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Research Article

Dynamic Output Feedback Stabilization of Singular Fractional-Order Systems

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This paper is concerned with dynamic output feedback controller (DOFC) design problem for singular fractional-order systems with the fractional-order α satisfying $0 < \alpha < 2$. Based on the stability theory of fractional-order system, sufficient and necessary conditions are derived for the admissibility of the systems, which are more convenient to analytical design of stabilizing controllers than the existing results. A full-order DOFC is then synthesized based on the obtained conditions and the characteristics of Moore-Penrose inverse. Finally, a numerical example is presented to show the effectiveness of the proposed methods.

1. Introduction

During the past years, the study on fractional-order systems has become a hot research topic since they can concisely and precisely characterize many real-world physical systems with the introduction of fractional-order calculus [1, 2]. Up to date, considerable attention has been devoted to the stability analysis and controller design for fractional-order systems and many results have been published in the literature (see, e.g., [3–9] and the references therein).

On the other hand, singular systems have been extensively studied in the past years due to their applications in economics, circuits, and many other fields (see, e.g., [10–14] and the references therein). The research on singular fractional-order systems is much more complicated than that for state-space systems, because it requires considering not only stability, but also regularity and impulse elimination, while the latter do not appear in regular ones. Very recently, the study on singular fractional-order systems has received much attention. For example, sufficient and necessary condition of admissibility for fractional-order singular system was proposed in [15]. However, it is noted that the conditions obtained in [15] are not convenient for matrix transformation and may cause difficulty in controller design. In addition, only systems with fractional-order α belonging to $0 < \alpha < 1$ are considered. Based on the obtained stability results in [15] and matrix's singular value decomposition (SVD), state and static output feedback controller design methods were proposed in [16] for singular fractional-order systems with fractional-order $0 < \alpha < 1$. In [17], the problem of robust stabilization for a class of uncertain singular fractional-order systems with fractional commensurate order $0 < \alpha < 2$ was studied. However, the systems in [17] are required to be normalizable, that is, rank $\begin{bmatrix} E & B \end{bmatrix} = n$, which is very restrictive. In [18], the problem of robust stability and stabilization of interval uncertain descriptor fractional-order systems with fractional-order $1 \leq \alpha < 2$ was studied. However, similarly as in [17], only state-feedback controller was designed, which means that the obtained method cannot be applied to system with state being unavailable. Up to now, the problem of stability analysis and controller design for singular fractional-order systems has not been fully investigated, which is very challenging and of great importance. This motivates us to carry out this study.

In this paper, we will study the dynamic output control problem for singular fractional-order systems with fractional commensurate order $0 < \alpha < 2$. The main contributions of this paper can be summarized as follows.

(1) New necessary and sufficient conditions will be derived for the stability of the focused systems, which

are more convenient to analytical design of stabilizing controllers than the existing results.

(2) The desired full DOFC is designed in a whole framework. It is practically important to stabilize systems by output feedback controller since it is usually hard to sense all the states and feed them back. The effectiveness and applicability of proposed results are verified by a numerical example.

Notations. $\mathbb{R}^{m\times n}$ is the set of all $m \times n$ real matrices. The superscripts "T" and "+" represent the transpose and the Moore-Penrose inverse, respectively, and "$*$" denotes the term that is induced by symmetry. $\mathrm{Sym}\{X\}$ is used to denote $X^T + X$, $\otimes$ stands for the Kronecker product, and D^α represents initialized αth order differintegration.

2. Problem Formulation and Preliminaries

Consider the following singular fractional-order system:

$$\begin{aligned} ED^\alpha x(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where $0 < \alpha < 2$ is the fractional commensurate order; $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^q$ are the state vector, the control input, and the measurable output, respectively. The matrix $E \in \mathbb{R}^{n\times n}$ may be singular and it is assumed that $\mathrm{rank}(E) = r \le n$. The system matrices A, B, C, and D are real matrices with appropriate dimensions.

In this paper, the Caputo definition is adopted for fractional derivatives of order α of function $f(t)$ since this Laplace transform allows using initial values of classical integral-order derivatives with clear physical interpretations, which is defined as follows:

$$D^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(\alpha - m)} \int_0^t \frac{f^{(m)}\tau}{(t-\tau)^{\alpha+1-m}} d\tau, \tag{2}$$

where the fractional-order $m - 1 < \alpha \le m$, $m \in N$, and the gamma function $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$.

