Research Article

Parameter Estimation and Joint Confidence Regions for the Parameters of the Generalized Lindley Distribution

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We deal with the problem of estimating the parameters of the generalized Lindley distribution. Besides the classical estimator, inverse moment and modified inverse estimators are proposed and their properties are investigated. A condition for the existence and uniqueness of the inverse moment and modified inverse estimators of the parameters is established. Monte Carlo simulations are conducted to compare the estimators' performances. Two methods for constructing joint confidence regions for the two parameters are also proposed and their performances are discussed. A real example is presented to illustrate the proposed methods.

1. Introduction

Lindley [1] originally introduced the Lindley distribution to illustrate a difference between fiducial distribution and posterior distribution. This distribution is becoming increasingly popular for modeling lifetime data and has a wide applicability in survival and reliability as closed forms for the survival and hazard functions and good flexibility of fit. Its density function is given by

$$f(t) = \frac{\lambda^2}{1+\lambda}(1+t)e^{-\lambda t}, \quad t, \lambda > 0. \tag{1}$$

We denoted this by writing $LD(\lambda)$. The Lindley distribution is a mixture of an exponential distribution with scale λ and a gamma distribution with shape 2 and scale λ, where the mixing proportion is $p = \lambda/(1+\lambda)$.

Ghitany et al. [2] provided a comprehensive treatment of the statistical properties of the Lindley distribution. Mazucheli and Achcar [3] used the Lindley distribution as a good alternative to analyze lifetime data within the competing risks approach as compared with the use of standard exponential or even the Weibull distribution commonly used in this area. Krishna and Kumar [4] considered the reliability estimation in Lindley distribution with progressively type II right censored sample. Al-Mutairi et al. [5] dealt with the estimation of the stress-strength parameter when the variables

are independent Lindley random variables with different shape parameters.

Some researchers have proposed and studied new classes of distributions based on the Lindley distribution. See, for example, Sankaran [6], Ghitany et al. [7], Bakouch et al. [8], Shanker et al. [9], and Ghitany et al. [10]. In this paper, we focus on the generalized Lindley distribution (GLD) introduced by Nadarajah et al. [11]. It has the attractive feature of allowing for monotonically decreasing, monotonically increasing, and bath tub shaped hazard rate functions while not allowing for constant hazard rate functions. It has better hazard rate properties than the gamma, lognormal, and the Weibull distributions.

The cumulative distribution function and the probability density function are, respectively, given by

$$F(x; \lambda, \alpha) = \left[1 - \frac{1+\lambda+\lambda x}{1+\lambda}e^{-\lambda x}\right]^{\alpha}, \quad x > 0, \tag{2}$$

$$f(x; \lambda, \alpha)$$
$$= \frac{\alpha\lambda^2(x+1)e^{-\lambda x}}{\lambda+1}\left(1 - \frac{e^{-\lambda x}(\lambda+\lambda x+1)}{\lambda+1}\right)^{\alpha-1}, \tag{3}$$
$$x > 0,$$

where $\alpha > 0$ and $\lambda > 0$ are two parameters. We denote this distribution as $GLD(\lambda, \alpha)$. When $\alpha = 1$, the generalized

Lindley distribution reduces to the one parameter Lindley distribution.

Singh et al. [12] developed the Bayesian estimation for the generalized Lindley distribution under squared error and general entropy loss functions in case of complete sample of observations. Singh et al. [13] considered the generalized Lindley distribution and proposed the progressive type II censoring scheme which allows the removal of the live units from a life-test with beta-binomial probability law during the execution of the experiment.

Nadarajah et al. [11] considered the classical maximum-likelihood estimation of the parameters of a generalized Lindley distribution. The results showed that the bias is not satisfied especially for a small or even moderate sample size. As for the moment estimates, two nonlinear equations need to be solved simultaneously and the existence and uniqueness of the roots are not clear and guaranteed.

In this paper, we consider the problem of estimating the two parameters of the generalized Lindley distribution. We propose inverse moment and modified inverse moment estimators and study their properties. The conditions of the existence and uniqueness of the estimators are established. Monte Carlo simulations are used to compare the performances of the estimators. We also investigate the methods for constructing joint confidence regions for the two parameters and study their performances.

The rest of this paper is organized as follows. In Section 2, we briefly review the classical maximum-likelihood estimation of the parameters of the generalized Lindley distribution. In Section 3, the moment estimator is discussed. In Section 4, we propose two new methods of estimating the parameters and study their properties. Joint confidence regions for the two parameters are proposed in Section 5. Section 6 conducts simulations to assess the methods. Finally, in Section 7, a real example is presented to illustrate the proposed methods.

2. Maximum-Likelihood Estimation

In this section, we briefly review the MLEs of the parameters of GLD distribution. Let $X_1, X_2, \ldots, X_n$ be a random sample from $\mathrm{GLD}(\lambda, \alpha)$ with pdf and cdf as (3) and (2), respectively. The log-likelihood function is given by

$$
\begin{aligned}
L(\lambda, \alpha) = {} & (\alpha - 1) \sum_{i=1}^{n} \log \left[1 - \frac{e^{-\lambda x_i}(\lambda x_i + \lambda + 1)}{\lambda + 1} \right] \\
& - \lambda \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \log(x_i + 1) + n \log(\alpha) \\
& + 2n \log(\lambda) - n \log(\lambda + 1).
\end{aligned} \tag{4}
$$

The score equations are thus as follows:

$$
\begin{aligned}
& \frac{\partial L(\lambda, \alpha)}{\partial \lambda} \\
& = -(\alpha - 1) \sum_{i=1}^{n} \frac{\lambda x_i ((\lambda + 1) x_i + \lambda + 2)}{(\lambda + 1)((\lambda + 1)(1 - e^{\lambda x_i}) + \lambda x_i)} \\
& \quad - \sum_{i=1}^{n} x_i + \frac{n(\lambda + 2)}{\lambda(1 + \lambda)},
\end{aligned} \tag{5}
$$

$$
\frac{\partial L(\lambda, \alpha)}{\partial \alpha} = \sum_{i=1}^{n} \log \left[1 - \frac{e^{-\lambda x_i}(\lambda x_i + \lambda + 1)}{\lambda + 1} \right] + \frac{n}{\alpha}. \tag{6}
$$

From (6) we obtain the MLE of α as a function of λ:

$$
\widehat{\alpha} = -\frac{n}{\sum_{i=1}^{n} \log \left[1 - e^{-\lambda x_i}(\lambda x_i + \lambda + 1)/(\lambda + 1) \right]}. \tag{7}
$$

The MLE of λ is the root of the following equation:

$$
\begin{aligned}
G(\lambda) = {} & \frac{n}{\sum_{i=1}^{n} \log \left[1 - e^{-\lambda x_i}(\lambda x_i + \lambda + 1)/(\lambda + 1) \right]} \\
& \cdot \sum_{i=1}^{n} \frac{\lambda x_i ((\lambda + 1) x_i + \lambda + 2)}{(\lambda + 1)((\lambda + 1)(1 - e^{\lambda x_i}) + \lambda x_i)} \\
& + \sum_{i=1}^{n} \frac{\lambda x_i ((\lambda + 1) x_i + \lambda + 2)}{(\lambda + 1)((\lambda + 1)(1 - e^{\lambda x_i}) + \lambda x_i)} - \sum_{i=1}^{n} x_i \\
& + \frac{n(\lambda + 2)}{\lambda(1 + \lambda)} = 0.
\end{aligned} \tag{8}
$$

Such nonlinear equation does not have closed form solution. We can apply numerical method such as Newton-Raphson method to compute λ.

3. Moment Estimation of Parameters

In this section, we discuss the moment estimation (MOM) of the parameters of GLD distribution. Let $X_1, X_2, \ldots, X_n$ be a random sample from $\mathrm{GLD}(\lambda, \alpha)$ with pdf and cdf as (3) and (2), respectively. $x_1, x_2, \ldots, x_n$ are the observed values. Let $m_1 = (1/n) \sum_{i=1}^{n} x_i$, $m_2 = (1/n) \sum_{i=1}^{n} x_i^2$. m_1, m_2 are sample moments. For the population moments, we need the following lemma (Nadarajah et al. [11]).

Lemma 1. Let $K(\alpha, \lambda, n) = \int_0^{\infty} x^n (x+1) e^{-\lambda x} (1 - e^{-\lambda x} (\lambda + \lambda x + 1)/(\lambda + 1))^{\alpha - 1} dx$. One has

$$
\begin{aligned}
K(\alpha, \lambda, n) = {} & \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{j+1} \binom{\alpha - 1}{i} \binom{i}{j} \binom{j+1}{k} \\
& \cdot \frac{(-1)^i \lambda^j \Gamma(n+1+k)}{(1+\lambda)^j (\lambda i + \lambda)^{n+1+k}}.
\end{aligned} \tag{9}
$$

Let X denote a GLD random variable. It follows that

$$
\mathbb{E} X^n = \frac{\alpha \lambda^2}{1 + \lambda} K(\alpha, \lambda, n). \tag{10}
$$

By equating the population moments with the sample moments, we obtain

$$
\frac{\alpha \lambda^2}{1 + \lambda} K(\alpha, \lambda, 1) = m_1, \tag{11}
$$

$$
\frac{\alpha \lambda^2}{1 + \lambda} K(\alpha, \lambda, 2) = m_2. \tag{12}
$$

The method of moments estimators is the roots of the two equations. Similar to the MLEs, such nonlinear equations do not have closed form solutions. We can apply numerical method such as Newton-Raphson method to determine the roots.

