Hindawi
Mathematical Problems in Engineering
Volume 2017, Article ID 1378740, 12 pages
https://doi.org/10.1155/2017/1378740

Research Article

(2 + 1)-Dimensional Coupled Model for Envelope Rossby Solitary Waves and Its Solutions as well as Chirp Effect

Xin Chen,[1] Hongwei Yang,[1,2] Min Guo,[1] and Baoshu Yin[3,4]

[1] *Shandong University of Science and Technology, Qingdao, Shandong 266590, China*
[2] *Key Laboratory of Meteorological Disaster of Ministry of Education, Nanjing University of Information Science and Technology, Nanjing 210044, China*
[3] *Key Laboratory of Ocean Circulation and Waves, Institute of Oceanology, Chinese Academy of Sciences, Qingdao 266071, China*
[4] *Function Laboratory for Ocean Dynamics and Climate, Qingdao National Laboratory for Marine Science and Technology, Qingdao 266237, China*

Correspondence should be addressed to Hongwei Yang; hwyang1979@163.com

Received 22 June 2017; Accepted 24 August 2017; Published 18 October 2017

Academic Editor: Jian G. Zhou

Using the method of multiple scales and perturbation method, a set of coupled models describing the envelope Rossby solitary waves in (2 + 1)-dimensional condition are obtained, also can be called coupled NLS (CNLS) equations. Following this, based on trial function method, the solutions of the NLS equation are deduced. Moreover, the modulation instability of coupled envelope Rossby waves is studied. We can find that the stable feature of coupled envelope Rossby waves is decided by the value of S. Finally, learning from the concept of chirp in the optical soliton communication field, we study the chirp effect caused by nonlinearity and dispersion in the propagation of Rossby waves.

1. Introduction

Wave phenomenon exists widely in nature. As a special and important branch of waves, Rossby solitary waves have important theoretical significance and research value. Meanwhile, with the intercross and penetration of different knowledge, the Rossby solitary waves theory has applied to many other fields successfully, such as physical oceanography, atmospheric physics, hydraulic engineering, communication engineering, and thermal power engineering. In the case of application of solitary waves in engineering, the application of optical soliton in communication engineering is the most representative. Meanwhile, the Rossby waves theory was widely used in the study of mesoscale eddies and the interaction of large and medium scale motions. Rossby solitary waves in the westerly shear flow were first found by Long [1]. Afterward, Benny [2] amplified this research and found that velocity and amplitude of Rossby waves were proportional and depicted the importance of nonlinearity. In view of the barotropic fluid and stratified fluid model, the KdV and mKdV equation are also generated to describe the generation and evolution of Rossby solitary waves by Redekopp [3]. Compared to KdV equation, the mKdV equation is more suitable to express the condition with stronger perturbation. With the development of solitary waves, a variety of equation models for describing the Rossby solitary waves such as ILW-Burgers equation and ZK-Burgers equation were discussed by Yang et al. [4, 5]. Moreover, the generation and evolution of solitary waves in different topography condition and different fluid depths were discussed. Recently, the Rossby parameters β along with the changes of latitude were discussed by Luo [6] and β plane approximation was obtained. These Rossby solitary waves are called classical solitary waves which related to the KdV-type equations. Later, many researchers have studied the Rossby wave equation in many aspects [7, 8], such as integrable system [9, 10], the integrable coupling of equations [11], and Hamiltonian structures [12]. Unlike the KdV-type equation, the nonlinear Schrödinger equations were used to study the evolution of envelope classical Rossby solitary waves. From the end of the 70's to the 80's, driven by the study of atmospheric blocking dynamics [13], nonlinear Rossby wave theory had been developing rapidly and gradually formed Rossby solitary waves theory and dipole waves theory. In addition, beside the above two theories, envelope Rossby solitary waves also dropped in the research scope of the theme. The envelope Rossby solitons in the barotropic shear and uniform flows were first investigated by Benney [14] and Yamagata [15]. Afterward, Luo [16] tried to use this envelope Rossby solitons to explain atmospheric blocking phenomenon. Later, dissipative NLS equation in rotational stratified fluids and its solution were obtained by Shi et al. [17]. As we all know, using the nonlinear Schrödinger equation describing the Rossby solitary waves, we can introduce the concept of chirp in optical soliton communication [18], to study the influence of dispersion and nonlinearity on solitary waves propagation process. In the optical soliton communication field, the concept of chirp [19, 20] is the phenomenon that the central wavelength shifts when the pulse is transmitted. It is helpful to analyze the propagation characteristics and the formation mechanism of solitary waves.

In the domain of solitary wave models, it is necessary to obtain the exact solutions of solitary wave models by all kinds of solving methods and analyze the feature of solitary waves in propagation process based on the exact solutions. Many methods to solve the equations are proposed, such as traveling wave method [21], Darboux transformation method [22–24], Hirota method [25, 26], homogeneous balance method [27], Jacobi elliptic function method [28], Symmetry method [29, 30], Rational solutions [31–33]; meanwhile the features of equations are also discussed [34–36]. In this paper, we plan to adopt the trial function method and derive the exact solution of model. The difference between the two-dimensional and three-dimensional model will be given and some features of three-dimensional NLS equation will be discussed.