For system (1), we are interested in designing a DOFC of the following form:

$$\begin{aligned} ED^\alpha x_k(t) &= A_k x_k(t) + B_k y(t), \\ u(t) &= C_k x_k(t), \end{aligned} \tag{3}$$

where $x_k(t) \in \mathbb{R}^n$ is the state vector of the controller. The matrices A_k, B_k, and C_k are the controller matrices to be determined.

Augmenting the model of system (1) to include the states of DOFC (3), the closed-loop system is governed by

$$E_c D^\alpha \eta(t) = A_c \eta(t), \tag{4}$$

where $\eta(t) = \left[x^T(t) \;\; x_k^T(t)\right]^T$ and

$$\begin{aligned} E_c &= \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}, \\ A_c &= \begin{bmatrix} A & BC_k \\ B_k C & A_k + B_k D C_k \end{bmatrix}. \end{aligned} \tag{5}$$

The unforced singular fractional-order system of (1) can be written as

$$ED^\alpha x(t) = Ax(t). \tag{6}$$

Definition 1 (see [15]). For singular fractional-order (6), the pair (E, A) is said to be regular if there exists a constant scalar $\lambda \in \mathbb{C}$ such that the pseudopolynomial $\det(s^\gamma E - A)$ is not identically zero.

Lemma 2 (see [15]). (1) *For singular fractional-order (6), the pair (E, A) is regular if and only if there exist two nonlinear matrices Q and P such that*

$$\begin{aligned} QEP &= \begin{bmatrix} I_r & 0 \\ 0 & N \end{bmatrix}, \\ QAP &= \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}, \end{aligned} \tag{7}$$

where A_1 and N are in Jordan canonical forms and N is nilpotent.

(2) *For singular fractional-order (6), the pair (E, A) is impulse-free if* $\deg(\det(s^\gamma E - A)) = \mathrm{rank}(E)$*; that is, $N = 0$ in (7).*

Lemma 3 (see [19]). *For fractional-order, linear system with order $\alpha : d^\alpha x/dt^\alpha = Ax$ is asymptotically stable if and only if*

$$\left|\arg(\mathrm{spec}(A))\right| > \alpha\frac{\pi}{2}, \tag{8}$$

where $A \in \mathbb{R}^{n\times n}$ is a deterministic real matrix and $\mathrm{spec}(A)$ *is the spectrum of A.*

The regularity and absence of impulses of the pair (E, A) ensure the existence and uniqueness of an impulse-free solution to system (6). Based on Lemmas 2 and 3, we have the following lemma.

Lemma 4. (1) *Suppose the pair (E, A) is regular and impulse-free; then the solution to (6) exists and is impulse-free and unique on* $[0, \infty)$.

(2) *Suppose the pair (E, A) is regular and impulse-free; system (6) is asymptotically stable if* $|\arg(\mathrm{spec}(A_1))| > \alpha(\pi/2)(|\arg(\mathrm{spec}(E, A))| > \alpha(\pi/2))$.

Proof. Noting the regularity and absence of impulses of the pair (E, A) and using the decomposition as in [15] and Lemma 3, the desired result follows immediately. □

Throughout the paper, we will use the following notion of admissibility for singular fractional-order system.

Definition 5 (see [15]). The singular fractional-order system (6) is said to be admissible, if it is regular, impulse-free, and asymptotically stable.

The necessary and sufficient conditions for the stability of state-space fractional-order systems with the fractional-order α belonging to $0 < \alpha < 2$ are provided by the two following lemmas.

Lemma 6 (see [20]). *Let $A \in R^{n\times n}$ and $1 \le \alpha < 2$. The fractional-order system $D^\alpha x(t) = Ax(t)$ is asymptotically stable (i.e., $|\arg(\mathrm{spec}(A))| > \alpha(\pi/2)$), if and only if there exists a symmetric matrix $P > 0$ such that*

$$\begin{bmatrix} \left(PA^T + AP^T\right)\sin\frac{\pi}{2}\alpha & \left(PA^T - AP^T\right)\cos\frac{\pi}{2}\alpha \\ \left(PA^T - AP^T\right)\cos\frac{\pi}{2}\alpha & \left(PA^T + AP^T\right)\sin\frac{\pi}{2}\alpha \end{bmatrix} < 0, \quad (9)$$

where $\mathrm{spec}(A)$ *is the spectrum of all eigenvalues of A.*

Lemma 7 (see [21]). *Let $A \in R^{n\times n}$ and $0 < \alpha < 1$. The fractional-order system $D^\alpha x(t) = Ax(t)$ is asymptotically stable (i.e., $|\arg(\mathrm{spec}(A))| > \alpha(\pi/2)$), if and only if there exist two real symmetric positive definite matrices $P_{k1} \in R^{n\times n}$, $k = 1, 2$, and two skew-symmetric matrices $P_{k2} \in R^{n\times n}$, $k = 1, 2$, such that*

