Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2016, Article ID 9617283, 11 pages
http://dx.doi.org/10.1155/2016/9617283

Research Article

Adaptive Integral Sliding Mode Stabilization of Nonholonomic Drift-Free Systems

Waseem Abbasi[1,2] and Fazal ur Rehman[1]

[1] *Department of Electrical Engineering, Capital University of Science and Technology (CUST), Kahuta Road, Express Highway, Islamabad 44000, Pakistan*
[2] *Department of Electrical Engineering, The University of Lahore (UOL), Japan Road, Express Highway, Islamabad 44000, Pakistan*

Correspondence should be addressed to Waseem Abbasi; waseemabbasi97@gmail.com

Received 23 June 2016; Revised 28 September 2016; Accepted 19 October 2016

Academic Editor: Muhammad N. Akram

This article presents adaptive integral sliding mode control algorithm for the stabilization of nonholonomic drift-free systems. First the system is transformed, by using input transform, into a special structure containing a nominal part and some unknown terms which are computed adaptively. The transformed system is then stabilized using adaptive integral sliding mode control. The stabilizing controller for the transformed system is constructed that consists of the nominal control plus a compensator control. The compensator control and the adaptive laws are derived on the basis of Lyapunov stability theory. The proposed control algorithm is applied to three different nonholonomic drift-free systems: the unicycle model, the front wheel car model, and the mobile robot with trailer model. The controllability Lie algebra of the unicycle model contains Lie brackets of depth one, the model of a front wheel car contains Lie brackets of depths one and two, and the model of a mobile robot with trailer contains Lie brackets of depths one, two, and three. The effectiveness of the proposed control algorithm is verified through numerical simulations.

1. Introduction

Designing feedback control laws for the stabilization of mechanical control systems has been an interesting subject for researchers in the field of control theory. These systems have attracted intensive attention from the control community because of their wide practical applications in robotics, industry, and automobiles. Due to mechanical design and configuration, these systems are classified into two categories: holonomic and nonholonomic. In holonomic systems, the control input degrees are equal to total degrees of freedom, whereas, nonholonomic systems have less controllable degrees of freedom as compared to total degrees of freedom and have restricted mobility due to the presence of nonholonomic constraints. Roger Brockett showed that the nonholonomic systems cannot be stabilized by continuous static state feedback laws [1]. Later on Murray et al. showed that the dependence of the stabilizing control on time is essential [2].

To solve this problem, different control approaches have been presented in the literature. A detailed survey of stabilization of nonholonomic systems can be found in [3] and a survey of underactuated mechanical systems is given in [4]. In the literature, several control techniques have been developed for stabilization of nonholonomic systems. Some of these include discontinuous time-invariant techniques [5–8], time-varying techniques [9–12], adaptive techniques [13, 14], and sliding mode control technique [15–20]. Sliding mode control (SMC) is a special nonlinear control technique. The objective of the SMC technique is to force the system states to a certain surface, known as the sliding manifold. Once the surface is reached, the system is forced to remain on it thereafter.

The main disadvantage of the SMC is the requirement of discontinuous control law across the sliding manifold. In practical systems, this leads to an undesirable phenomenon called chattering. The closed loop dynamics of the system in

SMC depends only on the design parameters of the switching sliding manifold. Sliding mode control also offers several advantages such as simplicity, fast response, and robustness to external disturbance and parameter variation.

Our objective in this article is to propose a scheme for the construction of stabilizing control for nonholonomic mechanical systems. The suggested sliding mode controller can stabilize systems, which do not fulfill Brockett's necessary conditions, as the sliding mode control is inherently discontinuous. Since sliding mode control is insensitive towards model errors, parametric uncertainties, and other disturbances; therefore, it is extensively used. Sliding surface will show system's behavior when the system reaches the sliding manifold [21–24].

The integral sliding mode control guarantees the robustness of the motion in the whole state space [25, 26] because of eliminating the reaching phase. Since the reaching phase is eliminated, therefore the robustness of the system can be guaranteed throughout the system response, starting from the initial time instance. The integral sliding mode control combines the nominal control that stabilizes the nominal system and a discontinuous control that rejects the uncertainty.

The control algorithm presented in this paper is general and applicable to a large class of nonholonomic control systems without drift. The proposed algorithm is applied to three different nonholonomic drift-free systems: the unicycle model, the front wheel car model, and the mobile robot with trailer model. The effectiveness of the proposed algorithm is verified through numerical simulations.

The rest of the article is organized as follows. Section 2 presents problem formulation. Section 3 presents the proposed control methodology in its general form. Section 4 presents application examples of the unicycle model, the front wheel car model, and the car with trailer model. Section 5 presents simulation results for the application examples, and finally Section 6 concludes the paper.

2. Problem Formulation

2.1. Mathematical Model of Nonholonomic System. The kinematic model for a drift-free nonholonomic system is given as

$$\dot{x} = \sum_{i=1}^{m} G_i(x) u_i, \quad x \in \Re^n, \tag{1}$$

where $G_i(x)$ are linearly independent vector fields on $\Re^n$, u_i are locally bounded in t, and piece-wise continuous control functions are defined on the interval $[0, \infty)$. These systems are difficult to control as revealed by the fact that linearization of system (1) is uncontrollable. The most difficult issue from a theoretical viewpoint is the design of feedback laws that can stabilize these systems about an equilibrium position.

2.2. Problem Statement. Given a desired set point $x_{\mathrm{des}} \in \Re^n$, construct a feedback strategy in presence of the control $u_i : \Re^n \to \Re,\ i = 1, 2, \ldots, m$, so that the desired set point x_{des} is an attractive set for (1), such that $x(t; 0, x_0) \to x_{\mathrm{des}}$, as $t \to \infty$ for any initial condition.

Generally, by appropriate translation of coordinate system, $x_{\mathrm{des}} = 0$ can be achieved.

2.3. Some Assumptions. For steering control problem, the systems described by (1) must satisfy the following conditions:

(P1) The vector fields $G_1(x), \ldots, G_m(x)$ are linearly independent.

(P2) System (1) satisfies the Lie algebra rank condition (LARC) for accessibility, where Lie algebra, $L(G_1, \ldots, G_m)(x)$, spans $\Re^n$ at each point $x \in \Re^n$.

