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Research Article

The Numerical Analysis of Two-Sided Space-Fractional Wave Equation with Improved Moving Least-Square Ritz Method

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A numerical analysis of the space-fractional wave equation is carried out by the improved moving least-square Ritz (IMLS-Ritz) method. The trial functions for the space-fractional wave equation are constructed by the IMLS approximation. By the Galerkin weak form, the energy functional is formulated. Employing the Ritz minimization procedure, the final algebraic equations system is obtained. In this numerical analysis, the applicability and efficiency of the IMLS-Ritz method are examined by some example problems. Comparing the numerical results with the analytical solutions, the stability and accuracy of the IMLS-Ritz method are also presented.

1. Introduction

Due to extensive use in the fields of dynamics [1], fluid mechanics [2], viscoelasticity [3], materials [4], hydrology [5], biology [6, 7], porous media [8], physics [9, 10], engineering [11, 12], and so on, fractional partial differential equations have become a hot research topic. Consequently, scholars pay much attention to the analytical solutions of fractional partial differential equation (PDE) of physical interest. Unfortunately, a fractional PDE has no exact solution in many cases owing to complex series or special functions. So it is extremely important and necessary to resort to numerical solutions.

The fractional diffusion-wave equation has been an interesting topic to invest in during the past decades. There is already some important progress for the fractional diffusion equation or advection-diffusion equations. Deng [13–15] presented the numerical method for fractional diffusion equations, Liu et al. [16] used the difference method for space-time fractional equation and presented the stability and convergence, Meerschaert and Tadjeran [17] applied the finite difference approximation for space-fractional equations, Meng [18] put forward a new approach for solving fractional partial differential equations, Sousa [19] developed numerical approximations for fractional diffusion equations via splines, Zhou and Wu [20] proposed the finite element multigrid method for the boundary value problem of fractional advection dispersion equation. Nigmatullin [21] presented the fractional diffusion equation to describe the diffusion in porous media. Mainardi [22, 23] has shown that the FWE describes the propagation of mechanical diffusive waves in viscoelastic media. Much study has been done for the time fractional PDE in [24–26]. In fact, most of the works focus on the time fractional PDE. In another latest paper, Sweilam et al. considered a 1D fractional wave equation in [27], Deng et al. used the alternating direction implicit algorithm for the space-fractional equation in [28], Jia and Wang used the fast finite difference methods for space-fractional PDE with fractional derivative boundary conditions in [29], and Guan and Gunzburger applied the finite element method (FEM) for the space-time fractional PDE in [30].

In the past few decades, meshless methods have already been a hot research topic in computational mechanics. Meshless methods also become important and powerful tools to research and analyze kinds of PDE. Researchers have presented many kinds of meshless methods, including the diffuse element method (DEM) [31], the smoothed particle hydrodynamics (SPH) method [32], the reproducing kernel particle method (RKPM) [33], the element-free Galerkin (EFG) method [34–37], the meshless local Petrov-Galerkin method (MLPG) [38–40], the hp-meshless cloud method [41], boundary element-free method (BEFM) [42, 43], the complex variable meshless method [44, 45], radius basis functions (RBF) [46], and LBIE method [47]. Now, meshless methods are widely used in various fields.

The MLS approximation originated from data fitting [48]. In recent years, the MLS technique is often used for analysis of solid mechanics problems in the meshless method or EFG method [48]. Unfortunately, the final algebraic equations system obtained by EFG method may be ill-conditioned. Thus, the ill-conditioned algebraic equations system should be considered in the MLS approximation. Unless we solve the equation, it is hard to confirm whether the algebraic equations system is ill-conditioned or not. Therefore, it could lead to poor or erroneous numerical results. In order to overcome this problem, Cheng and Chen proposed the IMLS approximation [49]. In the IMLS approximation, the orthogonal function system is chosen as the basis function, and the resulting algebraic equation system is not ill-conditioned any more. Based on the IMLS approximation, the boundary element-free method is presented for elasticity, fracture, elastodynamics, and potential problems [50–53]. The IEFG method is a combination of IMLS approximation and EFG method [54–57]. The IEFG method needs fewer nodes than the conventional EFG method. Hence, the IEFG method is bounded to increase the computational speed and has higher accuracy than the EFG method.