$$\begin{gathered} \sum_{i=1}^{2}\sum_{j=1}^{2}\mathrm{Sym}\left\{\Gamma_{ij}\otimes\left(AP_{ij}\right)\right\} < 0, \\ \begin{bmatrix} P_{11} & P_{12} \\ -P_{12} & P_{11} \end{bmatrix} > 0, \\ \begin{bmatrix} P_{21} & P_{22} \\ -P_{22} & P_{21} \end{bmatrix} > 0, \end{gathered} \quad (10)$$

where

$$\begin{gathered} \Gamma_{11} = \begin{bmatrix} \sin\left(\frac{\pi}{2}\alpha\right) & -\cos\left(\frac{\pi}{2}\alpha\right) \\ \cos\left(\frac{\pi}{2}\alpha\right) & \sin\left(\frac{\pi}{2}\alpha\right) \end{bmatrix}, \\ \Gamma_{12} = \begin{bmatrix} \cos\left(\frac{\pi}{2}\alpha\right) & \sin\left(\frac{\pi}{2}\alpha\right) \\ -\sin\left(\frac{\pi}{2}\alpha\right) & \cos\left(\frac{\pi}{2}\alpha\right) \end{bmatrix}, \\ \Gamma_{21} = \begin{bmatrix} \sin\left(\frac{\pi}{2}\alpha\right) & \cos\left(\frac{\pi}{2}\alpha\right) \\ -\cos\left(\frac{\pi}{2}\alpha\right) & \sin\left(\frac{\pi}{2}\alpha\right) \end{bmatrix}, \\ \Gamma_{22} = \begin{bmatrix} -\cos\left(\frac{\pi}{2}\alpha\right) & \sin\left(\frac{\pi}{2}\alpha\right) \\ -\sin\left(\frac{\pi}{2}\alpha\right) & -\cos\left(\frac{\pi}{2}\alpha\right) \end{bmatrix}. \end{gathered} \quad (11)$$

3. Admissibility Analysis of Singular Fractional-Order Systems

In this section, we will investigate the admissibility of closed-loop system (3) with fractional-order α belonging to $0 < \alpha < 1$ and $1 \le \alpha < 2$.

Theorem 8. *Closed-loop system (4) with fractional-order α belonging to $0 < \alpha < 1$ is admissible, if and only if there exist nonsingular matrices $P_{k1} \in R^{2n\times 2n}$ and $P_{k2} \in R^{2n\times 2n}$, $k = 1, 2$, such that*

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\mathrm{Sym}\left\{\Gamma_{ij}\otimes\left(A_cP_{ij}\right)\right\} < 0, \quad (12)$$

$$\begin{gathered} \begin{bmatrix} E_cP_{11} & E_cP_{12} \\ -E_cP_{12} & E_cP_{11} \end{bmatrix} \ge 0, \\ \begin{bmatrix} E_cP_{21} & E_cP_{22} \\ -E_cP_{22} & E_cP_{21} \end{bmatrix} \ge 0, \end{gathered} \quad (13)$$

$$\begin{gathered} E_cP_{11} = P_{11}^TE_c^T \ge 0, \\ E_cP_{21} = P_{21}^TE_c^T \ge 0, \end{gathered} \quad (14)$$

$$\begin{gathered} E_cP_{12} + P_{12}^TE_c^T = 0, \\ E_cP_{22} + P_{22}^TE_c^T = 0, \end{gathered} \quad (15)$$

where Γ_{ij}, $i = 1, 2$ and $j = 1, 2$, satisfy (11).

Proof.

