_{0}^{t}d_{\infty}^{2}\left(\widetilde{f}_{n}\left(s,y_{n}\left(s\right)\right),\widetilde{f}\left(s,x\left(s\right)\right)\right)ds\\ &\quad+4t\mathbb{E}\int_{0}^{t}d_{\infty}^{2}\left(f_{n}\left(s,y_{n}\left(s\right)\right),f\left(s,x\left(s\right)\right)\right)ds\\ &\quad+8\ell\sum_{k=1}^{\ell}\mathbb{E}\int_{0}^{t}\left\|h_{n}^{k}\left(s,y_{n}\left(s\right)\right)-h^{k}\left(s,x\left(s\right)\right)\right\|^{2}ds\\ &\leqslant 8t\mathbb{E}\int_{0}^{t}d_{\infty}^{2}\left(\widetilde{f}_{n}\left(s,y_{n}\left(s\right)\right),\widetilde{f}_{n}\left(s,x\left(s\right)\right)\right)ds\\ &\quad+8t\mathbb{E}\int_{0}^{t}d_{\infty}^{2}\left(\widetilde{f}_{n}\left(s,x\left(s\right)\right),\widetilde{f}\left(s,x\left(s\right)\right)\right)ds\\ &\quad+8t\mathbb{E}\int_{0}^{t}d_{\infty}^{2}\left(f_{n}\left(s,y_{n}\left(s\right)\right),f_{n}\left(s,x\left(s\right)\right)\right)ds\\ &\quad+8t\mathbb{E}\int_{0}^{t}d_{\infty}^{2}\left(f_{n}\left(s,x\left(s\right)\right),f\left(s,x\left(s\right)\right)\right)ds\\ &\quad+16\ell\sum_{k=1}^{\ell}\mathbb{E}\int_{0}^{t}\left\|h_{n}^{k}\left(s,y_{n}\left(s\right)\right)-h_{n}^{k}\left(s,x\left(s\right)\right)\right\|^{2}ds\\ &\quad+16\ell\sum_{k=1}^{\ell}\mathbb{E}\int_{0}^{t}\left\|h_{n}^{k}\left(s,x\left(s\right)\right)-h^{k}\left(s,x\left(s\right)\right)\right\|^{2}ds\\ &\leqslant A_{n}\\ &\quad+16L\left(\widetilde{T}+\ell^{2}\right)\int_{0}^{t}\mathbb{E}\sup_{s\in[0,u]} d_{\infty}^{2}\left(y_{n}\left(u\right),x\left(u\right)\right)ds, \end{aligned} \tag{58}$$

where

$$A_n = 8\widetilde{T}\mathbb{E}\int_{\widetilde{I}} d_\infty^2\left(\widetilde{f}_n(s,x(s)),\widetilde{f}(s,x(s))\right)ds + 8\widetilde{T}\mathbb{E}\int_{\widetilde{I}} d_\infty^2\left(f_n(s,x(s)),f(s,x(s))\right)ds + 16\ell\sum_{k=1}^{\ell}\mathbb{E}\int_{\widetilde{I}}\left\|h_n^k(s,x(s))-h^k(s,x(s))\right\|^2 ds. \tag{59}$$

Invoking Gronwall's inequality we obtain

$$\mathbb{E}\sup_{u\in[0,t]} d_\infty^2\left(y_n(u),x(u)\right) \leqslant A_n e^{16L(\widetilde{T}+\ell^2)t} \quad \text{for every } t\in\widetilde{I}. \tag{60}$$

The thesis follows by assumptions (56). □

4. Application to Bipartite Set-Valued Stochastic Differential Equations

In this part of the paper we present some results concerning bipartite set-valued stochastic differential equations. We do this because set-valued analysis constitutes a branch of research also in context of set-valued differential equations [36]. We discuss only main issues without including proofs. This is because the results presented here are parallel to those established in Section 3 for bipartite fuzzy stochastic differential equations. All the inference methods are similar to those contained in preceding section.

By the bipartite set-valued stochastic differential equations written in their integral form we mean the following equations:

$$X(t) \overset{I\ P.1}{=} \left[\left(X_0+\int_0^t \widetilde{F}(s,X(s))\,ds\right) \ominus \left((-1)\int_0^t F(s,X(s))\,ds\right)\right] + \left\{\sum_{k=1}^{\ell}\int_0^t H^k(s,X(s))\,dW^k(s)\right\}, \tag{61}$$

where X_0 is a set-valued random variable, $\ell\in\mathbb{N}$, $H^1,H^2,\ldots,H^\ell : I\times\Omega\times\mathscr{K}(\mathbb{R}^d)\to\mathbb{R}^d$, and $W^1,W^2,\ldots,W^\ell$ are the independent one-dimensional $\{\mathscr{A}_t\}_{t\in I}$-Brownian motions. The first and the second integral in (61) are the set-valued stochastic Lebesgue integral, while the next integrals are the $\mathbb{R}^d$-valued stochastic Itô integrals.

Denote $\widetilde{T}\in(0,T]$, $\widetilde{I}=[0,\widetilde{T}]$.