As far as is known, the space-fractional PDE has never been analyzed and researched by the IMLS-Ritz method. We try to consider the one-dimensional two-sided space-fractional equation with left and right Riemann-Liouville fractional derivatives. The IMLS-Ritz method for the two-sided space-fractional wave equation is put forward. In this paper, the IMLS approximation is used to approximate displacement field, the penalty method is applied to impose the boundary conditions, and Ritz minimization procedure is used to obtain the final algebraic equation system. In order to verify the validity and stability of the proposed method, numerical examples are presented compared with existing results available in extant literature.

2. IMLS Shape Functions

The local approximation is defined in the IMLS approximation [49] as follows:

$$u^h(\mathbf{x}, \overline{\mathbf{x}}) = \sum_{j=1}^{m} p_j(\overline{\mathbf{x}})\, a_j(\mathbf{x}) \equiv \mathbf{p}^{\mathrm{T}}(\overline{\mathbf{x}})\, \mathbf{a}(\mathbf{x}), \tag{1}$$

where m is the number of bases, $p_j(\overline{\mathbf{x}})$ are monomial basis functions, and $a_j(\mathbf{x})$ are corresponding coefficients. We can define the following quadratic form:

$$J = \sum_{i=1}^{n} w(\mathbf{x} - \mathbf{x}_i)\left[u^h(\mathbf{x}, \mathbf{x}_i) - u(\mathbf{x}_i)\right]^2, \tag{2}$$

where $w(\mathbf{x} - \mathbf{x}_i)$ are compact weight functions and $\mathbf{x}_i$ are the nodes. Equation (2) can be expressed in the matrix form

$$J = (\mathbf{pa} - \mathbf{u})^{\mathrm{T}}\, \mathbf{W}(\mathbf{x})\, (\mathbf{pa} - \mathbf{u}), \tag{3}$$

where

$$\begin{gathered}
\mathbf{u}^{\mathrm{T}} = (u_1, u_2, \ldots, u_n), \\
\mathbf{p} = \begin{bmatrix} p_1(\mathbf{x}_1) & p_2(\mathbf{x}_1) & \cdots & p_m(\mathbf{x}_1) \\ p_1(\mathbf{x}_2) & p_2(\mathbf{x}_2) & \cdots & p_m(\mathbf{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ p_1(\mathbf{x}_n) & p_2(\mathbf{x}_n) & \cdots & p_m(\mathbf{x}_n) \end{bmatrix}, \\
\mathbf{W}(\mathbf{x}) = \begin{bmatrix} w(\mathbf{x} - \mathbf{x}_1) & 0 & \cdots & 0 \\ 0 & w(\mathbf{x} - \mathbf{x}_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w(\mathbf{x} - \mathbf{x}_n) \end{bmatrix}.
\end{gathered} \tag{4}$$

In order to find the coefficients $\mathbf{a}(\mathbf{x})$, we take the extremum of J by

$$\frac{\partial J}{\partial \mathbf{a}} = \mathbf{A}(\mathbf{x})\, \mathbf{a}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\, \mathbf{u} = 0 \tag{5}$$

which will get the following equation system:

$$\mathbf{A}(\mathbf{x})\, \mathbf{a}(\mathbf{x}) = \mathbf{B}(\mathbf{x})\, \mathbf{u}. \tag{6}$$

If the functions $p_1(\mathbf{x}), p_2(\mathbf{x}), \ldots, p_m(\mathbf{x})$ meet with the following conditions

$$\begin{gathered}
(p_k, p_j) = \sum_{i=1}^{n} w_i p_k(\mathbf{x}_i)\, p_j(\mathbf{x}_i) = \begin{cases} 0 & k \neq j \\ A_k & k = j \end{cases} \\
(k, j = 1, 2, \ldots, m),
\end{gathered} \tag{7}$$

it will be called a weighted orthogonal function set with a weight function $\{w_i\}$ with points $\{\mathbf{x}_i\}$. The orthogonal function set $\mathbf{p} = (p_i)$ can be obtained by using the Schmidt method,

$$\begin{gathered}
p_1 = 1, \\
p_i = r^{i-1} - \sum_{k=1}^{i-1} \frac{(r^{i-1}, p_k)}{(p_k, p_k)} p_k, \quad i = 2, 3, \ldots.
\end{gathered} \tag{8}$$

Equation (6) can be rewritten as

$$\begin{bmatrix} (p_1, p_1) & 0 & \cdots & 0 \\ 0 & (p_2, p_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (p_m, p_m) \end{bmatrix} \begin{bmatrix} a_1(\mathbf{x}) \\ a_2(\mathbf{x}) \\ \vdots \\ a_m(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} (p_1, u_I) \\ (p_2, u_I) \\ \vdots \\ (p_m, u_I) \end{bmatrix}. \tag{9}$$