4. Inverse Moment Estimation of Parameters

Unlike the regular method of moments, the idea of the inverse moment estimation (IME) is as follows: for a given random sample $X_1, \ldots, X_n$ from a distribution with unknown parameters, first transform the original sample to a quasisample $Y_1, \ldots, Y_n$, where Y_i contains the unknown parameters but its distribution does not depend on the unknown parameters; that is, Y_i is a pivot variable, $i = 1, \ldots, n$. The population moments of the new sample do not depend on the unknown parameters while the sample moments do. Let the population moments of the quasisample equal the sample moments and solve for the unknown parameters.

Let $X_1, \ldots, X_n$ form a sample from $\mathrm{GLD}(\lambda, \alpha)$ with pdf given in (3); it is known that $F(X_i)$, $i = 1, \ldots, n$, follow the uniform distribution $U(0, 1)$, and thus $-\log F(X_i)$, $i = 1, \ldots, n$, follow standard exponential distribution $\mathrm{Exp}(1)$. By the method of inverse moment estimation, we let

$$\frac{1}{n} \sum_{i=1}^{n} \left[-\log F(X_i) \right] = 1; \tag{13}$$

that is,

$$-\frac{\alpha}{n} \sum_{i=1}^{n} \log \left[1 - \frac{e^{-\lambda x_i} (\lambda x_i + \lambda + 1)}{\lambda + 1} \right] = 1. \tag{14}$$

Thus, the IME of α is obtained as a function of λ,

$$\widehat{\alpha} = -\frac{n}{\sum_{i=1}^{n} \log \left[1 - e^{-\lambda x_i} (\lambda x_i + \lambda + 1) / (\lambda + 1) \right]}, \tag{15}$$

which is identical to the MLE of α. In the following, we determine the IME of λ.

Lemma 2. *Let $Z_{(1)} \leq Z_{(2)} \leq \cdots \leq Z_{(n)}$ be the order statistics from the standard exponential distribution. Then, the random variables $W_1, W_2, \ldots, W_n$, where*

$$W_i = (n - i + 1) \left(Z_{(i)} - Z_{(i-1)} \right), \quad i = 1, 2, \ldots, n \tag{16}$$

with $Z_{(0)} \equiv 0$, are independent and follow standard exponential distributions.

Proof. The proof can be found by Arnold et al. [14]. $\square$

Lemma 3. *Let $W_1, W_2, \ldots, W_n$ be iid standard exponential variables, $S_i = W_1 + \cdots + W_i$, $U_i = (S_i/S_{i+1})^i$, $i = 1, 2, \ldots, n-1$, $U_n = W_1 + \cdots + W_n$; then*

(1) *$U_1, U_2, \ldots, U_n$ are independent;*

(2) *$U_1, U_2, \ldots, U_{n-1}$ follow the uniform distribution $U(0, 1)$;*

(3) *$2U_n$ follows $\chi^2(2n)$.*

Proof. The proof can be found by Wang [15]. $\square$

For the sample $X_1, \ldots, X_n$ from $\mathrm{GLD}(\lambda, \alpha)$, considering the order statistics $X_{(1)} \leq \cdots \leq X_{(n)}$, we have that

$$-\log F(X_{(n)}) \leq \cdots \leq -\log F(X_{(1)}) \tag{17}$$

are n-order statistics from standard exponential distribution. Let $Z_{(i)} = -\alpha \log[1 - e^{-\lambda X_{(n-i+1)}} (\lambda X_{(n-i+1)} + \lambda + 1)/(\lambda + 1)] \equiv -\alpha \log G(X_{(n-i+1)})$, $i = 1, \ldots, n$, where $G(x) = 1 - e^{-\lambda x} (\lambda x + \lambda + 1)/(\lambda + 1)$. Thus, $Z_{(1)} \leq Z_{(2)} \leq \cdots \leq Z_{(n)}$ are the first n-order statistics from the standard exponential distribution. By Lemma 2, $W_i = (n - i + 1)(Z_{(i)} - Z_{(i-1)})$, $i = 1, 2, \ldots, n$, form a sample from standard exponential distribution.

Let $S_i = W_1 + \cdots + W_i$, $U_i = (S_i/S_{i+1})^i$, $i = 1, 2, \ldots, n-1$, and $U_n = W_1 + \cdots + W_n$; by Lemma 3, we have

$$-2 \sum_{i=1}^{n-1} \log U_i = -2 \sum_{i=1}^{n-1} i \log \left(\frac{S_i}{S_{i+1}} \right) = 2 \sum_{i=1}^{n-1} \log \left(\frac{S_n}{S_i} \right)$$
$$\sim \chi^2 (2n - 2), \tag{18}$$

where

$$\frac{S_n}{S_i} = \frac{W_1 + \cdots + W_n}{W_1 + \cdots + W_i} = \frac{Z_{(1)} + Z_{(2)} + \cdots + Z_{(n)}}{Z_{(1)} + Z_{(2)} + \cdots + Z_{(i-1)} + (n - i + 1) Z_{(i)}}$$
$$= \frac{\log G(X_{(n)}) + \log G(X_{(n-1)}) + \cdots + \log G(X_{(1)})}{\log G(X_{(n)}) + \cdots + \log G(X_{(n-i+2)}) + (n - i + 1) \log G(X_{(n-i+1)})}. \tag{19}$$

Note that the mean of $\chi^2(2n - 2)$ is $2n - 2$. Thus, we obtain an inverse moment equation for λ as follows:

$$\sum_{i=1}^{n-1} \log \left[\frac{\log G(X_{(n)}) + \log G(X_{(n-1)}) + \cdots + \log G(X_{(1)})}{\log G(X_{(n)}) + \cdots + \log G(X_{(n-i+2)}) + (n - i + 1) \log G(X_{(n-i+1)})} \right] = n - 1. \tag{20}$$

Solve the equation and we obtain the inverse estimate $\widehat{\lambda}_{\mathrm{IME}}$ of λ. Plugging $\widehat{\lambda}_{\mathrm{IME}}$ into (15), we obtain the inverse estimate

$\widehat{\alpha}_{\mathrm{IME}}$. In addition, considering that the mode of $\chi^2(2n-2)$ is $2n-4$, we can obtain a modified equation for λ:

$$\sum_{i=1}^{n-1} \log\left[\frac{\log G(X_{(n)}) + \log G(X_{(n-1)}) + \cdots + \log G(X_{(1)})}{\log G(X_{(n)}) + \cdots + \log G(X_{(n-i+2)}) + (n-i+1)\log G(X_{(n-i+1)})}\right] = n - 2. \tag{21}$$

Solve the equation and we obtain the modified inverse estimate $\widehat{\lambda}_{\mathrm{MIME}}$ of λ. Plugging $\widehat{\lambda}_{\mathrm{MIME}}$ into (15), we obtain the modified inverse estimate $\widehat{\alpha}_{\mathrm{MIME}}$. Unlike the moment estimation, here we only need to solve one nonlinear equation instead of two equations.

In the following, we prove the existence and uniqueness of the root in (20) and (21).

Lemma 4. *The following limits hold:*

(1) *One has*

$$\lim_{\lambda \to 0} \frac{\log G(a)}{\log G(b)}$$

$$= \lim_{\lambda \to 0} \frac{\log\left[1 - e^{-\lambda a}(\lambda a + \lambda + 1)/(\lambda + 1)\right]}{\log\left[1 - e^{-\lambda b}(\lambda b + \lambda + 1)/(\lambda + 1)\right]} = 1, \tag{22}$$

$$\text{for } a > 0, b > 0.$$

(2) *One has*

$$\lim_{\lambda \to \infty} \frac{\log G(a)}{\log G(b)}$$

$$= \lim_{\lambda \to \infty} \frac{\log\left[1 - e^{-\lambda a}(\lambda a + \lambda + 1)/(\lambda + 1)\right]}{\log\left[1 - e^{-\lambda b}(\lambda b + \lambda + 1)/(\lambda + 1)\right]} = 0, \tag{23}$$

$$\text{for } a > b > 0.$$

(3) *One has*

$$\lim_{\lambda \to \infty} \frac{\log G(a)}{\log G(b)}$$

$$= \lim_{\lambda \to \infty} \frac{\log\left[1 - e^{-\lambda a}(\lambda a + \lambda + 1)/(\lambda + 1)\right]}{\log\left[1 - e^{-\lambda b}(\lambda b + \lambda + 1)/(\lambda + 1)\right]} \tag{24}$$

$$= +\infty, \quad \text{for } b > a > 0.$$

Lemma 5. *For $t > 0$, $f(t) = (\lambda + t + 2 - e^t[\lambda - (\lambda + 1)t + 2])/(\lambda - (\lambda + 1)e^t + t + 1)$ is a decreasing function of t.*