3. The Proposed Control Algorithm

Step 1. Write system (1) in the following form:

$$\begin{aligned} \dot{x}_1 &= g_1(x, u), \\ \dot{x}_2 &= g_2(x, u), \\ &\vdots \\ \dot{x}_{n-1} &= g_{n-1}(x, u), \\ \dot{x}_n &= g_n(x, u), \end{aligned} \tag{2}$$

where $g_i : \Re^n \times \Re^m \to \Re$ are nonlinear functions.

Step 2. Using the input transformation, transform system (2) into the following form:

$$\begin{aligned} \dot{x}_1 &= h_1(x), \\ \dot{x}_2 &= h_2(x), \\ &\vdots \\ \dot{x}_{n-1} &= h_{n-1}(x), \\ \dot{x}_n &= v, \end{aligned} \tag{3}$$

where $h_i : \Re^n \times \Re^m \to \Re$ are nonlinear function and v is the new input.

After some manipulation, system (3) can be rewritten as

$$\begin{aligned} \dot{x}_1 &= x_2 + F_1, \\ \dot{x}_2 &= x_3 + F_2, \\ &\vdots \\ \dot{x}_{n-1} &= x_n + F_{n-1}, \\ \dot{x}_n &= v, \end{aligned} \tag{4}$$

where $F_i = -x_{i+1} + h_i(x)$.

Step 3. Assume that F_i are uncertainties in the system. Let $\widehat{F}_i$, $i = 1, \ldots, n$ be an estimate of F_i, $i = 1, \ldots, n-1$, respectively. Apply the function approximation technique [27] to represent F_i and their estimates $\widehat{F}_i$ as $F_i = w_i^T \varphi_i(t)$ and $\widehat{F}_i = \widehat{w}_i^T \varphi_i(t)$.

$\varphi_i(t) = [\varphi_{i1}(t)\ \varphi_{i2}(t)\ \cdots\ \varphi_{in}(t)]^T$ is the function of basis vector and $w_i = [w_{i1}\ w_{i2}\ \cdots\ w_{in}]^T$ is a vector of weightings. Let $\widehat{w}_i = [\widehat{w}_{i1}\ \widehat{w}_{i2}\ \cdots\ \widehat{w}_{in}]^T$ be estimate of $w_i = [w_{i1}\ w_{i2}\ \cdots\ w_{in}]^T$. Therefore, we can estimate F_i by estimating the weight vector w_i; that is, $\widehat{F}_i = \widehat{w}_i^T\varphi_i(t)$. Define $\widetilde{w}_i = w_i - \widehat{w}_i$; then system (4) can be written as

$$\begin{aligned}
\dot{x}_1 &= x_2 + \widehat{w}_1^T\varphi_1(t) + \widetilde{w}_1^T\varphi_1(t),\\
\dot{x}_2 &= x_3 + \widehat{w}_2^T\varphi_1(t) + \widetilde{w}_2^T\varphi_1(t),\\
&\vdots\\
\dot{x}_{n-1} &= x_n + \widehat{w}_{n-1}^T\varphi_{n-1}(t) + \widetilde{w}_{n-1}^T\varphi_{n-1}(t),\\
\dot{x}_n &= v.
\end{aligned} \tag{5}$$

Step 4. Choose the nominal system for (5) as

$$\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= x_3,\\
&\vdots\\
\dot{x}_{n-1} &= x_n,\\
\dot{x}_n &= v_0.
\end{aligned} \tag{6}$$

Step 5. Define the sliding surface for nominal system (6) as

$$s_0 = x_1 + \sum_{i=2}^{n-1} c_i x_i + x_n, \tag{7}$$

where $c_i > 0$ are chosen in such a way that s_0 becomes Hurwitz polynomial. Then

$$\dot{s}_0 = \dot{x}_1 + \sum_{i=2}^{n-1} c_i\dot{x}_i + \dot{x}_n = x_2 + \sum_{i=2}^{n-1} c_i x_{i+1} + v_0. \tag{8}$$

Choose

$$v_0 = -x_2 - \sum_{i=2}^{n-1} c_i x_{i+1} - k\,\mathrm{sign}(s_0), \quad k > 0. \tag{9}$$

We have $\dot{s}_0 = -k\,\mathrm{sign}(s_0)$. Therefore, nominal system (6) is asymptotically stable.

Step 6. Define the sliding surface for system (5) as

$$s = s_0 + z = x_1 + \sum_{i=2}^{n-1} c_i x_i + x_n + z, \tag{10}$$

where z is an integral term. To avoid the reaching phase, choose $z(0)$ such that $s(0) = 0$. Choose $v = v_0 + v_s$, where v_0 is the nominal input and v_s is compensator term. Then

$$\begin{aligned}
\dot{s} &= \dot{x}_1 + \sum_{i=2}^{n-1} c_i\dot{x}_i + \dot{x}_n + \dot{z}\\
&= x_2 + \widehat{w}_i^T\varphi_1(t) + \widetilde{w}_i^T\varphi_2(t)\\
&\quad + \sum_{i=2}^{n-1} c_i\left\{x_{i+1} + \widehat{w}_i^T\varphi_i(t) + \widetilde{w}_i^T\varphi_i(t)\right\} + v_0 + v_s + \dot{z}\\
&= x_2 + \sum_{i=2}^{n-1} c_i x_{i+1} + v_0 + v_s + \dot{z}\\
&\quad + \sum_{i=1}^{n-1}\left\{\widehat{w}_i^T\varphi_i(t) + \widetilde{w}_i^T\varphi_i(t)\right\},
\end{aligned} \tag{11}$$

where $c_1 = 1$.

Step 7. Choose a Lyapunov function as

$$V = \frac{1}{2}s^2 + \frac{1}{2}\sum_{i=1}^{n-1}\widetilde{w}_i^T\widetilde{w}_i. \tag{12}$$

Design the adaptive laws for $\widetilde{w}_i$ & $\widehat{w}_i$, $i = 1,\dots,n$ and compute v_s such that $\dot{V} < 0$.

Theorem 1. *Choose a Lyapunov function as*

$$V = \frac{1}{2}s^2 + \frac{1}{2}\sum_{i=1}^{n-1}\widetilde{w}_i^T\widetilde{w}_i. \tag{13}$$

The following adaptive laws for $\widetilde{w}_i$ & $\widehat{w}_i$ and the value of v_s will guarantee the time derivative of V in (13) to be strictly negative (i.e., $\dot{V} < 0$)

$$\begin{aligned}
\dot{z} &= -x_2 - \sum_{i=2}^{n-1} c_i x_{i+1} - v_0,\\
v_s &= -\sum_{i=1}^{n-1} c_i\widehat{w}_i^T\varphi_i(t) - ks,\\
\dot{\widetilde{w}}_i &= -c_i s\varphi_i(t) - k_i\widetilde{w}_i,\\
\dot{\widehat{w}}_i &= -\dot{\widetilde{w}}_i,
\end{aligned} \tag{14}$$

where k and $k_i > 0$, $i = 1,\dots,n-1$.