The coefficients $a_i(\mathbf{x})$ can be easily founded as follows:

$$a_i(\mathbf{x}) = \frac{(p_i, u_I)}{(p_i, p_i)}; \quad (i = 1, 2, \ldots, m); \tag{10}$$

that is,

$$\mathbf{a}(\mathbf{x}) = \overline{\mathbf{A}}(\mathbf{x})\,\mathbf{B}(\mathbf{x})\,\mathbf{u}, \tag{11}$$

where

$$\overline{\mathbf{A}}(\mathbf{x}) = \begin{bmatrix} \frac{1}{(p_1, p_1)} & 0 & \cdots & 0 \\ 0 & \frac{1}{(p_2, p_2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{(p_m, p_m)} \end{bmatrix}. \tag{12}$$

From (1), $u^h(\mathbf{x})$ is expressed as

$$u^h(\mathbf{x}) = \overline{\mathbf{\Phi}}(\mathbf{x})\,\mathbf{u} = \sum_{I=1}^{n} \overline{\Phi}_I(\mathbf{x})\,u_I, \tag{13}$$

where shape function $\overline{\mathbf{\Phi}}(\mathbf{x})$ is

$$\begin{aligned} \overline{\mathbf{\Phi}}(\mathbf{x}) &= \left(\overline{\Phi}_1(\mathbf{x}), \overline{\Phi}_2(\mathbf{x}), \ldots, \overline{\Phi}_n(\mathbf{x})\right) \\ &= \mathbf{p}^{\mathrm{T}}(\mathbf{x})\,\overline{\mathbf{A}}(\mathbf{x})\,\mathbf{B}(\mathbf{x}). \end{aligned} \tag{14}$$

Taking derivatives of (14), we will get derivatives of shape function

$$\overline{\Phi}_{I,i}(\mathbf{x}) = \sum_{j=1}^{m} \left[p_{j,i}\left(\overline{\mathbf{A}}\mathbf{B}\right)_{jI} + p_j\left(\overline{\mathbf{A}}_{,i}\mathbf{B} + \overline{\mathbf{A}}\mathbf{B}_{,i}\right)_{jI} \right]. \tag{15}$$

The cubic spline weight function is chosen as follows:

$$\begin{aligned} w_i &= w(\mathbf{x} - \mathbf{x}_i) \equiv w(r) \\ &= \begin{cases} \frac{2}{3} - 4r^2 + 4r^3, & r \le \frac{1}{2} \\ \frac{4}{3} - 4r + 4r^2 - \frac{4}{3}r^3, & \frac{1}{2} < r \le 1 \\ 0, & r > 1, \end{cases} \end{aligned} \tag{16}$$

where

$$r = \frac{d_i}{d_{m_i}}, \quad d_i = \left\|\mathbf{x} - \mathbf{x}_i\right\|, \; d_{m_i} = d_{\max} c_i, \tag{17}$$

where $d_{\max}$ is called a scaling parameter and distance c_i is chosen to make matrix $M(x)$ which is no longer singular.

3. IMLS-Ritz Formulation for the Two-Sided Space-Fractional Wave Equation

Consider the following two-sided space-fractional wave equation:

$$\begin{aligned} \frac{\partial^2 u(x,t)}{\partial t^2} &= c_+(x)\frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha} + c_-(x)\frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha} \\ &\quad + d(x,t), \quad a \le x \le b, \; 0 \le t \le T \end{aligned} \tag{18a}$$

with initial conditions

$$\begin{aligned} u(x,0) &= u_0(x), \\ u_t(x,0) &= u_1(x) \end{aligned} \tag{18b}$$

and boundary conditions

$$u(a,t) = u(b,t) = 0, \tag{18c}$$

where the parameter α describes the fractional order of spatial derivatives with $1 < \alpha \le 2$. Function $d(x,t)$ refers to a source term, and the coefficient functions $c_+(x) > 0$ and $c_-(x) > 0$ refer to transport related coefficients.

In order to establish the numerical approximation scheme, points $x_i = (i-1)\Delta x$, $i = 1, 2, 3, \ldots, N$, are considered, where $\Delta x = (b-a)/(N-1)$. $x_1 = a$ and $x_N = b$ are the boundary points; $t_j = n\Delta t$, $n = 0, 1, 2, 3, \ldots$, where Δt is the time interval.