Theorem 6. *Let $W_i = (n-i+1)(Z_{(i)} - Z_{(i-1)})$, $i = 1, 2, \ldots, n$, form a sample from standard exponential distribution, $S_i = W_1 + \cdots + W_i$; then for $t > 0$, equation $\sum_{i=1}^{n-1} \log(S_n/S_i) = t$ has a unique positive solution.*

Proof. By Lemma 4, we obtain

$$\lim_{\lambda \to 0} \frac{S_n}{S_i} = \lim_{\lambda \to 0} \frac{W_1 + \cdots + W_n}{W_1 + \cdots + W_i} = \lim_{\lambda \to 0} \frac{Z_{(1)} + Z_{(2)} + \cdots + Z_{(n)}}{Z_{(1)} + Z_{(2)} + \cdots + Z_{(i-1)} + (n-i+1)Z_{(i)}}$$

$$= \lim_{\lambda \to 0} \frac{\log G(X_{(n)}) + \log G(X_{(n-1)}) + \cdots + \log G(X_{(1)})}{\log G(X_{(n)}) + \cdots + \log G(X_{(n-i+2)}) + (n-i+1)\log G(X_{(n-i+1)})} \tag{25}$$

$$= \lim_{\lambda \to 0} \frac{\left[\log G(X_{(n)}) + \log G(X_{(n-1)}) + \cdots + \log G(X_{(1)})\right]/\log G(X_{(n)})}{\left[\log G(X_{(n)}) + \cdots + \log G(X_{(n-i+2)}) + (n-i+1)\log G(X_{(n-i+1)})\right]/\log G(X_{(n)})} = \frac{n}{n} = 1.$$

Thus, $\lim_{\lambda \to 0} \sum_{i=1}^{n-1} \log(S_n/S_i) = 0$. On the other hand,

$$\lim_{\lambda \to \infty} \frac{S_n}{S_i} = \lim_{\lambda \to \infty} \frac{\log G(X_{(n)}) + \log G(X_{(n-1)}) + \cdots + \log G(X_{(1)})}{\log G(X_{(n)}) + \cdots + \log G(X_{(n-i+2)}) + (n-i+1)\log G(X_{(n-i+1)})}$$

$$= 1 + \lim_{\lambda \to \infty} \frac{\log G(X_{(n-i)}) + \cdots + \log G(X_{(1)}) - (n-i)\log G(X_{(n-i+1)})}{\log G(X_{(n)}) + \cdots + \log G(X_{(n-i+1)}) + (n-i)\log G(X_{(n-i+1)})} \tag{26}$$

$$= 1 + \lim_{\lambda \to \infty} \frac{\left[\log G(X_{(n-i)}) + \cdots + \log G(X_{(1)}) - (n-i)\log G(X_{(n-i+1)})\right]/\log G(X_{(n-i+1)})}{\left[\log G(X_{(n)}) + \cdots + \log G(X_{(n-i+1)}) + (n-i)\log G(X_{(n-i+1)})\right]/\log G(X_{(n-i+1)})} = +\infty.$$

Thus, $\lim_{\lambda \to \infty} \sum_{i=1}^{n-1} \log(S_n/S_i) = \infty$. Therefore, for $t > 0$, equation $\sum_{i=1}^{n-1} \log(S_n/S_i) = t$ has one positive solution. For

the uniqueness of the solution, we consider the derivative of S_n/S_i with respect to λ. Note that, for $i = 1, \ldots, n$,

$$W_i = (n-i+1)\alpha\left[\log G\left(X_{(n-i+2)}\right) - \log G\left(X_{(n-i+1)}\right)\right],$$

$$\frac{dW_i}{d\lambda} = (n-i+1)\alpha\left\{\frac{\lambda X_{(n-i+2)}e^{-\lambda X_{(n-i+2)}}\left[(\lambda+1)X_{(n-i+2)}+\lambda+2\right]}{(\lambda+1)^2 G\left(X_{(n-i+2)}\right)} - \frac{\lambda X_{(n-i+1)}e^{-\lambda X_{(n-i+1)}}\left[(\lambda+1)X_{(n-i+1)}+\lambda+2\right]}{(\lambda+1)^2 G\left(X_{(n-i+1)}\right)}\right\} = W_i$$

$$\cdot \frac{\left\{\lambda X_{(n-i+2)}e^{-\lambda X_{(n-i+2)}}\left[(\lambda+1)X_{(n-i+2)}+\lambda+2\right]/(\lambda+1)^2 G\left(X_{(n-i+2)}\right) - \lambda X_{(n-i+1)}e^{-\lambda X_{(n-i+1)}}\left[(\lambda+1)X_{(n-i+1)}+\lambda+2\right]/(\lambda+1)^2 G\left(X_{(n-i+1)}\right)\right\}}{\log G\left(X_{(n-i+2)}\right) - \log G\left(X_{(n-i+1)}\right)}. \quad (27)$$

$$\left(\frac{S_n}{S_i}\right)' = \left(1 + \frac{W_{i+1}+\cdots+W_n}{W_1+\cdots+W_i}\right)' = \frac{1}{\left(\sum_{k=1}^{i}W_k\right)^2}\sum_{j=i+1}^{n}\sum_{k=1}^{i}\left[W_j'W_k - W_jW_k'\right] = \frac{1}{\lambda\left(\sum_{k=1}^{i}W_k\right)^2}\sum_{j=i+1}^{n}\sum_{k=1}^{i}W_jW_k\left[A(\lambda) - B(\lambda)\right],$$

where

$$A(\lambda)$$
$$= \frac{\lambda X_{(n-j+2)}e^{-\lambda X_{(n-j+2)}}\lambda\left[(\lambda+1)X_{(n-j+2)}+\lambda+2\right]/(\lambda+1)^2 G\left(X_{(n-j+2)}\right) - \lambda X_{(n-j+1)}e^{-\lambda X_{(n-j+1)}}\lambda\left[(\lambda+1)X_{(n-j+1)}+\lambda+2\right]/(\lambda+1)^2 G\left(X_{(n-j+1)}\right)}{\log G\left(X_{(n-j+2)}\right) - \log G\left(X_{(n-j+1)}\right)},$$

$$B(\lambda)$$
$$= \frac{\lambda X_{(n-k+2)}e^{-\lambda X_{(n-k+2)}}\lambda\left[(\lambda+1)X_{(n-k+2)}+\lambda+2\right]/(\lambda+1)^2 G\left(X_{(n-k+2)}\right) - \lambda X_{(n-k+1)}e^{-\lambda X_{(n-k+1)}}\lambda\left[(\lambda+1)X_{(n-k+1)}+\lambda+2\right]/(\lambda+1)^2 G\left(X_{(n-k+1)}\right)}{\log G\left(X_{(n-k+2)}\right) - \log G\left(X_{(n-k+1)}\right)}. \quad (28)$$

By Cauchy's mean-value theorem, for $j = i+1, \ldots, n$, $k = 1, \ldots, i$, there exist $\xi_1 \in (\lambda X_{(n-j+1)}, \lambda X_{(n-j+2)})$ and $\xi_2 \in (\lambda X_{(n-k+1)}, \lambda X_{(n-k+2)})$ such that

$$A(\lambda) = \frac{\lambda + \xi_1 + 2 - e_1^{\xi}\left[\lambda - (\lambda+1)\xi_1 + 2\right]}{\lambda - (\lambda+1)e_1^{\xi} + \xi_1 + 1},$$

$$B(\lambda) = \frac{\lambda + \xi_2 + 2 - e_2^{\xi}\left[\lambda - (\lambda+1)\xi_2 + 2\right]}{\lambda - (\lambda+1)e_2^{\xi} + \xi_2 + 1}. \quad (29)$$

Note that $\xi_1 < \xi_2$, by Lemma 5, $A(\lambda) - B(\lambda) > 0$, $(S_n/S_i)' > 0$, thus $\sum_{i=1}^{n-1}\log(S_n/S_i)$ is a strictly increasing function of λ, and equation $\sum_{i=1}^{n-1}\log(S_n/S_i) = t$ has a unique positive solution. $\square$

5. Joint Confidence Regions for λ and α

Let $X_1, X_2, \ldots, X_n$ form a sample from $GLD(\lambda, \alpha)$, and $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ are the order statistics. Let $Z_{(i)} = -\alpha\log[1 - e^{-\lambda X_{(n-i+1)}}(\lambda X_{(n-i+1)} + \lambda + 1)/(\lambda+1)] = -\alpha\log G(X_{(n-i+1)})$, $i = 1, \ldots, n$. Thus, $Z_{(1)} \leq Z_{(2)} \leq \cdots \leq Z_{(n)}$ are the first n-order statistics from the standard exponential distribution. By Lemma 2, $W_i = (n-i+1)(Z_{(i)} - Z_{(i-1)})$, $i = 1, 2, \ldots, n$, form a sample from standard exponential

distribution. Let $S_i = W_1 + \cdots + W_i$, $U_i = (S_i/S_{i+1})^i$, $i = 1, 2, \ldots, n-1$, and $U_n = W_1 + \cdots + W_n$. Hence

$$V = 2S_1 = 2W_1 = 2nZ_{(1)}$$

$$= -2n\alpha\log\left[1 - \frac{e^{-\lambda X_{(n)}}\left(\lambda X_{(n)} + \lambda + 1\right)}{\lambda + 1}\right]$$

$$\sim \chi^2(2),$$

$$U = 2\left(S_n - S_1\right) = 2\sum_{i=2}^{n}W_i$$

$$= 2\left[Z_{(1)} + \cdots + Z_{(n)} - nZ_{(1)}\right] \sim \chi^2(2n-2). \quad (30)$$

It is obvious that U and V are independent. Define

$$T_1 = \frac{U/(2n-2)}{V/2} = \frac{S_n - S_1}{(n-1)S_1} \sim F(2n-2, 2),$$

$$T_2 = U + V = 2S_n \sim \chi^2(2n). \quad (31)$$

We obtain that T_1 and T_2 are independent using the known bank-post office story in statistics.