Proof. Since

$$\begin{aligned}
\dot{V} &= s\dot{s} + \sum_{i=1}^{n-1}\widetilde{w}_i^T\dot{\widetilde{w}}_i = s\left\{x_2 + \widehat{w}_1^T\varphi_1(t) + \widetilde{w}_1^T\varphi_1(t)\right.\\
&\quad \left. + \sum_{i=2}^{n-1} c_i\left\{x_{i+1} + \widehat{w}_i^T\varphi_i(t) + \widetilde{w}_i^T\varphi_i(t)\right\} + v_0 + v_s + \dot{z}\right\}\\
&\quad + \sum_{i=1}^{n-1}\widetilde{w}_i^T\dot{\widetilde{w}}_i = s\left\{x_2 + \sum_{i=1}^{n-1}\left\{c_i x_{i+1} + c_i\widehat{w}_i^T\varphi_i(t)\right\} + v_0\right.\\
&\quad \left. + v_s + \dot{z}\right\} + \sum_{i=1}^{n-1}\widetilde{w}_i^T\left\{\dot{\widetilde{w}}_i + c_i s\varphi_i(t)\right\},
\end{aligned} \tag{15}$$

by using

$$\begin{aligned}
\dot{z} &= -x_2 - \sum_{i=2}^{n-1} c_i x_{i+1} - v_0,\\
v_s &= -\sum_{i=1}^{n-1} c_i\widehat{w}_i^T\varphi_i(t) - ks,
\end{aligned}$$

$$\dot{\widetilde{w}}_i = -c_i s\varphi_i(t) - k_i\widetilde{w}_i,$$
$$\dot{\hat{w}}_i = -\dot{\widetilde{w}}_i, \tag{16}$$

where k and $k_i > 0$, $i = 1, \ldots, n-1$, we have

$$\dot{V} = -ks^2 - \sum_{i=1}^{n-1} k_i \widetilde{w}_i^T \widetilde{w}_i. \tag{17}$$

Choosing

$$k_n = \min(k, k_1, \ldots, k_{n-1}), \tag{18}$$

we have

$$\dot{V} \leq -k_n \left(s^2 + \sum_{i=1}^{n-1} \widetilde{w}_i^T \widetilde{w}_i \right). \tag{19}$$

Using the chosen Lyapunov function

$$V = \frac{1}{2}s^2 + \frac{1}{2}\sum_{i=1}^{n-1} \widetilde{w}_i^T \widetilde{w}_i, \tag{20}$$

we can write

$$\dot{V} \leq -2k_n V,$$
$$\dot{V} \leq -\alpha V^{\beta}, \tag{21}$$

where $\alpha = 2k_n$ and $\beta = 1$.

From this we conclude that s & $\widetilde{w}_i \rightarrow 0$, $i = 1, \ldots, n$. Since $s \rightarrow 0$, therefore $x \rightarrow 0$. □

In the following section we illustrate the above algorithm by applying it to three different nonholonomic drift-free systems.

4. Application Examples

4.1. The Unicycle Model. A unicycle model or a two-wheel car model, shown in Figure 1, is basically a three-dimensional nonholonomic system having two inputs and three states with depth-one Lie bracket. A two-wheel car kinematic model is defined as [12]

$$\begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix} u_2. \tag{22}$$

Introducing a new set of state variables $x \overset{\text{def}}{=} [x_1, x_2, x_3]^T = [\theta, x, y]^T$ the kinematics model (22) can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ \cos x_1 \\ \sin x_1 \end{bmatrix} u_2 \tag{23}$$

or

$$\dot{x} = G_1(x) u_1 + G_2(x) u_2, \quad x \in \mathfrak{R}^3, \tag{24}$$

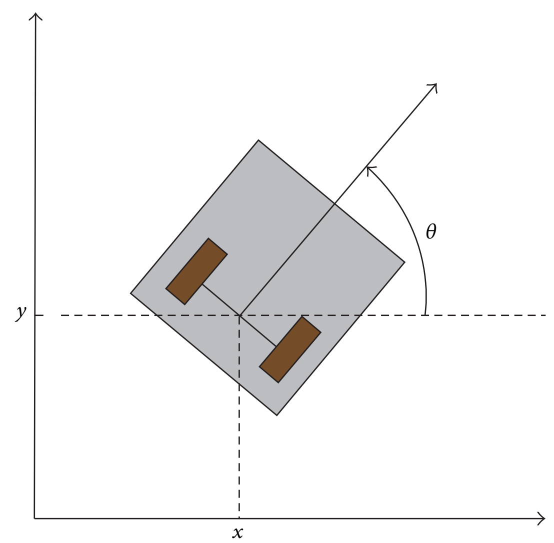

Figure 1: The unicycle model.

where

$$G_1(x) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$
$$G_2(x) = \begin{bmatrix} 0 \\ \cos x_1 \\ \sin x_1 \end{bmatrix}. \tag{25}$$

The kinematics model (22) satisfies the following assumptions:

(P1) The vector fields $G_1(x)$ and $G_2(x)$ are linearly independent.

(P2) System (24) satisfies the Lie algebra rank condition (LARC) for accessibility, where the Lie algebra, $L(G_1, G_2)(x)$, spans $\mathfrak{R}^3$ at each point $x \in \mathfrak{R}^3$.

To verify property (P2), it is sufficient to calculate the following Lie bracket of $G_1(x)$ and $G_2(x)$:

$$G_3(x) \overset{\text{def}}{=} [G_1, G_2](x) = \begin{bmatrix} 0 \\ -\sin x_1 \\ \cos x_1 \end{bmatrix}. \tag{26}$$

Then the LARC condition, namely, $\text{span}(G_1, G_2, G_3)(x) = \mathfrak{R}^3$, $\forall x \in \mathfrak{R}^3$, is satisfied.