The left-handed and right Riemann-Liouville fractional derivatives of order α are defined as [58]

$$\begin{aligned} \left(D_{a+}^{\alpha} f\right)(x) &= \frac{\partial^\alpha f(x,t)}{\partial_+ x^\alpha} \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{d_+ x^n}\int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}}dt \\ &\qquad \forall x \in [a,b], \; \alpha > 0 \\ \left(D_{b-}^{\alpha} f\right)(x) &= \frac{\partial^\alpha f(x,t)}{\partial_- x^\alpha} \\ &= \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^b \frac{f(t)}{(t-x)^{\alpha-n+1}}dt \\ &\qquad \forall x \in [a,b], \; \alpha > 0, \end{aligned} \tag{19}$$

where n is an integer such that $n - 1 < \alpha \le n$.

The weighted integral form of (18a) is obtained as follows:

$$\begin{aligned} \int_\Gamma w \cdot \left[\frac{\partial^2 u(x,t)}{\partial t^2} - c_+(x)\frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha} \right. \\ \left. - c_-(x)\frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha} - d(x,t) \right] d\Gamma = 0. \end{aligned} \tag{20}$$

Define the energy functional $\Pi(u)$ as

$$\Pi(u) = \int_\Gamma u \frac{\partial^2 u}{\partial t^2} d\Gamma - \int_\Gamma u \left(c_+ \frac{\partial^\alpha u}{\partial_+ x^\alpha} + c_- \frac{\partial^\alpha u}{\partial_- x^\alpha} + d \right) d\Gamma. \quad (21)$$

Due to the boundary condition, the modified energy functional becomes

$$\Pi^*(u) = \int_\Gamma u \frac{\partial^2 u}{\partial t^2} d\Gamma - \int_\Gamma u \left(c_+ \frac{\partial^\alpha u}{\partial_+ x^\alpha} + c_- \frac{\partial^\alpha u}{\partial_- x^\alpha} + d \right) d\Gamma + \frac{\alpha_1}{2} \int_{\Gamma_u} (u - \overline{u})^2 d\Gamma_u. \quad (22)$$

By (13), we can derive the approximate function as follows:

$$\begin{aligned} u^h(x,t) &= \sum_{I=1}^{n} \Phi_I(x) \cdot T_I(t) = \mathbf{\Phi}(x) \cdot \mathbf{T} \\ \frac{\partial^2 u(x,t)}{\partial t^2} &= \frac{\partial^2}{\partial t^2} \sum_{I=1}^{n} \Phi_I(x) \cdot T_I(t) \\ &= \sum_{I=1}^{n} \Phi_I(x) \cdot \frac{\partial^2 T_I(t)}{\partial t^2} = \mathbf{\Phi}(x) \ddot{\mathbf{T}}, \end{aligned} \quad (23)$$

where

$$\begin{aligned} \mathbf{\Phi}(x) &= (\Phi_1(x), \Phi_2(x), \dots, \Phi_n(x)) \\ \mathbf{T} &= (T_1(t), T_2(t), \dots, T_n(t))^{\mathrm{T}} \\ \ddot{\mathbf{T}} &= \left(\frac{\partial^2 T_1(t)}{\partial t^2}, \frac{\partial^2 T_2(t)}{\partial t^2}, \dots, \frac{\partial^2 T_n(t)}{\partial t^2} \right)^{\mathrm{T}}. \end{aligned} \quad (24)$$

Substituting (23) into (22), by applying the Ritz minimization procedure to $\Pi^*(u)$, we will derive

$$\begin{aligned} \frac{\partial \Pi^*(u)}{\partial \Delta} &= 0, \\ \Delta &= T_I(t), \frac{\partial^2 T_I(t)}{\partial t^2}, \quad I = 1, 2, \dots, n. \end{aligned} \quad (25)$$

The results can be expressed as

$$\mathbf{C}\ddot{\mathbf{T}} + \mathbf{K}\mathbf{T} = \mathbf{F}, \quad (26)$$

where

$$\begin{aligned} C &= \int_\Gamma \Phi^T(x) \Phi(x) d\Gamma \\ K &= \alpha_1 \Phi^T(x) \Phi(x) \Big|_{x=a, x=b} - \int_\Gamma \Phi^T(x) \left(c_+ \frac{\partial^\alpha \Phi(x)}{\partial_+ x^\alpha} + c_- \frac{\partial^\alpha \Phi(x)}{\partial_- x^\alpha} \right) d\Gamma \\ F_I &= -\int_\Gamma \Phi_I(x) d(x,t) d\Gamma. \end{aligned} \quad (27)$$