Sufficiency. Since $\mathrm{rank}(E_c) = 2r \le 2n$, there exist nonsingular matrices G and H such that

$$GE_cH = \begin{bmatrix} I_{2r} & 0 \\ 0 & 0 \end{bmatrix}. \quad (16)$$

Denote

$$\begin{gathered} GA_cH = \begin{bmatrix} A_{c1} & A_{c2} \\ A_{c3} & A_{c4} \end{bmatrix}, \\ H^{-1}P_{k1}G^T = \begin{bmatrix} P_{k11} & P_{k12} \\ P_{k13} & P_{k14} \end{bmatrix}, \\ H^{-1}P_{k2}G^T = \begin{bmatrix} P_{k21} & P_{k22} \\ P_{k23} & P_{k24} \end{bmatrix}, \\ k = 1, 2. \end{gathered} \quad (17)$$

From (14)-(15) and using the expressions in (16) and (17), it is easy to get $P_{k12} = 0$, $P_{k12} = 0$, and $P_{k22} = 0$, $k = 1, 2$. Premultiplying and postmultiplying inequality (12) by G and G^T, respectively, we obtain

$$\begin{aligned} &A_{c4}\left(P_{114} + P_{214} + P_{124} + P_{224}\right) \\ &\quad + \left(P_{114} + P_{214} + P_{124} + P_{224}\right)^T A_{c4}^T < 0 \end{aligned} \quad (18)$$

which implies A_{c4} is nonsingular. According to Theorem 10.1 of [22], the pair (E_c, A_c) is regular and impulse-free.

Next, we will show the asymptotical stability of system (4). Since A_{c4} is nonsingular, we set $\widehat{G} = \begin{bmatrix} I_{2r} & A_{c2}A_{c4}^{-1} \\ 0 & A_{c4}^{-1} \end{bmatrix} G$. It is easy to see that

$$\begin{aligned} \widehat{G}E_cH &= \begin{bmatrix} I_{2r} & 0 \\ 0 & 0 \end{bmatrix}, \\ \widehat{G}A_cH &= \begin{bmatrix} \widehat{A}_{c1} & 0 \\ \widehat{A}_{c3} & I \end{bmatrix}, \\ H^{-1}P_{k1}\widehat{G}^T &= \begin{bmatrix} \widehat{P}_{k11} & \widehat{P}_{k12} \\ \widehat{P}_{k13} & \widehat{P}_{k14} \end{bmatrix}, \\ H^{-1}P_{k2}\widehat{G}^T &= \begin{bmatrix} \widehat{P}_{k21} & \widehat{P}_{k22} \\ \widehat{P}_{k23} & \widehat{P}_{k24} \end{bmatrix}, \\ & k = 1, 2, \end{aligned} \tag{19}$$

where $\widehat{A}_{c1} = A_{c1} - A_{c2}A_{c4}^{-1}A_{c3}$ and $\widehat{A}_{c3} = A_{c4}^{-1}A_{c3}$.

From (12)–(15) and using the expression in (19), we obtain

$$\begin{aligned} \widehat{P}_{k11} &= \widehat{P}_{k11}^T > 0, \\ \widehat{P}_{k21} + \widehat{P}_{k21}^T &= 0, \\ & k = 1, 2, \\ \sum_{i=1}^{2}\sum_{j=1}^{2} \operatorname{Sym}\left\{\Gamma_{ij} \otimes \left(\widehat{A}_{c1}\widehat{P}_{ij1}\right)\right\} &< 0, \\ \begin{bmatrix} \widehat{P}_{111} & \widehat{P}_{121} \\ -\widehat{P}_{121} & \widehat{P}_{111} \end{bmatrix} &\geq 0, \\ \begin{bmatrix} \widehat{P}_{211} & \widehat{P}_{221} \\ -\widehat{P}_{221} & \widehat{P}_{211} \end{bmatrix} &> 0. \end{aligned} \tag{20}$$

Define $\begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} = H^{-1}\eta(t)$; closed-system (4) is the restricted system equivalent to

$$\begin{aligned} D^{\alpha}\eta_1(t) &= \widehat{A}_{c1}\eta_1(t), \\ 0 &= \widehat{A}_{c3}\eta_1(t) + \eta_2(t), \\ & 0 < \alpha < 1. \end{aligned} \tag{21}$$

By Lemma 6, we can derive from (20) that $|\arg(\operatorname{spec}(\widehat{A}_{c1}))| > \alpha(\pi/2)$, $\lim_{t\to\infty}\|\eta_1(t)\| = 0$, and

$$\lim_{t\to\infty}\|\eta_2(t)\| = \lim_{t\to\infty}\left\|\widehat{A}_{c3}\eta_1(t)\right\| = 0. \tag{22}$$

Therefore, by Definition 5, system (4) is admissible.