Let $F_\gamma(v_1, v_2)$ denote the percentile of F distribution with left-tail probability γ and v_1 and v_2 degrees of freedom. Let $\chi_\gamma^2(v)$ denote the percentile of χ^2 distribution with left-tail probability γ and v degrees of freedom.

By using the pivotal variables T_1 and T_2, a joint confidence region for the two parameters λ and α can be constructed as follows.

Theorem 7 (method 1). *Let $X_1, X_2, \ldots, X_n$ form a sample from $GLD(\lambda, \alpha)$; then, based on the pivotal variables T_1 and T_2, a $100(1 - \gamma)\%$ joint confidence region for the two parameters (λ, α) is determined by the following inequalities:*

$$\lambda_L \leq \lambda \leq \lambda_U$$

$$\frac{\chi^2_{(1-\sqrt{1-\gamma})/2}(2n)}{-2\sum_{i=1}^n \log\left[1 - e^{-\lambda X_{(i)}}\left(\lambda X_{(i)} + \lambda + 1\right)/(\lambda + 1)\right]} \leq \alpha \quad (32)$$

$$\leq \frac{\chi^2_{(1+\sqrt{1-\gamma})/2}(2n)}{-2\sum_{i=1}^n \log\left[1 - e^{-\lambda X_{(i)}}\left(\lambda X_{(i)} + \lambda + 1\right)/(\lambda + 1)\right]},$$

where λ_L is the root of λ for the equation $T_1 = F_{(1-\sqrt{1-\gamma})/2}(2n - 2, 2)$ and λ_U is the root of λ for the equation $T_1 = F_{(1+\sqrt{1-\gamma})/2}(2n - 2, 2)$.

Proof. $T_1 = (1/(n-1))\log[1 - e^{-\lambda X_{(n)}}(\lambda X_{(n)} + \lambda + 1)/(\lambda + 1)] + \cdots + \log[1 - e^{-\lambda X_{(1)}}(\lambda X_{(1)} + \lambda + 1)/(\lambda + 1)] - n\log[1 - e^{-\lambda X_{(n)}}(\lambda X_{(n)} + \lambda + 1)/(\lambda + 1)]/n\log[1 - e^{-\lambda X_{(n)}}(\lambda X_{(n)} + \lambda + 1)/(\lambda + 1)]$ is a function of λ and does not depend on α. From Theorem 6, we have $\lim_{\lambda \to 0} T_1 = (1/(n-1))\lim_{\lambda \to 0}(S_n/S_1 - 1) = 0$, $\lim_{\lambda \to \infty} T_1 = (1/(n-1))\lim_{\lambda \to \infty}(S_n/S_1 - 1) = \infty$, and $T_1' = (1/(n-1))(S_n/S_i)' > 0$. Therefore, for any $t > 0$, equation $T_1 = t$ has a unique positive root of λ:

$$\begin{aligned}
1 - \gamma &= \sqrt{1-\gamma}\sqrt{1-\gamma} = P\left(F_{(1-\sqrt{1-\gamma})/2}(2n - 2, 2)\right.\\
&\leq T_1 \leq F_{(1+\sqrt{1-\gamma})/2}(2n - 2, 2)\right) P\left(\chi^2_{(1-\sqrt{1-\gamma})/2}(2n)\right.\\
&\leq T_2 \leq \chi^2_{(1+\sqrt{1-\gamma})/2}(2n)\right)\\
&= P\left(F_{(1-\sqrt{1-\gamma})/2}(2n - 2, 2) \leq T_1\right.\\
&\leq F_{(1+\sqrt{1-\gamma})/2}(2n - 2, 2), \chi^2_{(1-\sqrt{1-\gamma})/2}(2n) \leq T_2\\
&\leq \chi^2_{(1+\sqrt{1-\gamma})/2}(2n)\right) = P\left(\lambda_L \leq \lambda\right.\\
&\leq \lambda_U, \frac{\chi^2_{(1-\sqrt{1-\gamma})/2}(2n)}{-2\sum_{i=1}^n \log G(X_{(i)})} \leq \alpha\\
&\leq \frac{\chi^2_{(1+\sqrt{1-\gamma})/2}(2n)}{-2\sum_{i=1}^n \log G(X_{(i)})}\right).
\end{aligned} \quad (33)$$

$\square$

On the other hand, by Lemma 3, we have

$$\begin{aligned}
T_3 &= -2\sum_{i=1}^{n-1}\log U_i = -2\sum_{i=1}^{n-1} i\log\left(\frac{S_i}{S_{i+1}}\right)\\
&= 2\sum_{i=1}^{n-1}\log\left(\frac{S_n}{S_i}\right) \sim \chi^2(2n - 2).
\end{aligned} \quad (34)$$

T_2 and T_3 are also independent. By using the pivotal variables T_2 and T_3, a joint confidence region for the two parameters λ and α can be constructed as follows.

Theorem 8 (method 2). *Let $X_1, X_2, \ldots, X_n$ form a sample from $GLD(\lambda, \alpha)$; then, based on the pivotal variables T_2 and T_3, a $100(1 - \gamma)\%$ joint confidence region for the two parameters (λ, α) is determined by the following inequalities:*

$$\lambda_L^* \leq \lambda \leq \lambda_U^*$$

$$\frac{\chi^2_{(1-\sqrt{1-\gamma})/2}(2n)}{-2\sum_{i=1}^n \log\left[1 - e^{-\lambda X_{(i)}}\left(\lambda X_{(i)} + \lambda + 1\right)/(\lambda + 1)\right]} \leq \alpha \quad (35)$$

$$\leq \frac{\chi^2_{(1+\sqrt{1-\gamma})/2}(2n)}{-2\sum_{i=1}^n \log\left[1 - e^{-\lambda X_{(i)}}\left(\lambda X_{(i)} + \lambda + 1\right)/(\lambda + 1)\right]},$$

where λ_L^* is the root of λ for the equation $T_3 = \chi^2_{(1-\sqrt{1-\gamma})/2}(2n - 2)$ and λ_U^* is the root of λ for the equation $T_3 = \chi^2_{(1+\sqrt{1-\gamma})/2}(2n - 2)$.

Proof. $T_3 = 2\sum_{i=1}^{n-1}\log(S_n/S_i)$ is a function of λ and does not depend on α. From Theorem 6, for any $s > 0$, equation $T_3 = s$ has a unique positive root of λ:

$$\begin{aligned}
1 - \gamma &= \sqrt{1-\gamma}\sqrt{1-\gamma} = P\left(\chi^2_{(1-\sqrt{1-\gamma})/2}(2n - 2) \leq T_3\right.\\
&\leq \chi^2_{(1+\sqrt{1-\gamma})/2}(2n - 2)\right) P\left(\chi^2_{(1-\sqrt{1-\gamma})/2}(2n) \leq T_2\right.\\
&\leq \chi^2_{(1+\sqrt{1-\gamma})/2}(2n)\right) = P\left(\chi^2_{(1-\sqrt{1-\gamma})/2}(2n - 2) \leq T_3\right.\\
&\leq \chi^2_{(1+\sqrt{1-\gamma})/2}(2n - 2), \chi^2_{(1-\sqrt{1-\gamma})/2}(2n) \leq T_2\\
&\leq \chi^2_{(1+\sqrt{1-\gamma})/2}(2n)\right) = P\left(\lambda_L^* \leq \lambda\right.\\
&\leq \lambda_U^*, \frac{\chi^2_{(1-\sqrt{1-\gamma})/2}(2n)}{-2\sum_{i=1}^n \log G(X_{(i)})} \leq \alpha\\
&\leq \frac{\chi^2_{(1+\sqrt{1-\gamma})/2}(2n)}{-2\sum_{i=1}^n \log G(X_{(i)})}\right).
\end{aligned} \quad (36)$$

$\square$

6. Simulation Study

6.1. Comparison of the Four Estimation Methods. In this section, we conduct simulations to compare the performances of the MIMEs, IMEs, MLEs, and MOMs mainly with respect to their biases and mean squared errors (MSEs), for various sample sizes and for various true parametric values.