We note that the above researches commonly considered two-dimensional model or single (2 + 1)-dimensional model to reflect the evolution of envelope Rossby solitary waves. There are two disadvantages:

(1) The two-dimensional model can only be applied to describe the evolution of envelope Rossby solitary waves in a line.

(2) The velocity of the KdV-type soliton is larger than the real observation.

While, as we know, the (2 + 1)-dimensional model can be applied to reflect the evolution of envelope Rossby solitary waves in a plane, which is more suitable for the real ocean and atmosphere, in this paper, by using the method of multiple scales and perturbation method, starting from the barotropic atmospheric vorticity equation, we will derive the coupled (2 + 1)-dimensional nonlinear Schrödinger equations for envelope Rossby solitary waves in Section 2. Not only is the model (2 + 1)-dimensional and more suitable to describe the feature of two envelope Rossby solitary waves in a plane, particularly, but also it is a coupled model and can show the interaction process between two waves. Then, based on trial function method, we will deduce the solution of the CNLS equations group and the envelope solitary waves characteristics in Section 3. Thirdly, we study the modulation instability of Rossby waves trains in (2 + 1)-dimensional condition in Section 4. Finally, the concept of chirp in optical soliton communication is introduced, and the chirp effect caused by dispersion and nonlinearity is also discussed in Section 5.

2. Derivation of the (2 + 1)-Dimensional CNLS Equations

The tropical atmospheric motion is quasihorizontal and quasiconvergent. The governing equation is the quasigeostrophic barotropic vorticity equation [37].

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\overline{U}(y)\frac{\partial}{\partial x}\right)\nabla^2\psi+\left(\beta-\overline{U}''\right)\frac{\partial\psi}{\partial x}\\=-\epsilon J\left(\psi,\nabla^2\psi\right),\end{aligned}\tag{1}$$

and the boundary condition is

$$\frac{\partial\psi}{\partial x}=0,\quad \text{as } y=0,L_y,\tag{2}$$

where x and y are the local Cartesian coordinates pointing east and north. In this $\beta=\beta^* L^2/U$, β^* is Rossby parameter. U is characteristic velocity, L is characteristic length, and L_y is the width of the beta-channel. ϵ is small parameter, on behalf of nonlinear strength. J is the Jacobian of (A, B).

$$J(A,B)=\frac{\partial A}{\partial x}\frac{\partial B}{\partial y}-\frac{\partial A}{\partial y}\frac{\partial B}{\partial x},\tag{3}$$

and ∇^2 is Laplace operator

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.\tag{4}$$

In general, it is difficult to get analytic solution of (1). But, because of the nonlinear term with a small parameter ε, we use the multiple scales method and perturbation method to obtain the nonlinear solution of it. As a preliminary study, we will consider two waves.

Let us introduce the slow time and space variables

$$\begin{aligned}T_1&=\epsilon t,\\T_2&=\epsilon^2 t,\\X_1&=\epsilon x,\\X_2&=\epsilon^2 x,\\Y&=\epsilon y,\end{aligned}\tag{5}$$

so long time and space scales are defined as

$$\begin{aligned}\frac{\partial}{\partial x} &\longrightarrow \frac{\partial}{\partial x}+\epsilon\frac{\partial}{\partial X_1}+\epsilon^2\frac{\partial}{\partial X_2},\\ \frac{\partial}{\partial y} &\longrightarrow \frac{\partial}{\partial y}+\epsilon\frac{\partial}{\partial Y},\\ \frac{\partial}{\partial t} &\longrightarrow \frac{\partial}{\partial t}+\epsilon\frac{\partial}{\partial T_1}+\epsilon^2\frac{\partial}{\partial T_2}.\end{aligned} \tag{6}$$

Substituting (6) into (1), we obtain

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+\epsilon\frac{\partial}{\partial T_1}+\epsilon^2\frac{\partial}{\partial T_2}\right.\\ &\quad\left.+\overline{U}(y)\left(\frac{\partial}{\partial x}+\epsilon\frac{\partial}{\partial X_1}+\epsilon^2\frac{\partial}{\partial X_2}\right)\right]\left[\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right.\\ &\quad+2\epsilon\left(\frac{\partial^2}{\partial x\partial X_1}+\frac{\partial^2\psi}{\partial y\partial Y}\right)\\ &\quad+\epsilon^2\left(\frac{\partial^2\psi}{\partial X_1}+2\frac{\partial^2\psi}{\partial x\partial X_2}+\frac{\partial^2\psi}{\partial Y^2}\right)+2\epsilon^3\frac{\partial^2\psi}{\partial X_1\partial X_2}\\ &\quad\left.+\epsilon^4\frac{\partial^2\psi}{\partial X_2^2}\right]+\left(\beta-\overline{U}''\right)\left(\frac{\partial\psi}{\partial x}+\epsilon\frac{\partial\psi}{\partial X_1}+\epsilon^2\frac{\partial\psi}{\partial X_2}\right)\\ &\quad+\epsilon\left(\frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x}\right)=0,\\ &\qquad\qquad\frac{\partial\psi}{\partial x}=0,\quad y=0,L_y.\end{aligned} \tag{7}$$