4.1.1. Application of the Proposed Algorithm to the Unicycle Model

Step 1. The unicycle model given (24) can be rewritten as

$$\dot{x}_1 = u_1,$$
$$\dot{x}_2 = \cos x_1 u_2, \tag{27}$$
$$\dot{x}_3 = \sin x_1 u_2.$$

Step 2. Choose $u_1 = v$ and $u_2 = x_3/\cos x_1$, where, $x_1 \neq \pi/2$; then system (27) becomes

$$\begin{aligned}\dot{x}_1 &= v,\\ \dot{x}_2 &= x_3,\\ \dot{x}_3 &= x_3 \tan x_1.\end{aligned} \tag{28}$$

After some manipulation the above-mentioned system can be written as

$$\begin{aligned}\dot{x}_2 &= x_3,\\ \dot{x}_3 &= x_1 + F,\\ \dot{x}_1 &= v,\end{aligned} \tag{29}$$

where $F = -x_1 + x_3 \tan x_1$.

Step 3. Assume F as an uncertainty and let $\widehat{F}$ be an estimate of F. The estimate of F by function approximating technique [27] is $F = w^T\varphi$. Then $\widehat{F} = \widehat{w}^T\varphi$ and system (29) can be written as

$$\begin{aligned}\dot{x}_2 &= x_3,\\ \dot{x}_3 &= x_1 + \widehat{w}^T\varphi + \widetilde{w}^T\varphi,\\ \dot{x}_1 &= v.\end{aligned} \tag{30}$$

Step 4. Choose the nominal system for (30) as

$$\begin{aligned}\dot{x}_2 &= x_3,\\ \dot{x}_3 &= x_1,\\ \dot{x}_1 &= v_0.\end{aligned} \tag{31}$$

Step 5. Define the sliding surface for nominal system (31) as

$$s_0 = x_2 + 2x_3 + x_1. \tag{32}$$

Then

$$\dot{s}_0 = \dot{x}_2 + 2\dot{x}_3 + \dot{x}_1 = x_3 + 2x_1 + v_0. \tag{33}$$

By choosing

$$v_0 = -x_3 - 2x_1 - k\,\text{sign}(s_0), \quad k > 0, \tag{34}$$

we have

$$\dot{s}_0 = -k\,\text{sign}(s_0). \tag{35}$$

Therefore, nominal system (31) is asymptotically stable.

Step 6. Define the sliding surface for system (30) as

$$s = s_0 + z = x_2 + 2x_3 + x_1 + z. \tag{36}$$

Choose $v = v_0 + v_s$.

Then

$$\begin{aligned}\dot{s} &= \dot{x}_1 + 2\dot{x}_3 + \dot{x}_2 + \dot{z}\\ &= x_3 + 2x_2 + 2\widehat{w}^T\varphi + 2\widetilde{w}^T\varphi + v_0 + v_s + \dot{z}.\end{aligned} \tag{37}$$

Step 7. The adaptive laws for $\widetilde{w}, \widehat{w}$ and the value of v_s are as follows:

$$\begin{aligned}\dot{z} &= -x_3 - 2x_2 - v_0,\\ v_s &= -2\widehat{w}^T\varphi - ks,\\ \dot{\widetilde{w}} &= -2s\varphi - k_1\widetilde{w},\\ \dot{\widehat{w}} &= -\dot{\widetilde{w}},\end{aligned} \tag{38}$$

where k and $k_1 > 0$.

Give

$$\dot{V} = -ks^2 - k_1\widetilde{w}^T\widetilde{w}, \tag{39}$$

where $V = (1/2)s^2 + (1/2)\widetilde{w}^T\widetilde{w}$.

Choosing

$$k_2 = \min(k, k_1), \tag{40}$$

we have

$$\dot{V} \leq -k_2\left(s^2 + \widetilde{w}^T\widetilde{w}\right). \tag{41}$$

Using the chosen Lyapunov function we can write

$$\begin{aligned}\dot{V} &\leq -2k_2 V,\\ \dot{V} &\leq -\alpha V^{\beta},\end{aligned} \tag{42}$$

where $\alpha = 2k_2$ and $\beta = 1$.

From this we conclude that s & $\widetilde{w} \to 0$. Since $s \to 0$, therefore $x \to 0$.

Simulation results are shown in Figure 4.

4.2. The Front Wheel Car Model. A front wheel car model, shown in Figure 2, is basically a four-dimensional nonholonomic system having two inputs and four states with depth-two Lie bracket. A front wheel car kinematic model [6] can be defined as

$$\begin{bmatrix}\dot{\psi}\\ \dot{x}\\ \dot{\theta}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} u_1 + \begin{bmatrix}0\\ \cos\theta\\ \dfrac{1}{l}\tan\psi\\ \sin\theta\end{bmatrix} u_2. \tag{43}$$

Assuming that $l = 1$ and introducing a new set of state variables $x \overset{\text{def}}{=} (x_1, x_2, x_3, x_4) = (\psi, x, y, \theta)$ the kinematics model (43) can be written as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} u_1 + \begin{bmatrix}0\\ \cos x_4\\ \sin x_4\\ \tan x_1\end{bmatrix} u_2 \tag{44}$$

or

$$\dot{x} = G_1(x)\,u_1 + G_2(x)\,u_2, \quad x \in \mathfrak{R}^4, \tag{45}$$

where $G_1(x) = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ and $G_2(x) = \begin{bmatrix}0\\ \cos x_4\\ \sin x_4\\ \tan x_1\end{bmatrix}$.

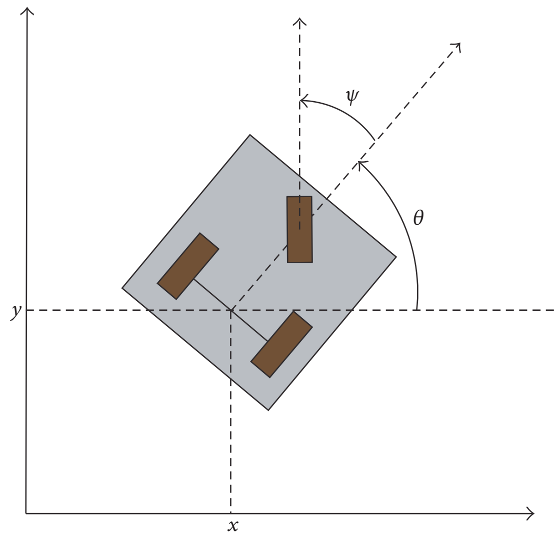

Figure 2: The front wheel car model.

The kinematics model (45) satisfies the following assumptions:

(P1) The vector fields $G_1(x)$ and $G_2(x)$ are linearly independent.