Definition 11. By a local solution (in the case $\widetilde{T}<T$) to (61) we mean a set-valued stochastic process $X:\widetilde{I}\times\Omega\to\mathscr{K}(\mathbb{R}^d)$ satisfying the following: (i) $X\in\mathscr{L}^2(\widetilde{I}\times\Omega,\mathscr{N};\mathscr{K}(\mathbb{R}^d))$, (ii) X is d_H-continuous, and (iii) it holds

$$X(t) \overset{\widetilde{I}\ P.1}{=} \left[\left(X_0+\int_0^t \widetilde{F}(s,X(s))\,ds\right) \ominus \left((-1)\int_0^t F(s,X(s))\,ds\right)\right] + \left\{\sum_{k=1}^{\ell}\int_0^t H^k(s,X(s))\,dW^k(s)\right\}. \tag{62}$$

If $\widetilde{T}=T$, then X is said to be a global solution to (61). A (local or global) solution $X:\widetilde{I}\times\Omega\to\mathscr{K}(\mathbb{R}^d)$ to (61) is unique iff $X(t)\overset{\widetilde{I}\ P.1}{=}Y(t)$, where $Y:\widetilde{I}\times\Omega\to\mathscr{K}(\mathbb{R}^d)$ is any other solution to (61).

We shall state an existence and uniqueness theorem for solutions to (61) under the following conditions:

(S0) $X_0\in\mathscr{L}^2(\Omega,\mathscr{A}_0,P;\mathscr{K}(\mathbb{R}^d))$,

(S1) the mappings $F,\widetilde{F}:(I\times\Omega)\times\mathscr{K}(\mathbb{R}^d)\to\mathscr{K}(\mathbb{R}^d)$ are $\mathscr{N}\otimes\mathscr{B}_{d_H}\mid\mathscr{B}_{d_H}$-measurable and $H^1,H^2,\ldots,H^\ell:(I\times\Omega)\times\mathscr{K}(\mathbb{R}^d)\to\mathbb{R}^d$ are $\mathscr{N}\otimes\mathscr{B}_{d_H}\mid\mathscr{B}(\mathbb{R}^d)$-measurable,

(S2) there exists a constant $L>0$ such that for $\gamma\times P$-a.a. (t,ω) and for every $A,B\in\mathscr{K}(\mathbb{R}^d)$ it holds

$$\max\left\{d_H^2(F(t,\omega,A),F(t,\omega,B)),\ d_H^2\left(\widetilde{F}(t,\omega,A),\widetilde{F}(t,\omega,B)\right),\ \left\|H^k(t,\omega,A)-H^k(t,\omega,B)\right\|^2\right\} \leqslant L d_H^2(A,B), \quad k=1,2,\ldots,\ell, \tag{63}$$

(S3) there exists $K\in L^1(I\times\Omega,\mathscr{B}(I)\otimes\mathscr{A},\gamma\times P;\mathbb{R})$ such that for $\gamma\times P$-a.a. (t,ω) it holds

$$\max\left\{d_H^2(F(t,\omega,\{0\}),\{0\}),\ d_H^2\left(\widetilde{F}(t,\omega,\{0\}),\{0\}\right),\ \left\|H^k(t,\omega,\{0\})\right\|^2\right\} \leqslant K(t,\omega), \quad k=1,2,\ldots,\ell, \tag{64}$$

(S4) there exists a constant $\widetilde{T}\in(0,T]$ such that the sequence $\{X_n\}_{n=0}^\infty$ of the set-valued mappings $X_n:\widetilde{I}\times\Omega\to\mathscr{K}(\mathbb{R}^d)$ described as

$$X_0(t)\overset{\widetilde{I}\ P.1}{:=}X_0, \tag{65}$$

and for $n=1,2,\ldots$

$$X_n(t) \overset{\widetilde{I}\ P.1}{=} \left[\left(X_0+\int_0^t \widetilde{F}(s,X_{n-1}(s))\,ds\right) \ominus \left((-1)\int_0^t F(s,X_{n-1}(s))\,ds\right)\right] + \left\{\sum_{k=1}^{\ell}\int_0^t H^k(s,X_{n-1}(s))\,dW^k(s)\right\}, \tag{66}$$

is well defined.

Using the sequence $\{X_n\}$ defined in (S4) and proceeding similarly like in the proof of Theorem 5 we are able to derive the following result.

Corollary 12. *Assume that X_0, F, $\widetilde{F}$, and H^k's satisfy conditions (S0)–(S4). Then the bipartite set-valued stochastic differential equation (61) has a unique (possibly local) solution.*

The theory of bipartite set-valued stochastic differential equations is well-posed. Below, by stating two corollaries, we indicate that the set-valued solution to (61) possesses properties of continuous dependence on initial set-valued random variable and coefficients F, $\widetilde{F}$, H^k's.