The random data X from the $GLD(\lambda, \alpha)$ distribution can be generated as follows:

$$X = \frac{-\lambda - 1 - W\left(e^{-\lambda-1}(\lambda + 1)\left(U^{1/\alpha} - 1\right)\right)}{\lambda}, \quad (37)$$

TABLE 1: Average relative estimates and MSEs of α.

n	Methods	$\alpha = 2.0$	$\alpha = 2.5$	$\alpha = 3.0$	$\alpha = 3.5$	$\alpha = 4.0$
30	MOM	1.2705 (0.4298)	1.2350 (0.3510)	1.2965 (0.5115)	1.2770 (0.4460)	1.3522 (0.6256)
	MLE	1.1310 (0.1459)	1.1703 (0.2029)	1.1590 (0.2240)	1.1693 (0.2267)	1.1911 (0.2888)
	IME	1.0956 (0.1267)	1.1292 (0.1728)	1.1154 (0.1906)	1.1234 (0.1930)	1.1395 (0.2400)
	MIME	1.0482 (0.1063)	1.0755 (0.1420)	1.0590 (0.1570)	1.0635 (0.1572)	1.0754 (0.1939)
40	MOM	1.1791 (0.2212)	1.1954 (0.2262)	1.2046 (0.2693)	1.2299 (0.3163)	1.2536 (0.3961)
	MLE	1.0878 (0.0934)	1.1175 (0.1220)	1.1164 (0.1237)	1.1359 (0.1596)	1.1034 (0.1464)
	IME	1.0631 (0.0838)	1.0894 (0.1098)	1.0858 (0.1089)	1.1025 (0.1412)	1.0703 (0.1315)
	MIME	1.0294 (0.0739)	1.0516 (0.0951)	1.0456 (0.0937)	1.0591 (0.1207)	1.0267 (0.1139)
50	MOM	1.1363 (0.1557)	1.1396 (0.1767)	1.1570 (0.2069)	1.1747 (0.2296)	1.1842 (0.2652)
	MLE	1.0746 (0.0701)	1.0971 (0.1027)	1.1064 (0.1102)	1.0979 (0.1059)	1.1064 (0.1289)
	IME	1.0553 (0.0648)	1.0747 (0.0922)	1.0823 (0.0994)	1.0726 (0.0960)	1.0783 (0.1145)
	MIME	1.0288 (0.0584)	1.0451 (0.0822)	1.0505 (0.0879)	1.0394 (0.0849)	1.0432 (0.1009)
60	MOM	1.1233 (0.1227)	1.1281 (0.1249)	1.1480 (0.1679)	1.1312 (0.1608)	1.1511 (0.1892)
	MLE	1.0592 (0.0587)	1.0646 (0.0601)	1.0663 (0.0695)	1.0832 (0.0849)	1.0877 (0.0911)
	IME	1.0431 (0.0546)	1.0468 (0.0548)	1.0470 (0.0637)	1.0625 (0.0786)	1.0656 (0.0827)
	MIME	1.0214 (0.0502)	1.0232 (0.0499)	1.0218 (0.0580)	1.0354 (0.0709)	1.0371 (0.0743)
80	MOM	1.0954 (0.0843)	1.0836 (0.0856)	1.0984 (0.0974)	1.1136 (0.1105)	1.1220 (0.1224)
	MLE	1.0579 (0.0426)	1.0556 (0.0480)	1.0624 (0.0553)	1.0533 (0.0515)	1.0700 (0.0651)
	IME	1.0459 (0.0400)	1.0426 (0.0452)	1.0485 (0.0517)	1.0385 (0.0484)	1.0539 (0.0612)
	MIME	1.0297 (0.0372)	1.0251 (0.0421)	1.0296 (0.0478)	1.0189 (0.0449)	1.0329 (0.0563)
100	MOM	1.0704 (0.0659)	1.0704 (0.0736)	1.0835 (0.0859)	1.0763 (0.0888)	1.0780 (0.0851)
	MLE	1.0332 (0.0299)	1.0383 (0.0344)	1.0504 (0.0428)	1.0344 (0.0370)	1.0478 (0.0482)
	IME	1.0244 (0.0288)	1.0286 (0.0330)	1.0393 (0.0407)	1.0230 (0.0356)	1.0353 (0.0459)
	MIME	1.0119 (0.0274)	1.0149 (0.0313)	1.0245 (0.0382)	1.0077 (0.0337)	1.0189 (0.0432)

where U follows uniform distribution over $[0, 1]$ and $W(a)$ giving the principal solution for w in $a = we^w$ is pronounced as Lambert W function; see Jodrá [16].

We obtain $\widehat{\lambda}_{\text{MLE}}$ by solving (8) and $\widehat{\alpha}_{\text{MLE}}$ by (7). $\widehat{\lambda}_{\text{MOM}}$ and $\widehat{\alpha}_{\text{MOM}}$ can be obtained by solving (11) and (12) simultaneously. $\widehat{\lambda}_{\text{IME}}$ and $\widehat{\lambda}_{\text{MIME}}$ can be obtained by solving (20) and (21), respectively. $\widehat{\alpha}_{\text{IME}}$ and $\widehat{\alpha}_{\text{MIME}}$ can be obtained from (15).

We consider sample sizes $n = 30, 40, 50, 60, 80, 100$ and $\alpha = 2.0, 2.5, 3.0, 3.5, 4.0$. We take $\lambda = 2$ in all our computations. For each combination of sample size n and parameter α, we generate a sample of size n from $\text{GLD}(\lambda = 2, \alpha)$ and estimate the parameters λ and α by the MLE, MOM, IME, and MIME methods. The average values of $\widehat{\alpha}/\alpha$ and $\widehat{\lambda}/2$ as well as the corresponding MSEs over 1000 replications are computed and reported.

Table 1 reports the average values of $\widehat{\alpha}/\alpha$ and the corresponding MSE is reported within parenthesis. Figures 1(a), 1(b), 1(c), and 1(d) show the relative biases and the MSEs of the four estimators of α for sample sizes $n = 40$ and $n = 80$. Figures 1(e) and 1(f) show the relative biases and the MSEs of the four estimators of α for $\alpha = 3.0$. The other cases are similar.

Table 2 reports the average values of $\widehat{\lambda}/\lambda = \widehat{\lambda}/2$ and the corresponding MSE is reported within parenthesis. Figures 2(a), 2(b), 2(c), and 2(d) show the relative biases and the MSEs of the four estimators of λ for sample sizes $n = 40$ and $n = 80$. Figures 2(e) and 2(f) show the relative biases and the MSEs
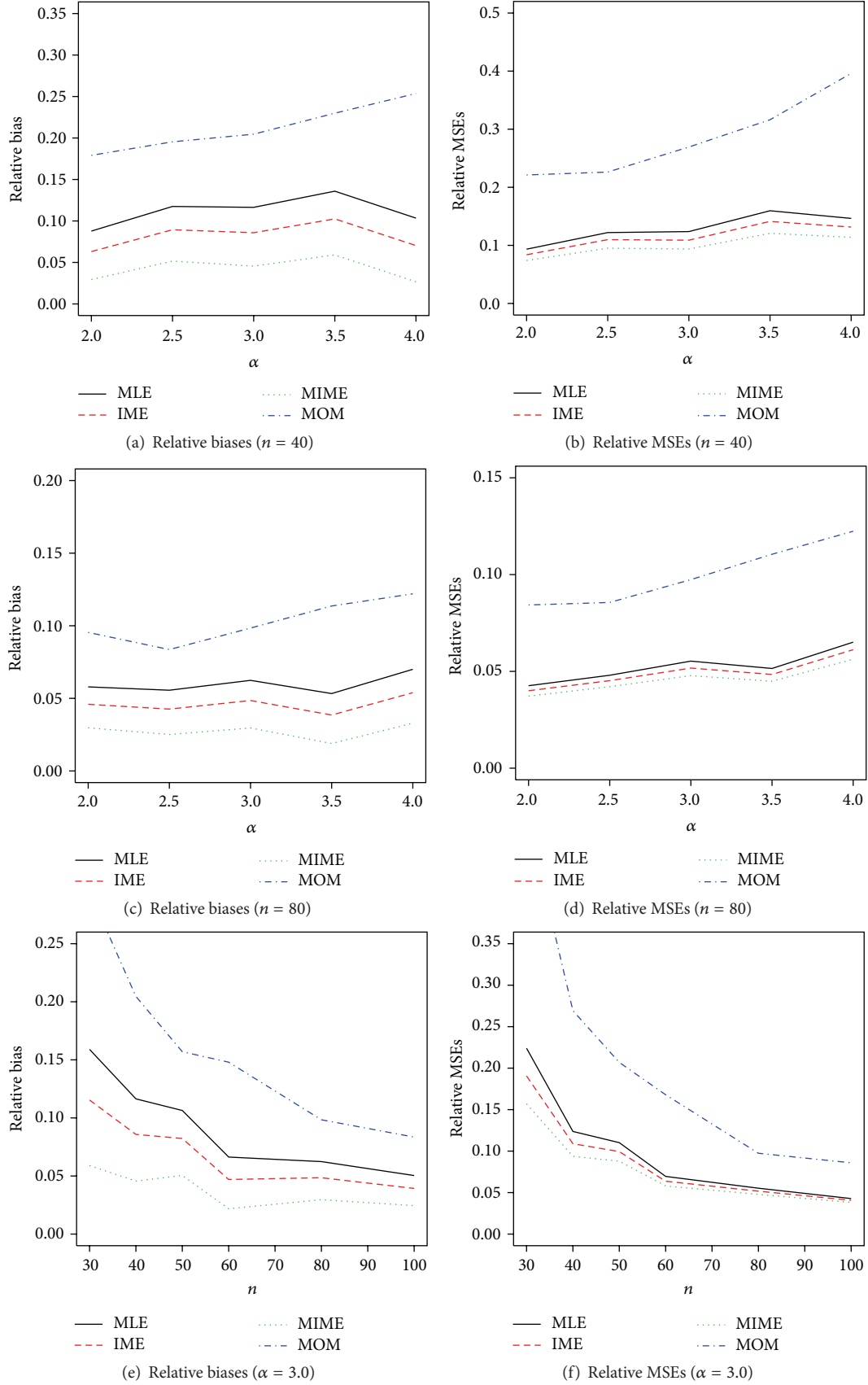
of the four estimators of λ for $\alpha = 3.0$. The other cases are similar.