(P2) System (45) satisfies the Lie algebra rank condition (LARC) for accessibility, where the Lie algebra, $L(G_1, G_2)(x)$, spans $\mathfrak{R}^4$ at each point $x \in \mathfrak{R}^4$.

To verify property (P2), it is sufficient to calculate the following Lie brackets of $G_1(x)$ & $G_2(x)$:

$$G_3(x) \stackrel{\text{def}}{=} [G_1, G_2](x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ (\sec x_1)^2 \end{bmatrix},$$
$$G_4(x) \stackrel{\text{def}}{=} [G_2, G_3](x) = \begin{bmatrix} 0 \\ -\sin x_4 (\sec x_1)^2 \\ \cos x_4 (\sec x_1)^2 \\ 0 \end{bmatrix}, \tag{46}$$

which satisfy the LARC condition: $\operatorname{span}(G_1, G_2, G_3, G_4)(x) = \mathfrak{R}^4$, $\forall x \in \mathfrak{R}^4$.

4.2.1. Application of the Proposed Algorithm to the Front Wheel Car Model

Step 1. The front wheel car model as given in (45) can be rewritten as

$$\begin{aligned} \dot{x}_1 &= u_1, \\ \dot{x}_2 &= \cos x_4 u_2, \\ \dot{x}_3 &= \sin x_4 u_2, \\ \dot{x}_4 &= \tan x_1 u_2. \end{aligned} \tag{47}$$

Step 2. Choose $u_1 = v$ and $u_2 = x_3/\cos x_4$, where $x_4 \neq \pi/2$, and then system (47) becomes

$$\begin{aligned} \dot{x}_1 &= v, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= x_3 \tan x_4, \\ \dot{x}_4 &= x_3 \tan x_1 \sec x_4, \end{aligned} \tag{48}$$

which can be rewritten as

$$\begin{aligned} \dot{x}_2 &= x_3, \\ \dot{x}_3 &= x_4 + F_3, \\ \dot{x}_4 &= x_1 + F_4, \\ \dot{x}_1 &= v, \end{aligned} \tag{49}$$

where

$$\begin{aligned} F_3 &= -x_4 + x_3 \tan x_4, \\ F_4 &= -x_1 + x_3 \tan x_1 \sec x_4. \end{aligned} \tag{50}$$

Step 3. Treat F_i, $i = 3, 4$ as uncertainties and let $\widehat{F}_i$, $i = 3, 4$ be an estimate of F_i, $i = 3, 4$, respectively. Using function approximation technique [27], we can approximate F_i, $i = 3, 4$ as $F_3 = w_3^T \varphi_3$, $F_4 = w_4^T \varphi_4$. Then $\widehat{F}_3 = \widehat{w}_3^T \varphi_3$ and $\widehat{F}_4 = \widehat{w}_4^T \varphi_4$.

Then system (49) can be written as

$$\begin{aligned} \dot{x}_1 &= x_3, \\ \dot{x}_3 &= x_4 + \widehat{w}_3^T \varphi_3 + \widetilde{w}_3^T \varphi_3, \\ \dot{x}_4 &= x_2 + \widehat{w}_4^T \varphi_4 + \widetilde{w}_4^T \varphi_4, \\ \dot{x}_2 &= v. \end{aligned} \tag{51}$$

Step 4. Choose the nominal system for (51) as

$$\begin{aligned} \dot{x}_1 &= x_3, \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= x_2, \\ \dot{x}_2 &= v_0. \end{aligned} \tag{52}$$

Step 5. Define the sliding surface for nominal system (52) as

$$s_0 = x_1 + 3x_3 + 3x_4 + x_2. \tag{53}$$

Then

$$\dot{s}_0 = \dot{x}_1 + 3\dot{x}_3 + 3\dot{x}_4 + \dot{x}_2 = x_3 + 3x_4 + 3x_2 + v_0. \tag{54}$$

By choosing

$$v_0 = -x_3 - 3x_4 - 3x_2 - k\,\mathrm{sign}\,(s_0), \quad k > 0, \tag{55}$$

we have

$$\dot{s}_0 = -k\,\mathrm{sign}\,(s_0). \tag{56}$$

Therefore, nominal system (52) is asymptotically stable.

Step 6. Define the sliding surface for system (51) as

$$s = s_0 + z = x_1 + 3x_3 + 3x_4 + x_2 + z. \tag{57}$$

Choose $v = v_0 + v_s$.

Then

$$\begin{aligned} \dot{s} &= \dot{x}_1 + 3\dot{x}_3 + 3\dot{x}_4 + \dot{x}_2 + \dot{z} \\ &= x_3 + 3x_4 + 3\widehat{w}_3^T\varphi_3 + 3\widetilde{w}_3^T\varphi_3 + 3x_2 + 3\widehat{w}_4^T\varphi_4 \\ &\quad + 3\widetilde{w}_4^T\varphi_4 + v_0 + v_s + \dot{z}. \end{aligned} \tag{58}$$

Step 7. The following adaptive laws for $\widetilde{w}_i, \widehat{w}_i$, $i = 3, 4$ and the value of v_s are chosen as

$$\begin{aligned} \dot{z} &= -x_3 - 3x_4 - 3x_2 - v_0, \\ v_s &= -3\widehat{w}_3^T\varphi_3 - 3\widehat{w}_4^T\varphi_4 - ks, \\ \dot{\widetilde{w}}_3 &= -3s\varphi_3 - k_1\widetilde{w}_3^T, \\ \dot{\widehat{w}}_3 &= -\dot{\widetilde{w}}_3, \\ \dot{\widetilde{w}}_4 &= -3s\varphi_4 - k_2\widetilde{w}_4^T, \\ \dot{\widehat{w}}_4 &= -\dot{\widetilde{w}}_4, \end{aligned} \tag{59}$$

where k, k_1 and $k_2 > 0$.