By the shifted Grünwald formula, we can discretize the Riemann-Liouville operator [18]

$$\frac{\partial^\alpha \Phi(\overline{x}_i)}{\partial_+ x^\alpha} = \frac{1}{h^\alpha} \sum_{j=0}^{i} w_j \Phi(\overline{x}_i - (j-1)h) + O(h) \quad i = 2, 3, \dots, M-1 \quad (28a)$$

$$\frac{\partial^\alpha \Phi(\overline{x}_i)}{\partial_- x^\alpha} = \frac{1}{h^\alpha} \sum_{j=0}^{M-i+1} w_j \Phi(\overline{x}_i + (j-1)h) + O(h) \quad i = 2, 3, \dots, M-1, \quad (28b)$$

where $\{\overline{x}_k\}$ ($k = 1, 2, \dots, M$, $M = 2N - 1$) is the set of nodes and Gauss points and w_j are the normalized Grünwald weights. The corresponding coefficients w_j can be easily calculated by iteration formula as follows:

$$\begin{aligned} w_0 &= 1, \\ w_j &= \left(1 - \frac{\alpha + 1}{k}\right) w_{j-1}. \end{aligned} \quad (29)$$

Substituting (28a), (28b), and (29) into (26) and discrete time by center difference method, we obtain

$$\mathbf{C} \frac{U_{n+1} - 2U_n + U_{n-1}}{\Delta t^2} + \mathbf{K} \frac{U_{n+1} + U_n}{2} = \frac{F_{n+1} + F_n}{2}, \quad (30)$$

where

$$\begin{aligned} K = {} & \alpha_1 \cdot \Phi^T(x) \Phi(x) \Big|_{x=a, x=b} - \int_\Gamma \Phi^T(x) \\ & \cdot \left(\frac{1}{h^\alpha} \left(\sum_{j=0}^{i} g_j \Phi(x_i - (j-1)h) \right.\right. \\ & \left.\left. + \sum_{j=0}^{M-i+1} g_j \Phi(x_i + (j-1)h) \right) \right) d\Gamma. \end{aligned} \quad (31)$$

The numerical solution of the space-fractional wave equation is obtained by iterative calculation.

4. Numerical Results

In order to verify the validity and correctness of the proposed IMLS-Ritz method for the space-fractional wave equation, examples are studied and the numerical results are presented. Note that, in all examples considered, the cubic spline function is chosen as weight function and the linear bases are chosen in this paper.

Example 1 (left-handed space-fractional wave equation). Consider the following left-handed space-fractional wave equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \Gamma(1.2)\, x^{1.8} \frac{\partial^{1.8} u(x,t)}{\partial x^{1.8}} + d(x,t) \quad 0 < x < 2 \quad (32)$$

Table 1: Maximum error for Example 1 at $t = 1$.

Δt	Δx	IMLS-Ritz method	Finite difference method
0.005	2^{-3}	0.0723	0.1128
0.005	2^{-4}	0.0483	0.0511
0.002	2^{-5}	0.0204	0.0270
0.002	2^{-6}	0.0115	0.0137

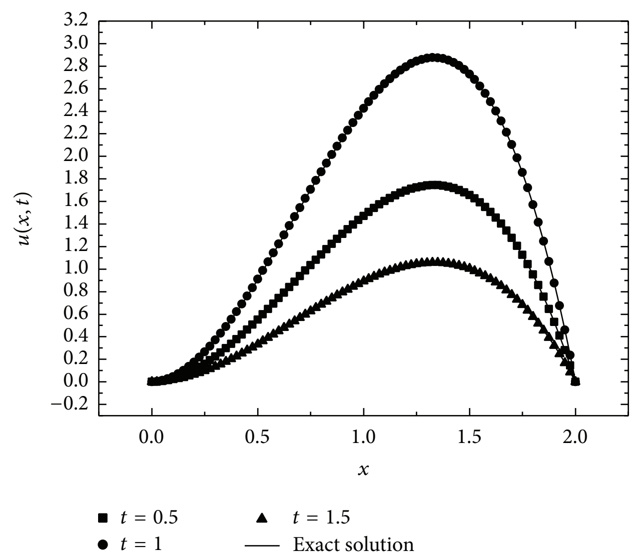

Figure 1: Numerical solution and exact solution of $u(x,t)$ when $t = 0.5$, 1, and 1.5 (Example 1).