From Tables 1 and 2, it is observed that for the four methods the average relative biases and the average relative MSEs decrease as sample size n increases as expected. The asymptotic unbiasedness of all the estimators is verified. The average MSEs of $\widehat{\alpha}/\alpha$ and $\widehat{\lambda}/\lambda = \widehat{\lambda}/2$ depend on the parameter α. For the four methods, the average relative MSEs of $\widehat{\lambda}/2$ decrease as α goes up. The average relative MSEs of $\widehat{\alpha}/\alpha$ increase as α goes up. Considering only MSEs, we can observe that the estimation of α's is more accurate for smaller values while the estimation of λ's is more accurate for larger values of α. MOM, MLE, and IME overestimate both of the two parameters α and λ. MIME overestimates only α.

As far as the biases and MSEs are concerned, it is clear that MIME works the best in all the cases considered for estimating the two parameters. Its performance is followed by IME, MLE, and MOM, especially for small sample sizes. The four methods are close for larger sample sizes.

Considering all the points, MIME is recommended for estimating both parameters of the $\text{GLD}(\lambda, \alpha)$ distribution. MOM is not suggested.

6.2. Comparison of the Two Joint Confidence Regions. In Section 5, two methods to construct the confidence regions of the two parameters λ and α are proposed. In this section, we conduct simulations to compare the two methods.

(a) Relative biases ($n = 40$)

(b) Relative MSEs ($n = 40$)

(c) Relative biases ($n = 80$)

(d) Relative MSEs ($n = 80$)

(e) Relative biases ($\alpha = 3.0$)

(f) Relative MSEs ($\alpha = 3.0$)

FIGURE 1: Average relative biases and MSEs of α.

(a) Relative biases ($n = 40$)

(b) Relative MSEs ($n = 40$)

(c) Relative biases ($n = 80$)

(d) Relative MSEs ($n = 80$)

(e) Relative biases ($\alpha = 3.0$)

(f) Relative MSEs ($\alpha = 3.0$)
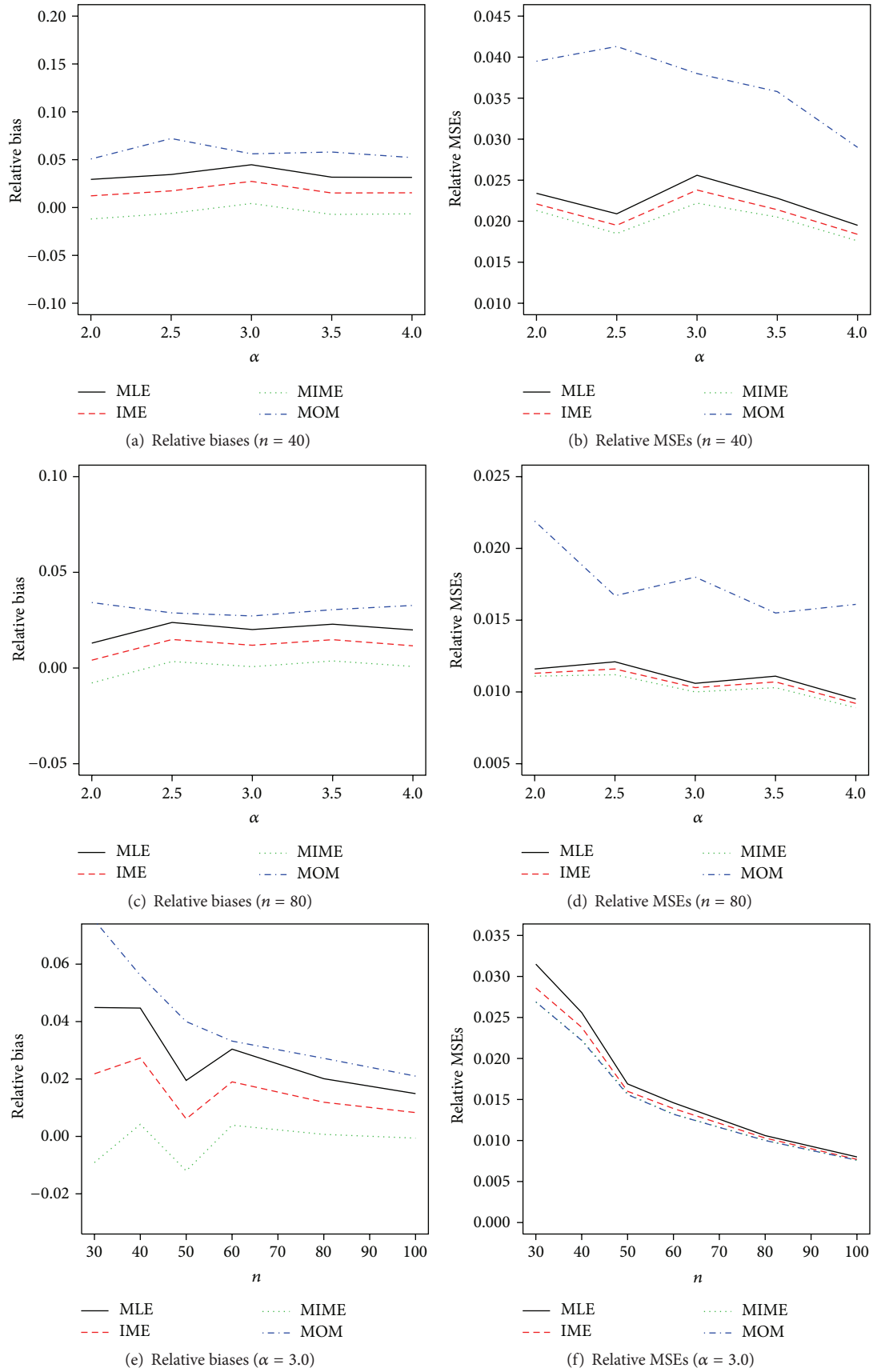
Figure 2: Average relative biases and MSEs of λ.