Give

$$\dot{V} = -ks^2 - k_1\widetilde{w}_3^T\widetilde{w}_3 - k_2\widetilde{w}_4^T\widetilde{w}_4, \tag{60}$$

where

$$V = \frac{1}{2}s^2 + \frac{1}{2}\widetilde{w}_3^T\widetilde{w}_3 + \frac{1}{2}\widetilde{w}_4^T\widetilde{w}_4. \tag{61}$$

Choosing

$$k_3 = \min(k, k_1, k_2), \tag{62}$$

we have

$$\dot{V} \le -k_3\left(s^2 + k_1\widetilde{w}_3^T\widetilde{w}_3 + k_2\widetilde{w}_4^T\widetilde{w}_4\right). \tag{63}$$

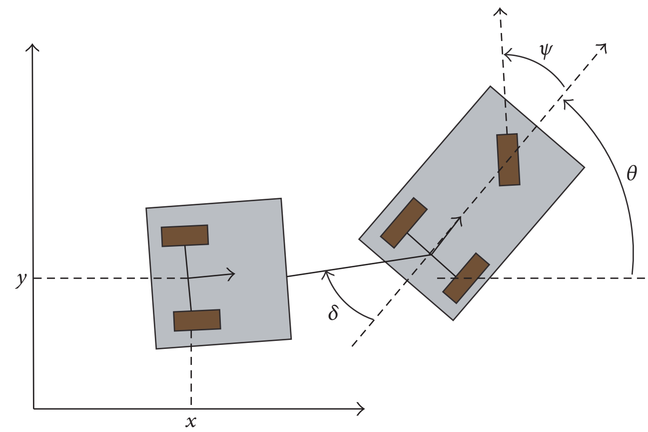

Figure 3: The mobile robot with trailer model.

Using the chosen Lyapunov function we can write

$$\begin{aligned} \dot{V} &\le -2k_3V, \\ \dot{V} &\le -\alpha V^{\beta}, \end{aligned} \tag{64}$$

where $\alpha = 2k_3$ and $\beta = 1$.

From this we conclude that $s, \widetilde{w}_3, \widetilde{w}_4 \to 0$. Since $s \to 0$, therefore $x \to 0$.

Simulation results are shown in Figure 5.

4.3. The Mobile Robot with Trailer Model. A car with trailer model, shown in Figure 3, is basically a five-dimensional nonholonomic system having two inputs and five states with depth-one, depth-two, and depth-three Lie brackets. A car with trailer kinematic model [8] can be defined as

$$\begin{aligned} \dot{x}_1 &= \cos x_3 \cos x_4 u_1, \\ \dot{x}_2 &= \cos x_3 \sin x_4 u_1, \\ \dot{x}_3 &= u_2, \\ \dot{x}_4 &= \frac{1}{l}\sin x_3 u_1, \\ \dot{x}_5 &= \frac{1}{d}\sin(x_4 - x_5)\cos x_3 u_1. \end{aligned} \tag{65}$$

By assuming $l = d = 1$, system (65) can be written in the following standard form:

$$\dot{x} = G_1(x)\,u_1 + G_2(x)\,u_2, \quad x \in \mathfrak{R}^5, \tag{66}$$

where

$$G_1(x) = \begin{bmatrix} \cos x_3 \cos x_4 \\ \cos x_3 \sin x_4 \\ 0 \\ \sin x_3 \\ \cos x_3 \sin(x_4 - x_5) \end{bmatrix},$$

$$G_2(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \tag{67}$$

It can be verified that system (66) satisfies the following assumptions which are necessary for steering problem.

(P1) The vectors $G_i(x)$, $i = 1, 2$ are linearly independent and have no singular point for all $x \in M \subseteq \mathfrak{R}^5$, where M is some manifold in $\mathfrak{R}^5$.

(P2) System (66) satisfies the Lie algebraic rank condition (LARC) for controllability, where the Lie algebra, $L(G_1, G_2)(x)$, spans $\mathfrak{R}^5$ at each point $x \in M \subseteq \mathfrak{R}^5$: that is, $\operatorname{span}(G_1(x), G_2(x), \ldots, G_5(x)) = \mathfrak{R}^5$, $\forall x \in M$.

To verify (P1) and (P2), calculate the linearly independent Lie brackets.

$$\begin{gathered} G_3(x) \stackrel{\text{def}}{=} [G_1, G_2](x) = \begin{bmatrix} \sin x_3 \cos x_4 \\ \sin x_3 \sin x_4 \\ 0 \\ -\cos x_3 \\ \sin x_3 \sin(x_4 - x_5) \end{bmatrix}, \\ G_4(x) \stackrel{\text{def}}{=} [G_1, [G_1, G_2]](x) = \begin{bmatrix} -\sin x_4 \\ \cos x_4 \\ 0 \\ 0 \\ \cos(x_4 - x_5) \end{bmatrix}, \\ G_5(x) \stackrel{\text{def}}{=} [G_1, [G_1, [G_1, G_2]]](x) \\ = \begin{bmatrix} -\sin x_3 \cos x_4 \\ -\sin x_3 \sin x_4 \\ 0 \\ 0 \\ -\sin x_3 \sin(x_4 - x_5) + \cos x_3 \end{bmatrix}. \end{gathered} \tag{68}$$

If the motion of system is restricted to manifold,

$$M \stackrel{\text{def}}{=} \left(x \in \mathfrak{R}^5 : |x_i| < \frac{\pi}{2},\ i = 3, 4 \right). \tag{69}$$

Then the Lie algebra rank condition, namely, $\operatorname{span}(G_1(x), G_2(x), \ldots, G_5(x)) = \mathfrak{R}^5$, $\forall x \in M$, is satisfied, hence guaranteeing that system (66) satisfies conditions (P1) and (P2) on the surface M.

4.3.1. Application of the Proposed Algorithm to the Mobile Robot with Trailer Model

Step 1. System (65) can be written as

$$\begin{aligned} \dot{x}_1 &= \cos x_3 \cos x_4 u_1, \\ \dot{x}_2 &= \cos x_3 \sin x_4 u_1, \\ \dot{x}_3 &= u_2, \\ \dot{x}_4 &= \sin x_3 u_1, \\ \dot{x}_5 &= \sin(x_4 - x_5) \cos x_3 u_1. \end{aligned} \tag{70}$$

Step 2. Choose $u_1 = x_2 / \cos x_3 \cos x_4$ and $u_2 = v$.
And, $x_3, x_4 \neq \pi/2$. Then system (70) can be written as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_2 \tan x_4, \\ \dot{x}_3 &= v, \\ \dot{x}_4 &= x_2 \tan x_3 \sec x_4, \\ \dot{x}_5 &= x_2 \sin(x_4 - x_5) \sec x_4. \end{aligned} \tag{71}$$