with initial conditions

$$u(x,0) = 4x^2(2-x), \quad u_t(x,0) = -4x^2(2-x) \tag{33}$$

with boundary conditions

$$u(0,t) = u(2,t) = 0, \tag{34}$$

where the source function is

$$d(x,t) = 4e^{-t}x^2(2-x) - 16e^{-t}x^2 + 20e^{-t}x^3. \tag{35}$$

The analytical solution is

$$u(x,t) = 4e^{-t}x^2(2-x). \tag{36}$$

Using IMLS-Ritz method to solve the equation with penalty factor $\alpha_1 = 10^7$, time step length $\Delta t = 0.001$, space step length $\Delta x = 0.0125$, and $d_{max} = 3.8$. Table 1 shows numerical results obtained by the IMLS-Ritz method. The maximum error at time $t = 1$ between the exact solution and the numerical solution and finite difference method [27] at different values of Δx and Δt is shown in Table 1. In Figure 1, the numerical and analytical solution are plotted at time $t = 0.5$,1, and 1.5, respectively. The surface of the numerical and analytical solution is plotted in Figures 2 and 3, respectively. Numerical results show that the IMLS-Ritz method is very effective and accurate.

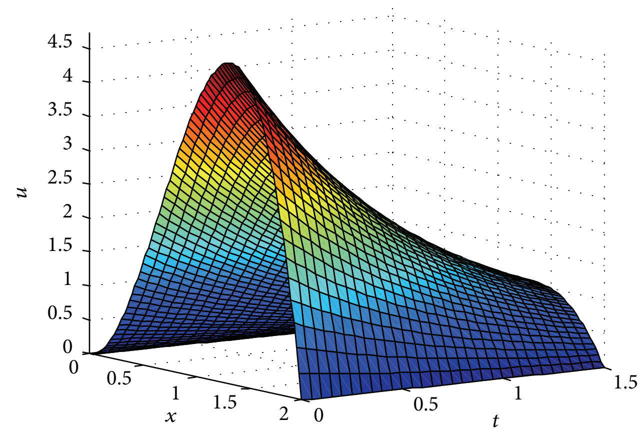

Figure 2: The surface of numerical solution with IMLS-Ritz method (Example 1).

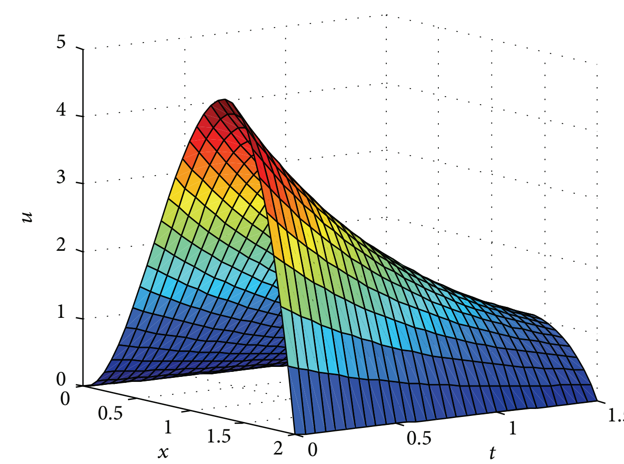

Figure 3: The surface of exact solution (Example 1).

Example 2 (two-sided space-fractional). Consider the following left-handed and right-handed space-fractional wave equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \Gamma(1.2)\, x^{1.8} \frac{\partial^{1.8} u(x,t)}{\partial_+ x^{1.8}} + \Gamma(1.2)(2-x)^{1.8} \frac{\partial^{1.8} u(x,t)}{\partial_- x^{1.8}} + d(x,t) \quad 0 < x < 2 \tag{37}$$

with initial conditions

$$u(x,0) = 4x^2(2-x)^2, \quad u_t(x,0) = -4x^2(2-x)^2 \tag{38}$$

with boundary conditions

$$u(0,t) = u(2,t) = 0, \tag{39}$$

Table 2: Maximum error for Example 2 at $t = 2$.