Necessity. Since (E_c, A_c) is admissible, there exist nonsingular matrices L and R such that

$$\begin{aligned} LE_cR &= \begin{bmatrix} I_{2r} & 0 \\ 0 & 0 \end{bmatrix}, \\ LA_cR &= \begin{bmatrix} \overline{A}_{c1} & 0 \\ 0 & I \end{bmatrix}. \end{aligned} \tag{23}$$

By Lemma 7, there exist two real symmetric positive definite matrices $\overline{P}_{k11} \in R^{2r\times 2r}$, $k = 1, 2$, and two skew-symmetric matrices $\overline{P}_{k21} \in R^{2r\times 2r}$, $k = 1, 2$, such that

$$\sum_{i=1}^{2}\sum_{j=1}^{2} \operatorname{Sym}\left\{\Gamma_{ij} \otimes \left(\overline{A}_{c1}\overline{P}_{ij1}\right)\right\} < 0, \tag{24}$$

$$\begin{aligned} \begin{bmatrix} \overline{P}_{111} & \overline{P}_{121} \\ -\overline{P}_{121} & \overline{P}_{111} \end{bmatrix} &> 0, \\ \begin{bmatrix} \overline{P}_{211} & \overline{P}_{221} \\ -\overline{P}_{221} & \overline{P}_{211} \end{bmatrix} &> 0. \end{aligned} \tag{25}$$

Define

$$\begin{aligned} P_{k1} &= R\begin{bmatrix} \overline{P}_{k11} & 0 \\ 0 & I \end{bmatrix}L^{-T}, \\ P_{k2} &= R\begin{bmatrix} \overline{P}_{k21} & 0 \\ 0 & 0 \end{bmatrix}L^{-T}, \\ & k = 1, 2. \end{aligned} \tag{26}$$

We can easily verify that the conditions in (12)–(15) hold. This completes the proof. □

Remark 9. Sufficient and necessary conditions are proposed for the admissibility of singular fractional-order systems with fractional-order α belonging to $0 < \alpha < 1$. The conditions are formulated in terms of nonstrict LMI with equality constraint. Compared with the results obtained in [15], the conditions proposed in Theorem 8 are more convenient to controller design.

By Lemma 6, we have the following theorem for the admissibility of closed-system (4) with fractional-order α belonging to $1 \leq \alpha < 2$.

Theorem 10. *Closed-loop system (4) with fractional-order α belonging to $1 \leq \alpha < 2$ is admissible, if and only if there exists nonsingular matrix $P_c \in R^{2n\times 2n}$ such that*

$$\begin{aligned} \operatorname{Sym}\{\Gamma \otimes (A_cP_c)\} &< 0, \\ E_cP_c = P_c^TE_c^T &\geq 0, \end{aligned} \tag{27}$$

where

$$\Gamma = \begin{bmatrix} \sin\frac{\pi}{2}\alpha & -\cos\frac{\pi}{2}\alpha \\ \cos\frac{\pi}{2}\alpha & \sin\frac{\pi}{2}\alpha \end{bmatrix}. \tag{28}$$

Proof. The proof is similar to that of Theorem 8; we omit it. □

Remark 11. To the author's best knowledge, it is for the first time that sufficient and necessary conditions are proposed for the admissibility of singular fractional-order systems with fractional-order $1 \le \alpha < 2$. The results obtained in Theorems 8 and 10 can be easily extended to singular fractional-order systems with norm-bounded parameter uncertainties (i.e., in [17]).

4. Dynamic Output Controller Design

In this section, we study the design of DOFC for singular fractional-order system (1) with fractional-order α belonging to $0 < \alpha < 1$ and $1 \le \alpha < 2$. The following lemma will be used in this section.

Lemma 12 (see [23]). *The matrix inequality* $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} > 0$ *holds, if and only if*

$$\begin{aligned} R &\ge 0, \\ Q - SR^{+}S^{T} &\ge 0, \\ S\left(I - RR^{+}\right) &= 0. \end{aligned} \tag{29}$$

Based on the conditions obtained in Theorem 8, the following theorem is presented to construct the desired controller.

Theorem 13. *Closed-system (3) with fractional-order α belonging to $0 < \alpha < 1$ is admissible, if there exist matrices X, Y, Θ, Ω, and Λ with appropriate dimensions such that*

$$\begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} = \begin{bmatrix} X^T & I \\ I & Y^T \end{bmatrix} \begin{bmatrix} E^T & 0 \\ 0 & E \end{bmatrix} \ge 0, \tag{30}$$

$$\operatorname{Sym}\{\Gamma_{11} \otimes \Psi\} + \operatorname{Sym}\{\Gamma_{21} \otimes \Psi\} < 0, \tag{31}$$

where

$$\Psi = \begin{bmatrix} AX + B\Theta & A \\ \Lambda & YA + \Omega C \end{bmatrix}. \tag{32}$$