Step 3. Assume F_i, $i = 2, 4, 5$ as uncertainties and let $\widehat{F}_i$, $i = 2, 4, 5$ be an estimate of F_i, $i = 2, 4, 5$, respectively. Approximate $F_i = w_i^T \varphi_i$, $i = 2, 4, 5$ and let $\widehat{F}_i = \widehat{w}_i^T \varphi_i$, $i = 2, 4, 5$, respectively. Then system (71) can be written as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_4 + \widehat{w}_2^T \varphi_2 + \widetilde{w}_2^T \varphi_2, \\ \dot{x}_4 &= x_5 + \widehat{w}_4^T \varphi_4 + \widetilde{w}_4^T \varphi_4, \\ \dot{x}_5 &= x_3 + \widehat{w}_5^T \varphi_5 + \widetilde{w}_5^T \varphi_5, \\ \dot{x}_3 &= v. \end{aligned} \tag{72}$$

Step 4. Choose the nominal system for (72) as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_4, \\ \dot{x}_4 &= x_5, \\ \dot{x}_5 &= x_3, \\ \dot{x}_3 &= v_0. \end{aligned} \tag{73}$$

Step 5. Define the Hurwitz sliding surface for nominal system (73) as

$$s_0 = x_1 + 4x_2 + 6x_4 + 4x_5 + x_3. \tag{74}$$

Then

$$\begin{aligned} \dot{s}_0 &= \dot{x}_1 + 4\dot{x}_2 + 6\dot{x}_4 + 4\dot{x}_5 + \dot{x}_3 \\ &= x_2 + 4x_4 + 6x_5 + 4x_3 + v_0. \end{aligned} \tag{75}$$

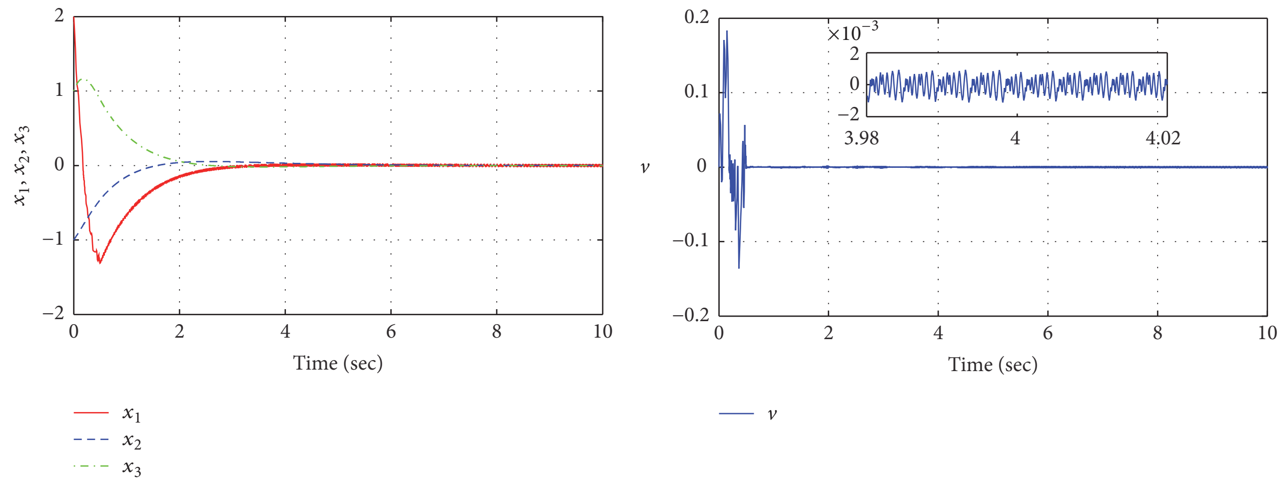

FIGURE 4: (a) Time response of the system states corresponding to initial condition $(x_1(0), x_2(0), x_3(0)) = (2, -1, 1)$. (b) Control input v.

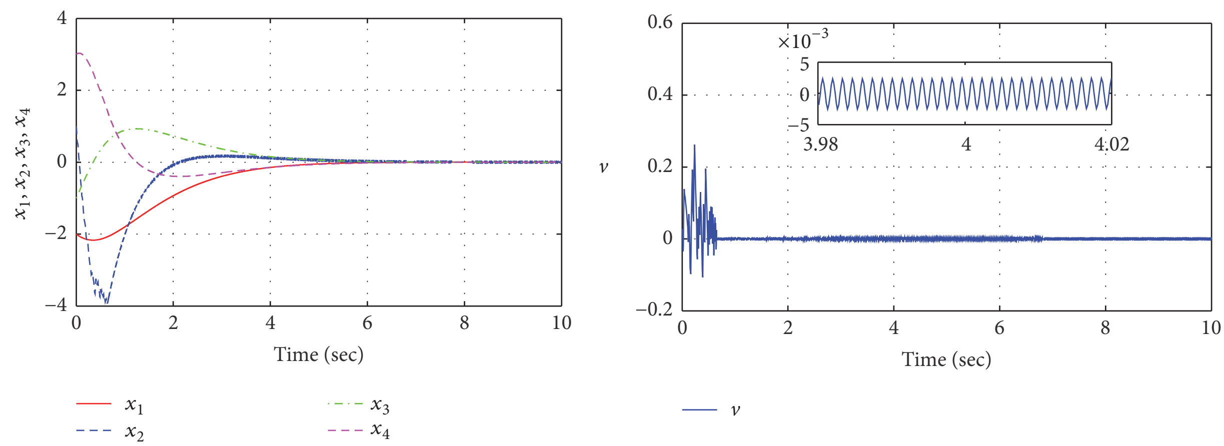

FIGURE 5: (a) Time response of the system states corresponding to initial condition $(x_1(0), \ldots, x_4(0)) = (-2, 1, -1, 3)$. (b) Control input v.

By choosing

$$v_0 = -x_2 - 4x_4 - 6x_5 - 4x_3 - k \operatorname{sign}(s_0), \quad k > 0, \tag{76}$$

we have

$$\dot{s}_0 = -k \operatorname{sign}(s_0). \tag{77}$$

Therefore, nominal system (73) is asymptotically stable.

Step 6. Define the sliding surface for system (72) as

$$s = s_0 + z = x_1 + 4x_2 + 6x_4 + 4x_5 + x_3 + z. \tag{78}$$

And choose $v = v_0 + v_s$.