Δt	Δx	IMLS-Ritz method	Finite difference method
0.020	2^{-3}	0.0382	0.0379
0.0066	2^{-4}	0.0135	0.0164
0.0050	2^{-6}	0.0037	0.0042
0.0033	2^{-5}	0.0079	0.0083

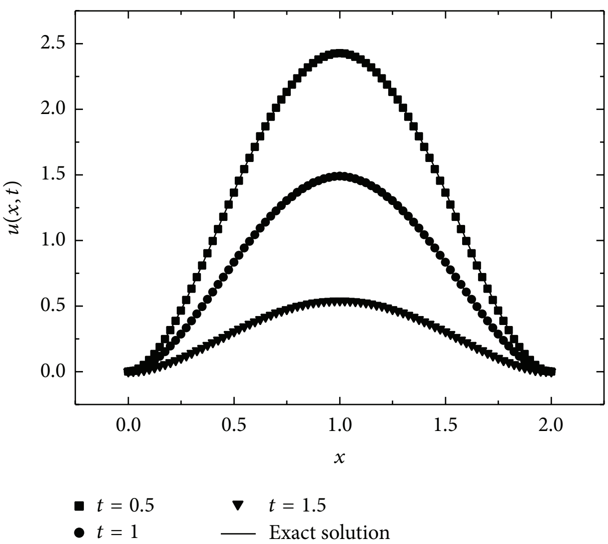

Figure 4: Numerical solution and exact solution of $u(x,t)$ when $t = 0.5$, 1, and 2 (Example 2).

where the source function is

$$d(x,t) = 4e^{-t}x^2(2-x)^2 - 32e^{-t}\Big[x^2 + (2-x)^2 - 2.5\left(x^3 + (2-x)^3\right) + \frac{25}{22}\left(x^4 + (2-x)^4\right)\Big]. \quad (40)$$

The exact solution is

$$u(x,t) = 4e^{-t}x^2(2-x)^2. \quad (41)$$

The IMLS-Ritz method is applied to solve the above equation with penalty factor $\alpha_1 = 10^7$ and time step length $\Delta t = 0.002$ and $d_{\max} = 3.8$. Table 2 shows numerical results obtained by the IMLS-Ritz method. The maximum error at time $t = 2$ between the exact solution and the numerical solution and finite difference method [27] at different values of Δx and Δt is shown in Table 2. In Figure 4, the numerical and analytical solution are plotted at time $t = 0.5$,1, and 2, respectively. The surface of the numerical and analytical solution is plotted in Figures 5 and 6, respectively. Numerical results show that the IMLS-Ritz method is very effective and accurate.

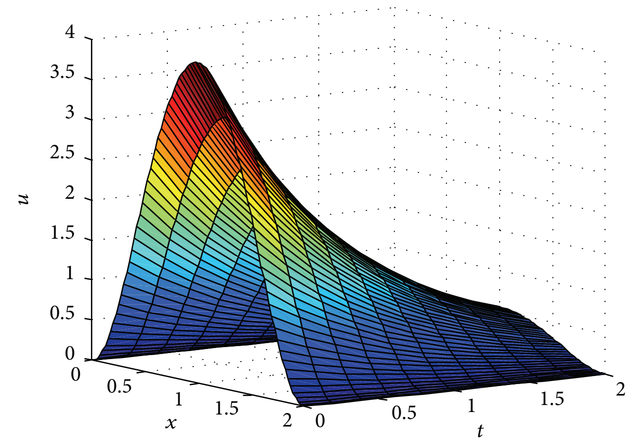

Figure 5: The surface of numerical solution with IMLS-Ritz method (Example 2).

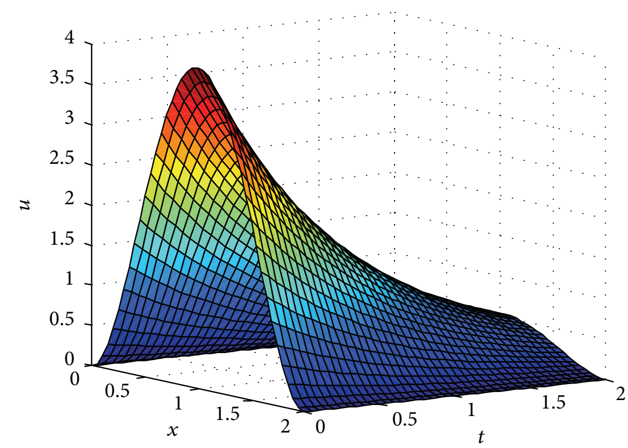

Figure 6: The surface of exact solution (Example 2).