Further

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+\overline{U}\frac{\partial}{\partial x}\right]\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)+\left(\beta-\overline{U}''\right)\frac{\partial\psi}{\partial x}\\ &\quad+\epsilon\left[\left(\frac{\partial}{\partial T_1}+\overline{U}\frac{\partial}{\partial X_1}\right)\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)+\left(\frac{\partial}{\partial t}\right.\right.\\ &\quad\left.+\overline{U}\frac{\partial}{\partial x}\right)\left(2\frac{\partial^2\psi}{\partial x\partial X_1}+2\frac{\partial^2\psi}{\partial y\partial Y}\right)+\left(\beta-\overline{U}''\right)\frac{\partial\psi}{\partial X_1}\\ &\quad\left.+\frac{\partial\psi}{\partial x}\left(\frac{\partial^3\psi}{\partial y\partial x^2}+\frac{\partial^3\psi}{\partial y^3}\right)-\frac{\partial\psi}{\partial y}\left(\frac{\partial^3\psi}{\partial x^3}+\frac{\partial^3\psi}{\partial x\partial y^2}\right)\right]\\ &\quad+\epsilon^2\left[\left(\frac{\partial}{\partial T_2}+\overline{U}\frac{\partial}{\partial X_2}\right)\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)+\left(\frac{\partial}{\partial T_1}\right.\right.\\ &\quad\left.+\overline{U}\frac{\partial}{\partial X_1}\right)\left(2\frac{\partial^2\psi}{\partial x\partial X_1}+2\frac{\partial^2\psi}{\partial y\partial Y}\right)+\left(\frac{\partial}{\partial t}+\overline{U}\frac{\partial}{\partial x}\right)\\ &\quad\cdot\left(\frac{\partial^2\psi}{\partial X_1^2}+\frac{\partial^2\psi}{\partial Y^2}\right)+\left(\beta-\overline{U}''\right)\frac{\partial\psi}{\partial X_2}\\ &\quad+\frac{\partial\psi}{\partial x}\left(2\frac{\partial^3\psi}{\partial y\partial x\partial X_1}+3\frac{\partial^3\psi}{\partial y^2\partial Y}+\frac{\partial^3\psi}{\partial x^2\partial Y}\right)\\ &\quad+\frac{\partial\psi}{\partial X_1}\left(\frac{\partial^3\psi}{\partial y\partial x^2}+\frac{\partial^3\psi}{\partial y^3}\right)-\frac{\partial\psi}{\partial y}\left(2\frac{\partial^3\psi}{\partial x^2\partial X_1}\right.\\ &\quad\left.\left.+2\frac{\partial^3\psi}{\partial y\partial x\partial Y}+\frac{\partial^3\psi}{\partial y^2\partial X_1}\right)-\frac{\partial\psi}{\partial Y}\left(\frac{\partial^3\psi}{\partial x^3}+\frac{\partial^3\psi}{\partial x\partial y^2}\right)\right]\\ &\quad+\epsilon^3\left[\left(\frac{\partial}{\partial t}+\overline{U}\frac{\partial}{\partial x}\right)\frac{\partial^2\psi}{\partial X_1\partial X_2}+\frac{\partial\psi}{\partial x}\left(\frac{\partial^3\psi}{\partial y\partial X_1^2}\right.\right.\\ &\quad\left.+\frac{\partial^3\psi}{\partial y\partial x\partial X_2}+2\frac{\partial^3\psi}{\partial x\partial Y\partial X_1}+3\frac{\partial^3\psi}{\partial y\partial Y^2}\right)\\ &\quad+\frac{\partial\psi}{\partial X_1}\left(2\frac{\partial^3\psi}{\partial y\partial x\partial X_1}+2\frac{\partial^3\psi}{\partial y^2\partial Y}+\frac{\partial^3\psi}{\partial Y\partial x^2}\right.\\ &\quad\left.\left.+\frac{\partial^3\psi}{\partial Y\partial y^2}\right)\right]+o\left(\epsilon^3\right)=0,\end{aligned} \tag{8}$$

and the boundary condition is

$$\frac{\partial\psi}{\partial x}=0,\quad \text{as } y=0,L_y. \tag{9}$$

The stream function ψ is expanded according to the small parameter ϵ

$$\psi=\psi^{(0)}+\epsilon\psi^{(1)}+\epsilon^2\psi^{(2)}+\epsilon^3\psi^{(3)}+\cdots, \tag{10}$$

and, substituting (9) into (8), we get the multiple order questions of the stream function ψ.