Then

$$\begin{aligned} \dot{s} &= \dot{x}_1 + 4\dot{x}_2 + 6\dot{x}_4 + 4\dot{x}_5 + \dot{x}_3 + \dot{z} \\ &= x_2 + 4x_4 + 4\widehat{w}_2^T\varphi_2 + 4\widetilde{w}_2^T\varphi_2 + 6x_5 + 6\widehat{w}_4^T\varphi_4 \\ &\quad + 6\widetilde{w}_4^T\varphi_4 + 4x_3 + 4\widehat{w}_5^T\varphi_5 + 4\widetilde{w}_5^T\varphi_5 + v_0 + v_s + \dot{z}. \end{aligned} \tag{79}$$

Step 7. The following adaptive laws for $\widetilde{w}_i$ & $\widehat{w}_i$ and the value of v_s

$$\begin{aligned} \dot{z} &= -x_2 - 4x_4 - 6x_5 - 4x_3 - v_0, \\ v_s &= -4\widehat{w}_2^T\varphi_2 - 6\widehat{w}_4^T\varphi_4 - 4\widehat{w}_5^T\varphi_5 - ks, \\ \dot{\widetilde{w}}_2 &= -4s\varphi_2 - k_1\widetilde{w}_2^T, \\ \dot{\widehat{w}}_2 &\approx -\dot{\widetilde{w}}_2, \\ \dot{\widetilde{w}}_4 &= -6s\varphi_4 - k_2\widetilde{w}_4^T, \\ \dot{\widehat{w}}_4 &\approx -\dot{\widetilde{w}}_4, \\ \dot{\widetilde{w}}_5 &= -4s\varphi_5 - k_3\widetilde{w}_5^T, \\ \dot{\widehat{w}}_5 &\approx -\dot{\widetilde{w}}_5, \end{aligned} \tag{80}$$

with k and $k_i > 0$, $i = 1, 2, 3$, result in

$$\dot{V} = -ks^2 - k_1\widetilde{w}_2^T\widetilde{w}_2 - k_2\widetilde{w}_4^T\widetilde{w}_4 - k_3\widetilde{w}_5^T\widetilde{w}_5, \tag{81}$$

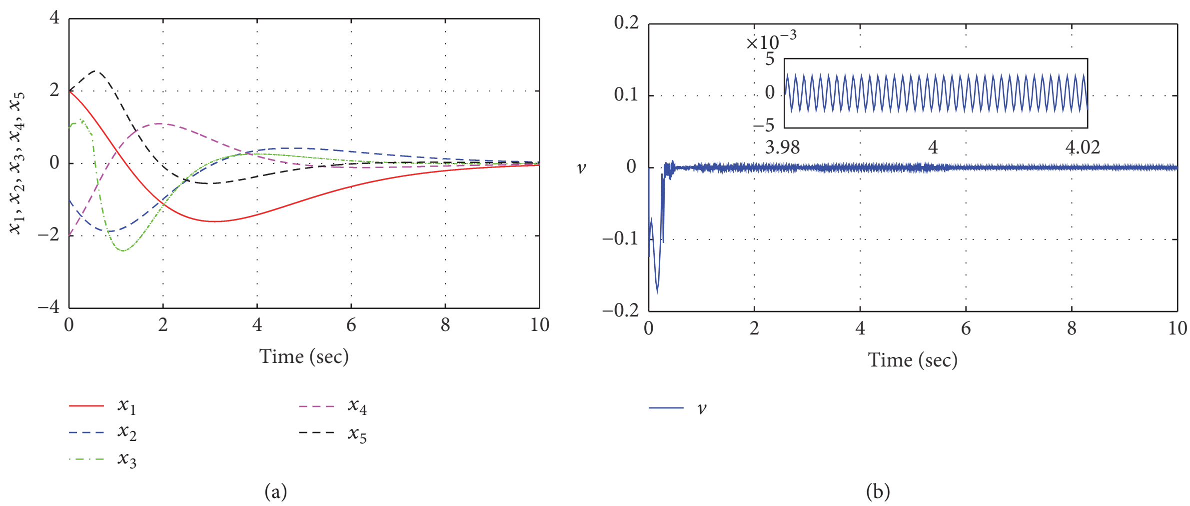

FIGURE 6: (a) Time response of the system states corresponding to initial condition $(x_1(0), \ldots, x_5(0)) = (2, -1, 1, -2, 2)$. (b) Control input v.

where

$$V = \frac{1}{2}s^2 + \frac{1}{2}\widetilde{w}_2^T\widetilde{w}_2 + \frac{1}{2}\widetilde{w}_4^T\widetilde{w}_4 + \frac{1}{2}\widetilde{w}_5^T\widetilde{w}_5. \quad (82)$$

Choosing

$$k_4 = \min(k, k_1, k_2, k_3), \quad (83)$$

we have

$$\dot{V} \le -k_4\left(s^2 + \widetilde{w}_2^T\widetilde{w}_2 + \widetilde{w}_4^T\widetilde{w}_4 + \widetilde{w}_5^T\widetilde{w}_5\right). \quad (84)$$

Using the chosen Lyapunov function we can write

$$\begin{aligned} \dot{V} &\le -2k_4 V, \\ \dot{V} &\le -\alpha V^{\beta}, \end{aligned} \quad (85)$$

where $\alpha = 2k_4$ and $\beta = 1$.

From this we conclude that $s, \widetilde{w}_2^T, \widetilde{w}_4^T$ and $\widetilde{w}_5^T \to 0$. Since $s \to 0$, therefore $x \to 0$. Simulation results are shown in Figures 4–6 for different initial conditions.

5. Simulation Results

Figures 4(a) and 4(b) show simulation results of the unicycle model and represent that the states and the control effort converge to zero and have settling time of 4 sec and 0.8 sec. Figures 5(a) and 5(b) show simulation results of the front wheel car model and represent that the states and the control effort converge to zero and have settling time of 6 sec and 1 sec. Figures 6(a) and 6(b) show simulation results for the car with trailer model and represent that the states and control effort converge to zero and have settling time of 10 sec and 0.4 sec. Simulation results show the effectiveness of the proposed scheme.

6. Conclusion

An adaptive integral sliding mode based control algorithm for the stabilization of nonholonomic drift-free control systems was presented. The objective was to steer the system from any arbitrary initial state to any desired state. The effectiveness of the method was tested on three different nonholonomic drift-free systems: the unicycle model, the front wheel car model, and the mobile robot with trailer model. The aim was to steer the systems to a desired value which was assumed to be zero. It is evident from the simulation results that the objective has been achieved. This method is general and can be employed to steer a variety of mechanical systems with nonholonomic constraints.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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