Moreover, if the above conditions in (30) and (31) are feasible, the system matrices of an admissible DOFC in the form of (3) are given by

$$\begin{aligned} A_k &= Y^{-1}\left[\Lambda - YAX - \Omega CX - \Omega D\Theta - YB\Theta\right] \\ &\quad \cdot \left(Y^{-1} - X\right)^{-1}, \\ B_k &= Y^{-1}\Omega, \\ C_k &= \Theta\left(Y^{-1} - X\right)^{-1}. \end{aligned} \tag{33}$$

Proof. By setting $P_{11} = P_{21} = P_c$ and $P_{12} = P_{22} = 0$ and using Theorem 8, sufficient conditions for the admissibility of closed-loop system (4) can be obtained as

$$\operatorname{Sym}\{\Gamma_{11} \otimes (A_c P_c)\} + \operatorname{Sym}\{\Gamma_{21} \otimes (A_c P_c)\} < 0, \tag{34}$$

$$E_c P_c = P_c^T E_c^T \ge 0. \tag{35}$$

Define

$$P_c = \begin{bmatrix} X & Y^{-1} - X \\ Y^{-1} - X & X - Y^{-1} \end{bmatrix}. \tag{36}$$

Without loss of generality, it is assumed that matrices Y and $(Y^{-1} - X)$ are nonsingular; if not, Y and $(Y^{-1} - X)$ may be perturbed by ΔY and ΔX, respectively, with sufficient small norm such that $Y + \Delta Y$ and $(Y^{-1} - (X + \Delta X))$ are nonsingular and satisfy (31). From (30) and by using the properties of Moore-Penrose inverse, we can get

$$\begin{aligned} EX &= X^T E^T \ge 0, \\ E^T Y &= Y^T E \ge 0, \\ EX &- E\left(E^T Y\right)^{+} E^T \\ &= EX - Y^{-T}\left(Y^T E\right)\left(E^T Y\right)^{+}\left(E^T Y\right) Y^{-1} \\ &= EX - Y^{-T}\left(E^T Y\right) Y^{-1} = EX - Y^{-T} E^T \\ &= \left(X - Y^{-1}\right)^T E^T = E\left(X - Y^{-1}\right) \ge 0. \end{aligned} \tag{37}$$

Moreover, from (37), we have

$$\begin{aligned} &EX - E\left(Y^{-1} - X\right)\left(E\left(X - Y^{-1}\right)\right)^{+} E\left(Y^{-1} - X\right) \\ &\quad = EX + E\left(Y^{-1} - X\right) = EY^{-1} \ge 0, \\ &E\left(Y^{-1} - X\right)\left[I - E\left(X - Y^{-1}\right)\left[E\left(X - Y^{-1}\right)\right]^{+}\right] \\ &\quad = E\left(Y^{-1} - X\right) + E\left(X - Y^{-1}\right)\left[E\left(X - Y^{-1}\right)\right]^{T} \\ &\quad \cdot \left[\left(E\left(X - Y^{-1}\right)\right)^{+}\right]^{T} = E\left(Y^{-1} - X\right) \\ &\quad + E\left(X - Y^{-1}\right)\left[\left(E\left(X - Y^{-1}\right)\right)^{+} E\left(X - Y^{-1}\right)\right]^{T} \\ &\quad = E\left(Y^{-1} - X\right) + E\left(X - Y^{-1}\right)\left(E\left(X - Y^{-1}\right)\right)^{+} \\ &\quad \cdot E\left(X - Y^{-1}\right) = E\left(Y^{-1} - X\right) + E\left(X - Y^{-1}\right) = 0. \end{aligned} \tag{38}$$

By Lemma 12, inequality (35) holds.

On the other hand, let $\Pi = \begin{bmatrix} I_n & 0 \\ Y & Y \end{bmatrix}$, and pre- and postmultiplying (34) by diag$\{\Pi, \Pi\}$ and its transpose, respectively, we obtain

$$\operatorname{Sym}\{\Gamma_{11} \otimes \Phi\} + \operatorname{Sym}\{\Gamma_{21} \otimes \Phi\} < 0, \tag{39}$$

where

$$\Phi = \begin{bmatrix} AX + BC_k\left(Y^{-1} - X\right) & A \\ Y\left(A + B_kC\right)X + Y\left(A_k + B_kDC_k + BC_k\right)\left(Y^{-1} - X\right) & YA + YB_kC \end{bmatrix}. \tag{40}$$

Considering the controller matrices given in (33), we obtain (34) from (39). This completes the proof. □

Remark 14. By using the characteristics of Moore-Penrose inverse, Theorem 13 provides sufficient conditions for the solvability of DOFC design problem for singular fractional-order systems with fractional-order α belonging to $0 < \alpha < 1$. The matrices of desired DOFC can be constructed through the solutions of LMIs (30)-(31).