Corollary 13. *Assume that X_0, $Y_0^{(n)}$ $(n \in \mathbb{N})$ satisfy (S0) and $F, \widetilde{F}, H^k$ $(k = 1, 2, \ldots, m)$ satisfy conditions (S1)–(S3). Assume that X_0, F, $\widetilde{F}$, H^k's satisfy (S4) and $Y_0^{(n)}$, F, $\widetilde{F}$, H^k's satisfy (S4) as well and this happens for a common $\widetilde{T} \in (0, T]$. Suppose that*

$$\mathbb{E} d_H^2 \left(Y_0^{(n)}, X_0\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{67}$$

Then for the unique solution $X : \widetilde{I} \times \Omega \to \mathcal{K}(\mathbb{R}^d)$ to (61) and the unique solutions $Y_n : \widetilde{I} \times \Omega \to \mathcal{F}(\mathbb{R}^d)$ to the bipartite set-valued stochastic differential equations

$$\begin{aligned} Y_n(t) &\overset{I\ P.1}{=} \left[\left(Y_0^{(n)} + \int_0^t \widetilde{F}(s, Y_n(s))\, ds\right)\right. \\ &\quad \left.\ominus \left((-1)\int_0^t F(s, Y_n(s))\, ds\right)\right] \\ &\quad + \left\{\sum_{k=1}^{\ell} \int_0^t H^k(s, Y_n(s))\, dW^k(s)\right\}, \quad n \in \mathbb{N}, \end{aligned} \tag{68}$$

it holds

$$\mathbb{E} \sup_{t \in \widetilde{I}} d_H^2 (Y_n(t), X(t)) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{69}$$

Denote by $X, Y_n : \widetilde{I} \times \Omega \to \mathcal{K}(\mathbb{R}^d)$ the solutions to bipartite set-valued stochastic differential equations (61) and

$$\begin{aligned} Y_n(t) &\overset{I\ P.1}{=} \left[\left(X_0 + \int_0^t \widetilde{F}_n(s, Y_n(s))\, ds\right)\right. \\ &\quad \left.\ominus \left((-1)\int_0^t F_n(s, Y_n(s))\, ds\right)\right] \\ &\quad + \left\{\sum_{k=1}^{\ell} \int_0^t H_n^k(s, Y_n(s))\, dW^k(s)\right\}, \quad n \in \mathbb{N}, \end{aligned} \tag{70}$$

respectively.

Corollary 14. *Assume that X_0 satisfies (S0) and F, F_n, $\widetilde{F}$, $\widetilde{F}_n$, H^k, H_n^k $(n \in \mathbb{N},\ k = 1, 2, \ldots, \ell)$ satisfy (S1)–(S3) with the same constant L and the same function K. Assume that condition (S4) is satisfied with the same constant $\widetilde{T} \in (0, T]$ for X_0, F, $\widetilde{F}$, H^k's and for X_0, F_n, $\widetilde{F}_n$, H_n^k's $(n \in \mathbb{N})$. Suppose that for every $A \in \mathcal{K}(\mathbb{R}^d)$ it holds*

$$\begin{aligned} &\mathbb{E} \int_{\widetilde{I}} d_H^2 (F_n(t, A), F(t, A))\, dt \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \\ &\mathbb{E} \int_{\widetilde{I}} d_H^2 \left(\widetilde{F}_n(t, A), \widetilde{F}(t, A)\right) dt \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \\ &\mathbb{E} \int_{\widetilde{I}} \left\| H_n^k(t, A) - H^k(t, A)\right\|^2 dt \longrightarrow 0 \\ &\qquad\qquad \text{as } n \longrightarrow \infty,\ k = 1, 2, \ldots, \ell. \end{aligned} \tag{71}$$

Then, for the unique solution $X : \widetilde{I} \times \Omega \to \mathcal{K}(\mathbb{R}^d)$ to (61) and the unique solutions Y_n to (70) we have

$$\mathbb{E} \sup_{t \in \widetilde{I}} d_H^2 (Y_n(t), X(t)) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{72}$$

Let us only mention that besides properties mentioned above the other properties like boundedness of nth approximation and boundedness of set-valued solution can also be stated.

5. Concluding Remarks

The paper introduces very first study on so-called *bipartite fuzzy stochastic differential equations*. Solutions of equations considered previously (cf. [26–30]) had a property that their trajectory values (the values are fuzzy sets) had either nondecreasing or nonincreasing diameter in time. Now, owing to new equations examined in this paper, we open a way to consider fuzzy stochastic differential equations with solutions that have trajectories of nonmonotone diameter of their values. Since seeking explicit solutions to such the equations is mostly without success, we provide a study on existence of a unique solution. This is achieved under conditions of Lipschitz coefficients of drift and diffusion. Then we indicate that solution is bounded and insensitive under small changes of coefficients and initial value. This confirms that the theory of the new equations investigated in this paper is well-posed. Finally, we show that all results achieved can be easily applied to bipartite set-valued stochastic differential equations.

The current study can be a starting point for some future investigations. For instance, from now on it is possible to speak on periodic diameter of solutions for fuzzy stochastic differential equations. Hence, a study in this direction would be interesting. Moreover, one can try to use some weaker assumptions (than Lipschitz conditions) imposed on coefficients to get existence of a unique solution; different kinds of stabilities of solutions are also of interest.

Competing Interests

The author declares that he has no competing interests.

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