First, one introduces an operator

$$L=\left[\frac{\partial}{\partial t}+\overline{U}\frac{\partial}{\partial x}\right]\nabla^2+\left(\beta-\overline{U}''\right)\frac{\partial}{\partial x}, \tag{11}$$

so that

$$\begin{aligned}&o\left(\varepsilon^0\right): L\left(\psi^{(0)}\right)\\ &\quad=\left(\frac{\partial}{\partial t}+\overline{U}\frac{\partial}{\partial x}\right)\nabla^2\psi^{(0)}+\left(\beta-\overline{U}''\right)\frac{\partial\psi^{(0)}}{\partial x},\\ &o\left(\varepsilon^1\right): L\left(\psi^{(1)}\right)\\ &\quad=\left(\frac{\partial}{\partial T_1}+\overline{U}\frac{\partial}{\partial X_1}\right)\left(\frac{\partial^2\psi^{(0)}}{\partial x^2}+\frac{\partial^2\psi^{(0)}}{\partial y^2}\right)\\ &\qquad+\left(\frac{\partial}{\partial t}+\overline{U}\frac{\partial}{\partial x}\right)\left(2\frac{\partial^2\psi^{(0)}}{\partial x\partial X_1}+2\frac{\partial^2\psi^{(0)}}{\partial y\partial Y}\right)\\ &\qquad+\left(\beta-\overline{U}''\right)\frac{\partial\psi^{(0)}}{\partial X_1}+\frac{\partial\psi^{(0)}}{\partial x}\left(\frac{\partial^3\psi^{(0)}}{\partial y\partial x^2}+\frac{\partial^3\psi^{(0)}}{\partial y^3}\right)\\ &\qquad-\frac{\partial\psi^{(0)}}{\partial y},\end{aligned}$$

$$
\begin{aligned}
o\left(\varepsilon^{2}\right):\ & L\left(\psi^{(2)}\right) \\
={}& \left(\frac{\partial}{\partial T_2}+\overline{U}\frac{\partial}{\partial X_2}\right)\left(\frac{\partial^2\psi^{(0)}}{\partial x^2}+\frac{\partial^2\psi^{(0)}}{\partial y^2}\right) \\
&+\left(\frac{\partial}{\partial T_1}+\overline{U}\frac{\partial}{\partial X_1}\right)\left(2\frac{\partial^2\psi^{(0)}}{\partial x\partial X_1}+2\frac{\partial^2\psi^{(0)}}{\partial y\partial Y}\right) \\
&+\left(\frac{\partial}{\partial t}+\overline{U}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2\psi^{(0)}}{\partial X_1^2}+\frac{\partial^2\psi^{(0)}}{\partial Y^2}\right) \\
&+\left(\beta-\overline{U}''\right)\frac{\partial\psi^{(0)}}{\partial X_2} \\
&+\frac{\partial\psi^{(0)}}{\partial x}\left(2\frac{\partial^3\psi^{(0)}}{\partial y\partial x\partial X_1}+3\frac{\partial^3\psi^{(0)}}{\partial y^2\partial Y}+\frac{\partial^3\psi^{(0)}}{\partial x^2\partial Y}\right) \\
&+\frac{\partial\psi^{(0)}}{\partial X_1}\left(\frac{\partial^3\psi^{(0)}}{\partial y\partial x^2}+\frac{\partial^3\psi^{(0)}}{\partial y^3}\right) \\
&-\frac{\partial\psi^{(0)}}{\partial y}\left(2\frac{\partial^3\psi^{(0)}}{\partial x^2\partial X_1}+2\frac{\partial^3\psi^{(0)}}{\partial y\partial x\partial Y}+\frac{\partial^3\psi^{(0)}}{\partial y^2\partial X_1}\right) \\
&+\frac{\partial\psi^{(0)}}{\partial Y}\left(\frac{\partial^3\psi^{(0)}}{\partial x^3}+\frac{\partial^3\psi^{(0)}}{\partial x\partial y^2}\right) \\
&-\left(\frac{\partial}{\partial T_1}+\overline{U}\frac{\partial}{\partial X_1}\right)\left(\frac{\partial^2\psi^{(1)}}{\partial x^2}-\frac{\partial^2\psi^{(1)}}{\partial y^2}\right) \\
&-\left(\frac{\partial}{\partial t}+\overline{U}\frac{\partial}{\partial x}\right)\left(2\frac{\partial^2\psi^{(1)}}{\partial x\partial X_1}+2\frac{\partial^2\psi^{(1)}}{\partial y\partial Y}\right) \\
&+\left(\beta-\overline{U}''\right)\frac{\partial\psi^{(1)}}{\partial X_1}-\frac{\partial\psi^{(1)}}{\partial x}\left(\frac{\partial^3\psi^{(1)}}{\partial y\partial x^2}+\frac{\partial^3\psi^{(1)}}{\partial y^3}\right) \\
&+\frac{\partial\psi^{(1)}}{\partial y}\left(\frac{\partial^3\psi^{(1)}}{\partial x^3}+\frac{\partial^3\psi^{(1)}}{\partial x\partial y^2}\right) \\
&\frac{\partial\psi}{\partial x}=0,\quad y=0,L_y.
\end{aligned}
\tag{12}
$$