The following theorem is proposed for the design of DOFC for singular fractional-order systems with fractional-order α belonging to $1 \le \alpha < 2$.

Theorem 15. *Closed-system (3) with fractional-order α belonging to $1 \le \alpha < 2$ is admissible, if there exist matrices X, Y, Θ, Ω, and Λ with appropriate dimensions such that*

$$\begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix}\begin{bmatrix} X & I \\ I & Y \end{bmatrix} = \begin{bmatrix} X^T & I \\ I & Y^T \end{bmatrix}\begin{bmatrix} E^T & 0 \\ 0 & E \end{bmatrix} \ge 0, \tag{41}$$

$$\operatorname{Sym}\{\Gamma \otimes \Psi\} < 0, \tag{42}$$

where

$$\Psi = \begin{bmatrix} AX + B\Theta & A \\ \Lambda & YA + \Omega C \end{bmatrix}. \tag{43}$$

Moreover, if the above conditions in (41)-(42) are feasible, the system matrices of an admissible DOFC in the form of (3) are given by

$$\begin{aligned} A_k &= Y^{-1}\left[\Lambda - YAX - \Omega CX - \Omega D\Theta - YB\Theta\right] \\ &\quad \cdot \left(Y^{-1} - X\right)^{-1}, \\ B_k &= Y^{-1}\Omega, \\ C_k &= \Theta\left(Y^{-1} - X\right)^{-1}. \end{aligned} \tag{44}$$

Proof. The desired result can be carried out by employing the same technique used as in Theorem 13. □

Remark 16. It is noted that the conditions obtained in Theorems 13 and 15 are nonstrict LMIs due to the matrix inequality constraints in (30) and (41), which will lead to numerical problems while checking the conditions. Substituting $X = \overline{X}E^T + \Phi_1 Z_1^T$ and $Y = \overline{Y}E + \Phi_2 Z_2^T$ into (25)-(26) and (41)-(42) can yield strict LMIs, where matrices $\overline{X} = \overline{X}^T > 0$ and $Y = \overline{Y}^T > 0$, matrices $Z_1 \in R^{n\times(n-r)}$ and $Z_2 \in R^{n\times(n-r)}$, $\Phi_1 \in R^{n\times(n-r)}$ is any matrix satisfying $E\Phi_1 = 0$, and $\Phi_2 \in R^{n\times(n-r)}$ is any matrix satisfying $E^T\Phi_2 = 0$.

5. Illustrative Example

Consider the following singular fractional-order system described in (1) with parameters as follows:

$$\begin{aligned} E &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ A &= \begin{bmatrix} -0.5 & 0.5 & 1 \\ -0.5 & -0.5 & -2 \\ 0 & 0 & 0 \end{bmatrix}, \\ B &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \\ C &= \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}, \\ D &= \begin{bmatrix} 1 & -1 \end{bmatrix}. \end{aligned} \tag{45}$$

Case I ($0 < \alpha < 1$). In this case, we choose $\alpha = 0.5$. Using Matlab/LMItool to solve LMIs in (30)-(31), we can get the DOFC parameters to be determined as follows:

$$\begin{aligned} A_k &= \begin{bmatrix} 264.8799 & -67.48 & 9.1465 \\ 213.7928 & -56.1459 & 7.5041 \\ 22.9430 & -7.5717 & 3.5849 \end{bmatrix}, \\ B_k &= \begin{bmatrix} 14.4704 \\ 10.7185 \\ 0.7995 \end{bmatrix}, \\ C_k &= \begin{bmatrix} -13.7721 & 2.1500 & -0.4889 \\ 7.5194 & -3.6967 & 3.1078 \end{bmatrix}. \end{aligned} \tag{46}$$

In addition, $|\arg(\operatorname{spec}(E_c, A_c))| = \pi > \pi/4$. By Definition 5, the obtained DOFC can asymptotically stabilize the singular fractional-order system with fractional-order $\alpha = 0.5$. Setting the initial conditions $x_0 = \begin{bmatrix} 0.5 & -0.5 & 1 \end{bmatrix}$, the time response of closed-loop system (4) is illustrated in Figure 1. From Figure 1, we can observe that the trajectories of fractional-order system (45) with fractional-order