TABLE 2: Average relative estimates and MSEs of λ.

n	Methods	$\alpha = 2.0$	$\alpha = 2.5$	$\alpha = 3.0$	$\alpha = 3.5$	$\alpha = 4.0$
30	MOM	1.0778 (0.0551)	1.0795 (0.0524)	1.0760 (0.0483)	1.0794 (0.0495)	1.0699 (0.0454)
	MLE	1.0596 (0.0381)	1.0516 (0.0357)	1.0449 (0.0315)	1.0529 (0.0335)	1.0460 (0.0315)
	IME	1.0356 (0.0343)	1.0276 (0.0323)	1.0218 (0.0286)	1.0302 (0.0308)	1.0233 (0.0288)
	MIME	1.0024 (0.0314)	0.9957 (0.0299)	0.9909 (0.0269)	0.9997 (0.0285)	0.9934 (0.0270)
40	MOM	1.0507 (0.0395)	1.0721 (0.0413)	1.0561 (0.0380)	1.0580 (0.0358)	1.0522 (0.0290)
	MLE	1.0294 (0.0234)	1.0345 (0.0209)	1.0447 (0.0256)	1.0317 (0.0228)	1.0315 (0.0195)
	IME	1.0122 (0.0221)	1.0174 (0.0195)	1.0273 (0.0238)	1.0152 (0.0214)	1.0154 (0.0184)
	MIME	0.9880 (0.0213)	0.9939 (0.0185)	1.0042 (0.0222)	0.9928 (0.0205)	0.9934 (0.0176)
50	MOM	1.0546 (0.0319)	1.0427 (0.0288)	1.0400 (0.0266)	1.0412 (0.0249)	1.0424 (0.0254)
	MLE	1.0338 (0.0202)	1.0307 (0.0198)	1.0195 (0.0169)	1.0292 (0.0179)	1.0327 (0.0162)
	IME	1.0197 (0.0190)	1.0165 (0.0187)	1.0061 (0.0160)	1.0161 (0.0169)	1.0193 (0.0154)
	MIME	1.0003 (0.0181)	0.9978 (0.0178)	0.9880 (0.0156)	0.9983 (0.0162)	1.0017 (0.0146)
60	MOM	1.0402 (0.0280)	1.0432 (0.0244)	1.0332 (0.0224)	1.0417 (0.0225)	1.0339 (0.0217)
	MLE	1.0230 (0.0159)	1.0181 (0.0151)	1.0304 (0.0146)	1.0226 (0.0140)	1.0247 (0.0129)
	IME	1.0109 (0.0152)	1.0067 (0.0145)	1.0190 (0.0139)	1.0113 (0.0134)	1.0132 (0.0122)
	MIME	0.9948 (0.0147)	0.9913 (0.0142)	1.0039 (0.0132)	0.9965 (0.0129)	0.9987 (0.0118)
80	MOM	1.0342 (0.0219)	1.0288 (0.0167)	1.0272 (0.0180)	1.0305 (0.0155)	1.0327 (0.0161)
	MLE	1.0130 (0.0116)	1.0238 (0.0121)	1.0201 (0.0106)	1.0229 (0.0111)	1.0199 (0.0095)
	IME	1.0041 (0.0113)	1.0149 (0.0116)	1.0119 (0.0103)	1.0148 (0.0107)	1.0116 (0.0092)
	MIME	0.9922 (0.0111)	1.0034 (0.0112)	1.0007 (0.0100)	1.0037 (0.0103)	1.0008 (0.0089)
100	MOM	1.0269 (0.0154)	1.0255 (0.0153)	1.0210 (0.0125)	1.0144 (0.0120)	1.0236 (0.0117)
	MLE	1.0129 (0.0087)	1.0135 (0.0095)	1.0149 (0.0080)	1.0150 (0.0080)	1.0133 (0.0075)
	IME	1.0059 (0.0085)	1.0067 (0.0092)	1.0083 (0.0077)	1.0084 (0.0077)	1.0069 (0.0073)
	MIME	0.9964 (0.0083)	0.9975 (0.0090)	0.9994 (0.0076)	0.9996 (0.0076)	0.9983 (0.0072)

First, we assess the precisions of the two methods of interval estimators for the parameter λ. We take sample sizes $n = 30, 40, 50, 60, 80, 100$ and $\alpha = 2.0, 2.5, 3.0, 3.5, 4.0$. We take $\lambda = 2$ in all our computations. For each combination of sample size n and parameter α, we generate a sample of size n from GLD$(\lambda = 2, \alpha)$ and estimate the parameter λ by the two proposed methods (32) and (35).

The mean widths as well as the coverage rates over 1000 replications are computed and reported. Here the coverage rate is defined as the rate of the confidence intervals that contain the true value $\lambda = 2$ among these 1,000 confidence intervals. The results are reported in Table 3.

It is observed that the mean widths of the intervals decrease as sample sizes n increase as expected. The mean widths of the intervals decrease as the parameter α increases. The coverage rates of the two methods are close to the nominal level 0.95. Considering the mean widths, the interval estimate of λ obtained in method 2 performs better than that obtained in method 1. Method 2 for constructing the interval estimate of λ is recommended.

Next we consider the two joint confidence regions and the empirical coverage rates and expected areas. The results of the methods for constructing joint confidence regions for (λ, α) with confidence level $\gamma = 0.95$ are reported in Table 4.

We can find that the mean areas of the joint regions decrease as sample sizes n increase as expected. The mean

areas of the joint regions increase as the parameter α increases. The coverage rates of the two methods are close to the nominal level 0.95. Considering the mean areas, the joint region of (λ, α) obtained in method 2 performs better than that obtained in method 1. Method 2 is recommended.

7. Real Illustrative Example

In this section, we consider a real lifetime data set (Gross and Clark [17]) and it shows the relief times of twenty patients receiving an analgesic. The dataset has been previously analyzed by Bain and Engelhardt [18], Kumar and Dharmaja [19], Nadarajah et al. [11], and so forth. The relief times in hours are shown as follows:

$$1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3,$$
$$1.7, 2.3, 1.6, 2. \tag{38}$$

Nadarajah et al. [11] fit the data with generalized Lindley distribution and showed that it can be a better model than those based on the gamma, lognormal, and the Weibull distributions. The MLEs of the parameters are $\widehat{\lambda}_{\text{MLE}} = 2.5395$ and $\widehat{\alpha}_{\text{MLE}} = 27.8766$ with log-likelihood value -16.4044. The Kolmogorov-Smirnov distance and its corresponding p value

TABLE 3: Results of the methods for constructing intervals for λ with confidence level 0.95.

n	Methods		$\alpha = 2.0$	$\alpha = 2.5$	$\alpha = 3.0$	$\alpha = 3.5$	$\alpha = 4.0$
30	(1)	Mean width	2.4282	2.4056	2.3482	2.3093	2.2885
		Coverage rate	0.947	0.947	0.941	0.944	0.951
	(2)	Mean width	1.3872	1.3424	1.2958	1.2705	1.2539
		Coverage rate	0.948	0.942	0.955	0.943	0.952
40	(1)	Mean width	2.2463	2.2405	2.2048	2.1733	2.1644
		Coverage rate	0.945	0.95	0.947	0.957	0.958
	(2)	Mean width	1.1937	1.1545	1.1209	1.0939	1.0828
		Coverage rate	0.947	0.938	0.938	0.958	0.948
50	(1)	Mean width	2.1506	2.107	2.0635	2.0356	2.0293
		Coverage rate	0.953	0.945	0.936	0.954	0.949
	(2)	Mean width	1.0566	1.0174	0.9953	0.9732	0.9563
		Coverage rate	0.953	0.945	0.943	0.947	0.95
60	(1)	Mean width	2.0555	2.0197	1.98	1.9502	1.9397
		Coverage rate	0.953	0.952	0.955	0.948	0.951
	(2)	Mean width	0.9627	0.9331	0.9021	0.8856	0.8692
		Coverage rate	0.94	0.957	0.952	0.942	0.952
80	(1)	Mean width	1.9277	1.8859	1.8485	1.8418	1.8412
		Coverage rate	0.958	0.959	0.956	0.961	0.941
	(2)	Mean width	0.831	0.7995	0.777	0.7599	0.7521
		Coverage rate	0.952	0.947	0.942	0.951	0.951
100	(1)	Mean width	1.8251	1.7883	1.7849	1.7729	1.7633
		Coverage rate	0.954	0.948	0.951	0.946	0.956
	(2)	Mean width	0.7423	0.7119	0.6947	0.6804	0.6684
		Coverage rate	0.937	0.961	0.946	0.948	0.947
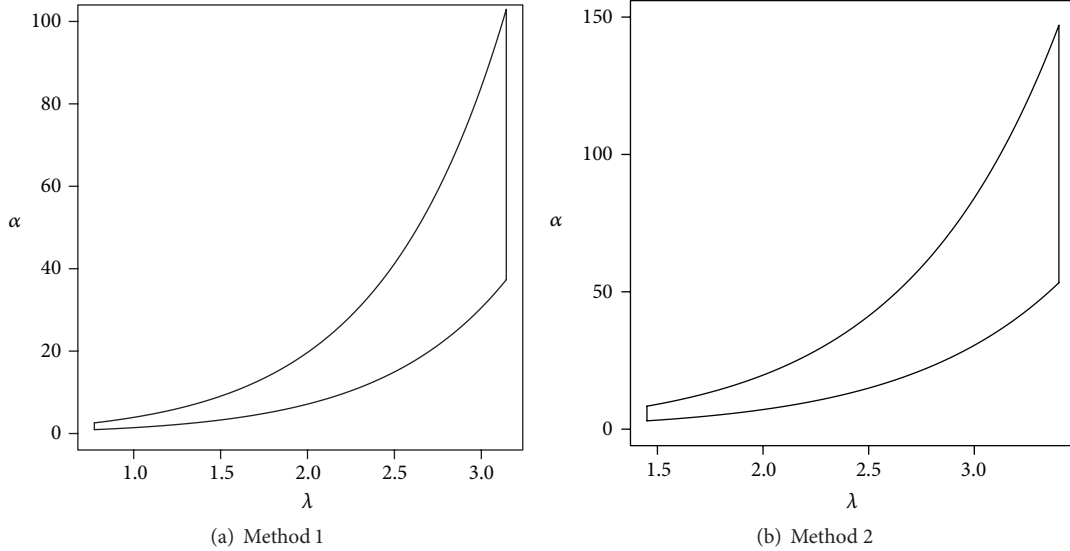
(a) Method 1

(b) Method 2

FIGURE 3: The 95% joint confidence region of (λ, α).

are $D = 0.1377$ and $p = 0.7941$, respectively. The MOMs of the parameters are $\widehat{\lambda}_{\mathrm{MOM}} = 2.1042$ and $\widehat{\alpha}_{\mathrm{MOM}} = 13.1280$.