Assume

$$
\begin{aligned}
\psi^{(0)}={}&\sum_{j=1}^{2}A_j\left(T_1,T_2,X_1,X_2,Y\right)\phi_j\left(y\right)e^{i(K_jx-\omega_jt)} \\
&+c.c.\ ,
\end{aligned}
\tag{13}
$$

and, substituting (13) into $L(\psi^{(0)})$, we have

$$
\begin{aligned}
L\left(\psi^{(0)}\right)={}&\left(\frac{\partial}{\partial t}+\overline{U}\frac{\partial}{\partial x}\right)\left[-\sum_{j=1}^{2}A_j\phi_jK_j^2e^{i(K_jx-\omega_jt)}\right. \\
&\left.+\sum_{j=1}^{2}A_j\frac{d^2\phi_j}{dy^2}e^{i(K_jx-\omega_jt)}\right]+\left(\beta-\overline{U}''\right) \\
&\cdot\sum_{j=1}^{2}A_j\phi_jiK_je^{i(K_jx-\omega_jt)}=\sum_{j=1}^{2}A_je^{i(K_jx-\omega_jt)}\left[i\omega_jK_j^2\phi_j\right. \\
&-\omega_j\frac{d^2\phi_j}{dy^2}-i\overline{U}K_j^3\phi_j+i\overline{U}K_j\frac{d^2\phi_j}{dy^2} \\
&\left.+iK_j\left(\beta-\overline{U}''\right)\phi_j\right]\equiv 0,
\end{aligned}
\tag{14}
$$

and further

$$
\begin{aligned}
&-\left(i\omega_j-i\overline{U}K_j\right)\frac{d^2\phi_j}{dy^2} \\
&\quad-\left[i\overline{U}K_j^3-i\omega_jK_j^2-iK_j\left(\beta-\overline{U}''\right)\right]\phi_j=0,
\end{aligned}
\tag{15}
$$

$$
\frac{d^2\phi_j}{dy^2}-\left[K_j^2-\frac{\overline{\beta}-\overline{U}''}{\overline{U}-C_j}\right]\phi_j=0,
\tag{16}
$$

and the boundary condition is

$$
\frac{\partial\psi}{\partial x}=0,\quad \text{as } y=0,L_y,
\tag{17}
$$

where

$$
C_j=\frac{\omega_j}{K_j}.
\tag{18}
$$

From (15), we can get the solution of ϕ_j:

$$
\begin{aligned}
&\phi_j\left(y\right)=\sqrt{\frac{2}{Ly_j}}\sin\left(my\right), \\
&\qquad m=\frac{n_1\pi}{Ly_j}\quad\left(n_1=\pm1,2,3,\ldots\right).
\end{aligned}
\tag{19}
$$

Formula (15) under the boundary condition poses a standard Sturm-Liouville problem. The effects of zonal flows on linear equation trapped waves were treated in detail by many researchers. The analytic solution of (15) can be obtained when $\overline{U}(y)$ takes some specific functions. Under normal circumstance one can only seek numerical solution, so we consider the higher order question

$$
\begin{aligned}
o\left(\epsilon^{(1)}\right):\ & L\left(\psi^{(1)}\right) \\
={}&-\left(\frac{\partial}{\partial T_1}+\overline{U}\frac{\partial}{\partial X_1}\right)\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right) \\
&-\left(\frac{\partial}{\partial t}+\overline{U}\frac{\partial}{\partial x}\right)\left(2\frac{\partial^2\psi}{\partial x\partial X_1}+2\frac{\partial^2\psi}{\partial y\partial Y}\right) \\
&-\left(\beta-\overline{U}''\right)\frac{\partial\psi}{\partial X_1}-\frac{\partial\psi}{\partial x}\left(\frac{\partial^3\psi}{\partial y\partial x^2}+\frac{\partial^3\psi}{\partial y^3}\right) \\
&+\frac{\partial\psi}{\partial y}\left(\frac{\partial^3\psi}{\partial x^3}+\frac{\partial^3\psi}{\partial x\partial y^2}\right),\quad \frac{\partial\psi}{\partial x}=0,\ y=0,L_y,
\end{aligned}
\tag{20}
$$

and, further, we get

$$L\left(\psi^{(1)}\right) = -\sum_{j=1}^{2} g_{1j} e^{2i(K_j x - \omega_j t)} - \sum_{j=1}^{2} g_{2j} e^{2i(K_j x - \omega_j t)} A_j^2 - i g_3 e^{i[(K_1+K_2)x - (\omega_1+\omega_2)t]} A_1 A_2 - i g_4 e^{i[(K_1-K_2)x - (\omega_1-\omega_2)t]} A_1 A_2, \tag{21}$$

where

$$\begin{aligned} g_{1j} &= \frac{\beta - \overline{U}''}{\overline{U} - C_j} \phi_j \left(\frac{\partial A_j}{\partial T_1} + C_{gj} \frac{\partial A_j}{\partial X_1} \right) - 2iK_j \left(\overline{U} - C_j\right) \frac{d\phi_j}{dy} \frac{\partial A_j}{\partial Y}, \\ g_{2j} &= K_j \left(\phi_j \frac{d}{dy} - \frac{d\phi_j}{dy} \right) \frac{d^2\phi_j}{dy^2}, \\ g_3 &= \left(K_1 \phi_1 \frac{d}{dy} - K_2 \frac{d\phi_1}{dy} \right) \left(\frac{d^2\phi_2}{dy^2} - K_2^2 \phi_2 \right) + \left(K_2 \phi_2 \frac{d}{dy} - K_1 \frac{d\phi_2}{dy} \right) \left(\frac{d^2\phi_1}{dy^2} - K_1^2 \phi_1 \right), \\ g_4 &= \left(K_1 \phi_1 \frac{d}{dy} + K_2 \frac{d\phi_1}{dy} \right) \left(\frac{d^2\phi_2}{dy^2} - K_2^2 \phi_2 \right) - \left(K_2 \phi_2 \frac{d}{dy} + K_1 \frac{d\phi_2}{dy} \right) \left(\frac{d^2\phi_1}{dy^2} - K_1^2 \phi_1 \right). \\ C_{gj} &= C_j + \frac{2K_j^2 \left(\overline{U} - C_j\right)^2}{\beta - \overline{U}''}, \end{aligned} \tag{22}$$