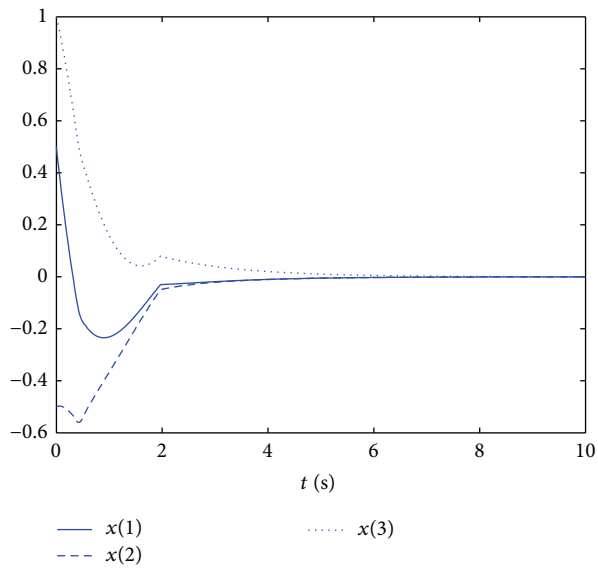

Figure 1: Time response of the closed-loop system with fractional-order $\alpha = 0.5$ under DOFC.

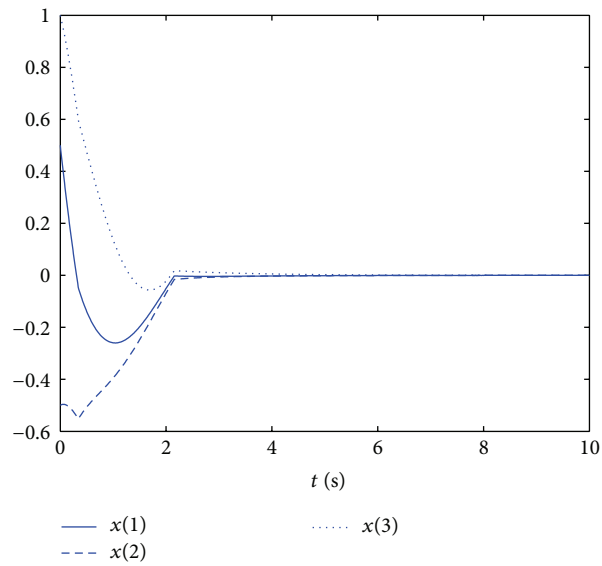

Figure 2: Time response of the closed-loop system with fractional-order $\alpha = 1.5$ under DOFC.

$\alpha = 0.5$ can asymptotically converge to zero under the designed DOFC controller.

Case II ($1 \leq \alpha < 2$). In this case, we choose $\alpha = 1.5$. Using Matlab/LMItool again to solve LMIs in (41)-(42), we can get the DOFC parameters to be determined as follows:

$$A_k = \begin{bmatrix} 264.8799 & -67.48 & 9.1465 \\ 213.7928 & -56.1459 & 7.5041 \\ 22.9430 & -7.5717 & 3.5849 \end{bmatrix},$$
$$B_k = \begin{bmatrix} 14.4704 \\ 10.7185 \\ 0.7995 \end{bmatrix}, \tag{47}$$
$$C_k = \begin{bmatrix} -13.7721 & 2.1500 & -0.4889 \\ 7.5194 & -3.6967 & 3.1078 \end{bmatrix}.$$

In this situation, $|\arg(\mathrm{spec}(E_c, A_c))| = \pi > 3\pi/4$. By Definition 5, the obtained DOFC can asymptotically stabilize the singular fractional-order system with fractional-order $\alpha = 1.5$. Setting the initial conditions $x_0 = [0.5 \ \ -0.5 \ \ 1]$, the time response of closed-loop system (4) is illustrated in Figure 2. From Figure 2, we can observer that the trajectories of fractional-order system (45) with fractional-order $\alpha = 1.5$ can asymptotically converge to zero under the designed DOFC controller.

6. Conclusion

In this paper, the problem of DOFC design for singular fractional-order systems with fractional-order α satisfying $0 < \alpha < 2$ has been investigated. Sufficient and necessary conditions for the admissibility of the systems have been derived, which are formulated in terms of LMIs. Based on the obtained conditions, the desired DOFC have been designed. An illustrative example has been given to show the effectiveness of the proposed method.

Competing Interests

The authors declare that they have no competing interests.

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