Using the methods proposed in Section 3, we obtain the following estimates:

$$\widehat{\lambda}_{\mathrm{IME}} = 2.3687,$$

$$\widehat{\alpha}_{\mathrm{IME}} = 21.7428,$$

$$\widehat{\lambda}_{\mathrm{MIME}} = 2.2671,$$

$$\widehat{\alpha}_{\mathrm{MIME}} = 18.7269. \tag{39}$$

TABLE 4: Results of the methods for constructing joint confidence regions for (λ, α) with confidence level $\gamma = 0.95$.

n	Methods		$\alpha = 2.0$	$\alpha = 2.5$	$\alpha = 3.0$	$\alpha = 3.5$	$\alpha = 4.0$
30	(1)	Mean area	8.0701	10.5512	15.5331	17.6929	21.8188
		Coverage rate	0.96	0.949	0.949	0.952	0.959
	(2)	Mean area	3.0543	3.7328	4.7094	5.2673	6.1423
		Coverage rate	0.964	0.956	0.95	0.949	0.952
40	(1)	Mean area	6.2578	8.0063	10.1501	12.8546	15.722
		Coverage rate	0.942	0.963	0.962	0.942	0.942
	(2)	Mean area	2.211	2.6583	3.1734	3.6342	4.174
		Coverage rate	0.949	0.951	0.956	0.948	0.953
50	(1)	Mean area	4.7221	6.71	8.016	10.0739	12.373
		Coverage rate	0.951	0.935	0.958	0.944	0.955
	(2)	Mean area	1.6762	2.0329	2.404	2.798	3.201
		Coverage rate	0.95	0.931	0.956	0.959	0.947
60	(1)	Mean area	4.3159	5.4371	7.1443	8.3226	10.2957
		Coverage rate	0.954	0.939	0.953	0.94	0.951
	(2)	Mean area	1.4017	1.6443	1.9985	2.2489	2.5635
		Coverage rate	0.954	0.943	0.952	0.944	0.96
80	(1)	Mean area	3.1997	4.2806	5.3322	6.392	7.7788
		Coverage rate	0.951	0.956	0.951	0.957	0.948
	(2)	Mean area	1.0114	1.2325	1.4396	1.6863	1.8538
		Coverage rate	0.947	0.956	0.955	0.951	0.942
100	(1)	Mean area	2.6803	3.4994	4.418	5.4509	6.3339
		Coverage rate	0.951	0.947	0.965	0.947	0.951
	(2)	Mean area	0.7938	0.9673	1.1306	1.3265	1.4548
		Coverage rate	0.947	0.945	0.962	0.943	0.964

In addition, based on method 1, the 95% joint confidence region for the parameters (λ, α) is given by the following inequalities:

$$0.7736 \le \lambda \le 3.1505$$

$$\frac{-11.3532}{\sum_{i=1}^{n} \log\left[1 - e^{-\lambda X_{(i)}}\left(\lambda X_{(i)} + \lambda + 1\right)/(\lambda + 1)\right]} \le \alpha \quad (40)$$

$$\le \frac{-31.3028}{\sum_{i=1}^{n} \log\left[1 - e^{-\lambda X_{(i)}}\left(\lambda X_{(i)} + \lambda + 1\right)/(\lambda + 1)\right]}.$$

Based on method 2, the 95% joint confidence region for the parameters (λ, α) is given by the following inequalities:

$$1.4503 \le \lambda \le 3.4078$$

$$\frac{-11.3532}{\sum_{i=1}^{n} \log\left[1 - e^{-\lambda X_{(i)}}\left(\lambda X_{(i)} + \lambda + 1\right)/(\lambda + 1)\right]} \le \alpha \quad (41)$$

$$\le \frac{-31.3028}{\sum_{i=1}^{n} \log\left[1 - e^{-\lambda X_{(i)}}\left(\lambda X_{(i)} + \lambda + 1\right)/(\lambda + 1)\right]}.$$

Figures 3(a) and 3(b) show the 95% joint confidence regions of (λ, α).

Considering the widths of λ, method 2 is suggested.

8. Conclusion

In this paper, we study the problem of estimating the two parameters of the generalized Lindley distribution introduced by Nadarajah et al. [11]. We propose the inverse moment estimator and modified inverse moment estimator and study their statistical properties. The existence and uniqueness of inverse moment and modified inverse moment estimates of the parameters are proved. Monte Carlo simulations are used to compare their performances. We also investigate the methods for constructing joint confidence regions for the two parameters and study their performances.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] D. V. Lindley, "Fiducial distributions and Bayes' theorem," *Journal of the Royal Statistical Society Series B: Methodological*, vol. 20, pp. 102–107, 1958.

[2] M. E. Ghitany, B. Atieh, and S. Nadarajah, "Lindley distribution and its application," *Mathematics and Computers in Simulation*, vol. 78, no. 4, pp. 493–506, 2008.

[3] J. Mazucheli and J. A. Achcar, "The lindley distribution applied to competing risks lifetime data," *Computer Methods and Programs in Biomedicine*, vol. 104, no. 2, pp. 188–192, 2011.

[4] H. Krishna and K. Kumar, "Reliability estimation in Lindley distribution with progressively type II right censored sample," *Mathematics & Computers in Simulation*, vol. 82, no. 2, pp. 281–294, 2011.

[5] D. K. Al-Mutairi, M. E. Ghitany, and D. Kundu, "Inferences on stress-strength reliability from Lindley distributions," *Communications in Statistics—Theory and Methods*, vol. 42, no. 8, pp. 1443–1463, 2013.

[6] M. Sankaran, "The discrete poisson-lindley distribution," *Biometrics*, vol. 26, no. 1, pp. 145–149, 1970.

[7] M. E. Ghitany, D. K. Al-Mutairi, and S. Nadarajah, "Zero-truncated Poisson-Lindley distribution and its application," *Mathematics and Computers in Simulation*, vol. 79, no. 3, pp. 279–287, 2008.

[8] H. S. Bakouch, B. M. Al-Zahrani, A. A. Al-Shomrani, V. A. Marchi, and F. Louzada, "An extended lindley distribution," *Journal of the Korean Statistical Society*, vol. 41, no. 1, pp. 75–85, 2012.

[9] R. Shanker, S. Sharma, and R. Shanker, "A two-parameter lindley distribution for modeling waiting and survival times data," *Applied Mathematics*, vol. 4, no. 2, pp. 363–368, 2013.

[10] M. E. Ghitany, D. K. Al-Mutairi, N. Balakrishnan, and L. J. Al-Enezi, "Power Lindley distribution and associated inference," *Computational Statistics & Data Analysis*, vol. 64, pp. 20–33, 2013.

[11] S. Nadarajah, H. S. Bakouch, and R. Tahmasbi, "A generalized Lindley distribution," *Sankhya B*, vol. 73, no. 2, pp. 331–359, 2011.

[12] S. K. Singh, U. Singh, and V. K. Sharma, "Bayesian estimation and prediction for the generalized lindley distribution under assymetric loss function," *Hacettepe Journal of Mathematics and Statistics*, vol. 43, no. 4, pp. 661–678, 2014.

[13] S. K. Singh, U. Singh, and V. K. Sharma, "Expected total test time and Bayesian estimation for generalized Lindley distribution under progressively Type-II censored sample where removals follow the beta-binomial probability law," *Applied Mathematics and Computation*, vol. 222, pp. 402–419, 2013.

[14] B. Arnold, N. Balakrishnan, and H. Nagaraja, *A First Course in Order Statistics*, Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa, USA, 1992.

[15] B. X. Wang, "Statistical inference of Weibull distribution," *Chinese Journal of Applied Probability & Statistics*, vol. 8, no. 4, pp. 357–364, 1992.

[16] P. Jodrá, "Computer generation of random variables with Lindley or Poisson–Lindley distribution via the Lambert W function," *Mathematics and Computers in Simulation*, vol. 81, no. 4, pp. 851–859, 2010.

[17] A. J. Gross and V. A. Clark, *Survival Distributions: Reliability Applications in the Biomedical Sciences*, vol. 11, Wiley, New York, NY, USA, 1975.

[18] L. J. Bain and M. Engelhardt, "Probability of correct selection of weibull versus gamma based on livelihood ratio," *Communications in Statistics—Theory and Methods*, vol. 9, no. 4, pp. 375–381, 2007.

[19] C. S. Kumar and S. H. Dharmaja, "On some properties of Kies distribution," *Metron*, vol. 72, no. 1, pp. 97–122, 2014.