and the special solution corresponding to the second, third, and forth item of (21) right side is

$$\begin{aligned} \psi_{2j}^{(1)} &= A_j^2 \phi_{2j} e^{2i(K_j x - \omega_j t)} + c.c, \\ \psi_3^{(1)} &= A_1 A_2 \phi_3 e^{i[(K_1+K_2)x - (\omega_1+\omega_2)t]} + c.c, \\ \psi_4^{(1)} &= A_1 A_2 \phi_4 e^{i[(K_1-K_2)x - (\omega_1-\omega_2)t]} + c.c, \end{aligned} \tag{23}$$

where ϕ_{2j}, ϕ_3, and ϕ_4 satisfied

$$\begin{aligned} &\frac{d^2\phi_{2j}}{dy^2} - \left(4K_j^2 - \frac{\beta - \overline{U}''}{\overline{U} - c_j} \right) \phi_{2j} = \frac{g_{2j}}{2\left(\overline{U}K_j - \omega_j\right)}, \\ &\frac{d^2\phi_3}{dy^2} - \left[\left(K_1 + K_2\right)^2 - \frac{\left(\beta - \overline{U}''\right)\left(K_1 + K_2\right)}{\overline{U}\left(K_1 + K_2\right) - \left(\omega_1 + \omega_2\right)} \right] \phi_3 = \frac{g_3}{\overline{U}\left(K_1 + K_2\right) - \left(\omega_1 + \omega_2\right)}, \\ &\frac{d^2\phi_4}{dy^2} - \left[\left(K_1 - K_2\right)^2 - \frac{\left(\beta - \overline{U}''\right)\left(K_1 - K_2\right)}{\overline{U}\left(K_1 - K_2\right) - \left(\omega_1 - \omega_2\right)} \right] \phi_4 = \frac{g_4}{\overline{U}\left(K_1 - K_2\right) - \left(\omega_1 - \omega_2\right)}, \end{aligned} \tag{24}$$

and boundary condition

$$\phi_{2j}\big|_{y=0} = \phi_{2j}\big|_{(y=Ly)} = \phi_3\big|_{y=0} = \phi_3\big|_{(y=Ly)} = \phi_4\big|_{y=0} = \phi_4\big|_{(y=Ly)}. \tag{25}$$

We assume the special solution corresponding to the first item of the right side of (20) is

$$\begin{aligned} \psi_{1j}^{(1)} &= \overline{\psi}_{1j}^{(1)} e^{i(K_j x - \omega_j t)} + c.c, \\ \overline{\psi}_{1j}^{(1)}\big|_{y=0} &= \overline{\psi}_{1j}^{(1)}\big|_{y=Ly} = 0, \end{aligned} \tag{26}$$

where $\psi_{1j}^{(1)}$ satisfied

$$\begin{aligned} &\frac{d^2\psi_{1j}^{(1)}}{dy^2} - \left(K_j^2 - \frac{\beta - \overline{U}''}{\overline{U} - C_j} \right) \psi_{1j}^{(1)} \\ &\quad = \frac{\beta - \overline{U}''}{\overline{U} - C_j} \phi_j \left(\frac{\partial A_j}{\partial T_1} + C_{gj} \frac{\partial A_j}{\partial X_1} \right) - 2iK_j \left(\overline{U} - C_j\right) \frac{d\phi_j}{dy} \frac{\partial A_j}{\partial Y}. \end{aligned} \tag{27}$$

The following operators are introduced:

$$\begin{aligned} \xi &= X_1 - C_{gj} T_1, \\ T_2 &= \varepsilon T_1. \end{aligned} \tag{28}$$

Substituting (28) into (27) and because ε is infinitesimal, we can ignore the first item

$$\begin{aligned} &\left(\overline{U} - C_{gj}\right) \frac{\partial}{\partial \xi} \left(\frac{\partial \overline{\psi}_{1j}}{\partial y^2} \right) + \beta \frac{\partial \overline{\psi}_{1j}}{\partial \xi} = -\frac{4K_j^2 m}{Ly_j} \left(\sin 2m_j y\right) \frac{\partial \left|A_j\right|^2}{\partial \xi}, \\ &\overline{\psi}_{1j}\big|_{y=0} = \overline{\psi}_{1j}\big|_{y=Ly} = 0. \end{aligned} \tag{29}$$