Example 3 (two-sided space-fractional). Consider the following left-handed and right-handed space-fractional wave equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^{1.8} u(x,t)}{\partial_+ x^{1.8}} + \frac{\partial^{1.8} u(x,t)}{\partial_- x^{1.8}}, \quad 0 < x < 5 \quad (42)$$

with initial conditions

$$u(x,0) = \sin(\pi x), \quad u_t(x,0) = 0 \quad (43)$$

with boundary conditions

$$u(0,t) = u(5,t) = 0. \quad (44)$$

Particularly, if the fractional order of spatial derivatives $\alpha = 2$, (42) will be a standard wave equation, and the exact solution to this problem in case $\alpha = 2$ is

$$u(x,t) = \sin(\pi x)\cos(2\pi t). \quad (45)$$

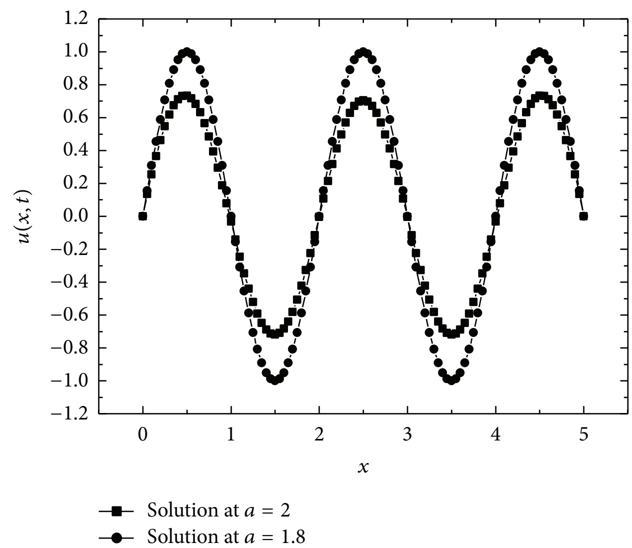

FIGURE 7: The numerical solution using IMLS-Ritz method with $\alpha = 2$ and $\alpha = 1.8$ at $t = 1$ (Example 3).

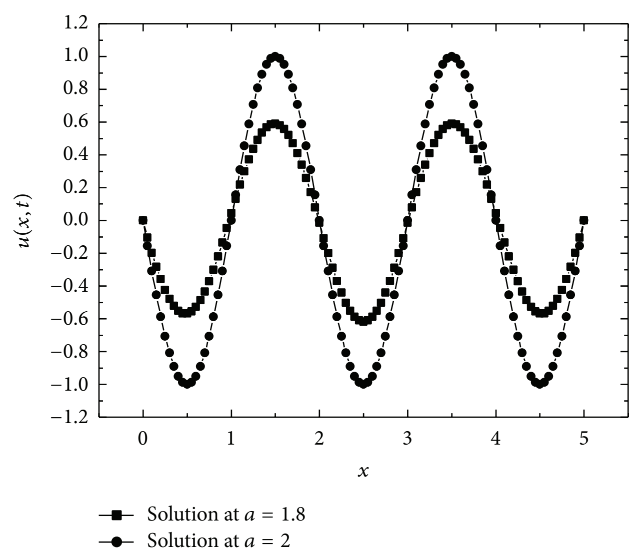

FIGURE 8: The numerical solution using IMLS-Ritz method with $\alpha = 2$ and $\alpha = 1.8$ at $t = 1.5$ (Example 3).

Figures 7 and 8 show the numerical results with IMLS-Ritz method with $\alpha = 2$ and $\alpha = 1.8$ at different times $t = 1$ and $t = 1.5$, respectively.

From these figures, it can be seen that the IMLS-Ritz method is very effective and accurate.

5. Conclusions

The meshless method for the two-sided space-fractional wave equation is put forward in this paper. In the present method, the IMLS approximation is employed to construct the shape functions. In the IMLS, the orthogonal function system with a weight function is chosen as the basis function. Through employing the Ritz minimization procedure to the energy expressions, the final algebraic equations system is obtained. The system obtained by IMLS technique will be not ill-conditioned any more, and the solution can be easily obtained without matrix inversion. Because of the simplicity of numerical implementation, the proposed IMLS-Ritz method will substitute for the difference method and the finite element method for solving space-fractional wave equation and other fractional partial differential equations.

Competing Interests

The authors declare that they have no competing interests.

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