Obviously the solution for ψ_{1j} may be expressed in the following form:

$$\overline{\psi}_{ij} = \frac{4K_j^2 \left(K_j^2 + m^2\right)^2 m \sin\left(2my\right) \left|A_j\right|^2}{\left[3m^2 \left(m^2 - 2K_j^2\right) - k_j^4\right] \beta L y_j}. \tag{30}$$

So we obtained the solution of (20):

$$\psi^{(1)} = \psi_{1j}^{(1)} + \psi_{2j}^{(1)} + \psi_3^{(1)} + \psi_4^{(1)} + \phi\left(y, T_1, X_1\right). \tag{31}$$

So as to obtain the solution of A, we continue to consider the question of $o(\varepsilon^2)$. Substituting (19) and (30) into $o(\varepsilon^2)$ and collecting the secular-producing items proportional to $e^{i(Kx-\omega t)}$, we have

$$\begin{aligned} L\left(\psi^{(2)}\right) = &-\sum_{j=1}^{2}\left\{\left(\frac{\partial A_j}{\partial T_2} + \overline{U}\frac{\partial A_j}{\partial X_2}\right)\left(\frac{d^2\phi_j}{dy^2} - K_j^2\phi_j\right)\right. \\ &+ i\phi_j\left(3\overline{U}K_j - \omega_j\right)\frac{\partial^2 A_j}{\partial X_1^2} + i\phi_j\left(\overline{U}K_j - \omega_j\right)\frac{\partial^2 A_j}{\partial Y^2} \\ &+ 2iK_j\phi_j\frac{\partial^2 A_j}{\partial T_1\partial X_1} + 2\frac{d\phi_j}{dy}\left(\frac{\partial^2 A_j}{\partial T_1\partial Y} + \overline{U}\frac{\partial^2 A_j}{\partial X_1\partial Y}\right) \\ &+ \left(\beta - \overline{U}''\right)\frac{\partial A_j}{\partial X_2}\phi_j + 8iK_j^3 m^2\left(K_j^2 + m^2\right)^2 \\ &\left.\cdot\frac{\left|A_j^2\right| A_j\cos\left(2my\right)}{\left[3m^2\left(m^2 - 2K_j^2\right) - K_j^4\right]\beta L y_j}\right\}, \\ &\frac{\partial\psi}{\partial x} = 0, \quad y = 0, L_y. \end{aligned} \tag{32}$$

For the sake of obtaining the evolution equation of A_j, we consider another class of nonhomogeneous solutions, assuming

$$L\left(\psi^{(2)}\right) = \sum_{j=1}^{2} F_j\left(T_1, T_2, X_1, X_2, Y\right) e^{i(K_j x - \omega_j t)} + c.c.\,, \tag{33}$$

introducing (33) into (32), where

$$\begin{aligned} F_1 &= f_1\frac{\partial A_1}{\partial T_2} + \alpha_1\frac{\partial A_1}{\partial X_2} + \beta_1\frac{\partial^2 A_1}{\partial Y^2} + f_{10}\frac{\partial^2 A_1}{\partial X_1^2} \\ &\quad + f_{11}\left|A_1\right|^2 A_1 + f_{12}\left|A_2\right|^2 A_1 + f_{13}A_1, \\ F_2 &= f_2\frac{\partial A_2}{\partial T_2} + \alpha_2\frac{\partial A_2}{\partial X_2} + \beta_2\frac{\partial^2 A_2}{\partial Y^2} + f_{20}\frac{\partial^2 A_2}{\partial X_1^2} \\ &\quad + f_{21}\left|A_1\right|^2 A_2 + f_{22}\left|A_2\right|^2 A_2 + f_{23}A_2. \end{aligned} \tag{34}$$

We assume two special nonhomogeneous solutions of (33) are

$$\begin{aligned} \psi_1^{(2)} &= \Phi_1^{(2)} e^{i(K_1 x - \omega_1 t)} + c.c.\,, \\ \psi_2^{(2)} &= \Phi_2^{(2)} e^{i(K_2 x - \omega_2 t)} + c.c.\,, \end{aligned} \tag{35}$$

and, substituting (35) into (34), we get

$$\frac{d^2\Phi_j^{(2)}}{dy^2} - \left(K_j - \frac{\beta - \overline{U}''}{\overline{U} - c_j}\right)\Phi_j^{(2)} = \frac{F_j}{K_j\left(\overline{U} - c_j\right)}, \tag{36}$$

multiplying (35) by φ_j, and integrating on y from 0 to L_y. The two sides of (35) are equal to zero, so we get the solution conditions:

$$\begin{aligned} &\left(\frac{\partial}{\partial T_2} + c_{g1}\frac{\partial}{\partial X_2}\right)A_1 - i\alpha_1\frac{\partial^2 A_1}{\partial X_1^2} - i\beta_1\frac{\partial^2 A_1}{\partial Y^2} \\ &\quad - i\left(\sigma_1\left|A_1\right|^2 + \gamma_{12}\left|A_2\right|^2 + \lambda_{1j}\left(X_1, Y, T_1\right)\right)A_1 = 0, \\ &\left(\frac{\partial}{\partial T_2} + c_{g2}\frac{\partial}{\partial X_2}\right)A_2 - i\alpha_2\frac{\partial^2 A_2}{\partial X_1^2} - i\beta_2\frac{\partial^2 A_2}{\partial Y^2} \\ &\quad - i\left(\sigma_2\left|A_2\right|^2 + \gamma_{21}\left|A_1\right|^2 + \lambda_{2j}\left(X_1, Y, T_1\right)\right)A_2 = 0. \end{aligned} \tag{37}$$

We get the evolution equations group of wave amplitude, that is, the coupled equations group. To simplify, we introduce the following transformation:

$$X = \frac{1}{\varepsilon}\left(X_2 - C_g T_2\right) = X_1 - C_g T_1, \quad T = T_2, \tag{38}$$

and then (37) can be written as

$$\begin{aligned} &i\frac{\partial A_1}{\partial T} + \alpha_1\frac{\partial^2 A_1}{\partial X^2} + \beta_1\frac{\partial^2 A_1}{\partial Y^2} \\ &\quad + \left(\sigma_1 * \left|A_1\right|^2 + r_{12} * \left|A_2\right|^2\right)A_1 = 0, \\ &i\left(\frac{\partial A_2}{\partial T} + Cg_2 * \frac{\partial A_2}{\partial X}\right) + \alpha_2\frac{\partial^2 A_2}{\partial X^2} + \beta_2\frac{\partial^2 A_2}{\partial Y^2} \\ &\quad + \left(\sigma_2 * \left|A_2\right|^2 + r_{21} * \left|A_1\right|^2\right)A_2 = 0, \end{aligned} \tag{39}$$

where

$$\begin{aligned} C_{g2}^* &= \frac{1}{\varepsilon}\left(C_{g1} - C_{g2}\right), \\ \sigma_1^* &= \sigma_1 + \lambda_{11}, \\ \sigma_2^* &= \sigma_2 + \lambda_{22}, \\ \gamma_{12}^* &= \gamma_{12} + \lambda_{12}, \\ \gamma_{21}^* &= \gamma_{21} + \lambda_{21}. \end{aligned} \tag{40}$$

In (38), coefficients α_1, α_2, β_1, and β_2 are dispersion coefficients, σ_1^*, σ_2^* are Landau coefficients, γ_{12}^*, γ_{21}^* are interaction coefficients, and from their expressions it can be seen that their values are related to the base flow function $\overline{U}(y)$. The (38) is called CNLS equations group.

3. The Solutions of the (2 + 1)-Dimensional CNLS Equations

In this chapter, we will discuss the solutions of the (2 + 1)-dimensional CNLS equation. Based on our experience, we should transform the coupled equations into two independent equations. Inspired by [38, 39], (39) can be written with the following form:

$$iA_{j,T} + \alpha_j A_{j,XX} + \beta_j A_{j,YY} + \gamma_j \left|A_j\right|^2 A_j = 0, \quad j = 1, 2, \tag{41}$$

where α_j, β_j represent the coefficients of dispersion term. γ_j is the nonlinear coupling term coefficient, which can be positive or negative. In order to obtain the traveling wave solutions of the CNLS equations, we define the transformation which is the complex number envelope solution:

$$A_j(T, X, Y) = \overline{A}_j(T, X, Y)\exp^{i\varphi(T,X,Y)}, \tag{42}$$

where $\overline{A}_j(T, X, Y)$ is the amplitude portion of the soliton solutions and $\varphi(T, X, Y)$ is the phase portion of the soliton solution, which is given as

$$\varphi(T, X, Y) = \omega T + KX + MY. \tag{43}$$

Substituting (42) into (41), let real part and imaginary part be zero, and we can get the following coupled equations:

$$\begin{aligned}
&\frac{\partial \overline{A}_j}{\partial T} + \alpha_j\left(2\frac{\partial \overline{A}_j}{\partial X}\frac{\partial \varphi}{\partial X} + \overline{A}_j\frac{\partial^2 \varphi}{\partial X^2}\right) \\
&\quad + \beta_j\left(2\frac{\partial \overline{A}_j}{\partial Y}\frac{\partial \varphi}{\partial Y} + \overline{A}_j\frac{\partial^2 \varphi}{\partial Y^2}\right) = 0, \\
&-\overline{A}_j\frac{\partial \varphi}{\partial T} + \alpha_j\left[\frac{\partial^2 \overline{A}_j}{\partial X^2} - \overline{A}_j\left(\frac{\partial \varphi}{\partial X}\right)^2\right] \\
&\quad + \beta_j\left[\frac{\partial^2 \overline{A}_j}{\partial Y^2} - \overline{A}_j\left(\frac{\partial^2 \varphi}{\partial Y}\right)^2\right] + \gamma_j\overline{A}_j^3 = 0.
\end{aligned} \tag{44}$$

Further, replacing φ with (43)

$$\begin{aligned}
&\frac{\partial \overline{A}_j}{\partial T} + 2\alpha_j K\frac{\partial \overline{A}_j}{\partial X} + 2\beta_j M\frac{\partial \overline{A}_j}{\partial Y} = 0, \\
&-\omega\overline{A}_j + \alpha_j\left(\frac{\partial^2 \overline{A}_j}{\partial X^2} - K^2\overline{A}_j\right) + \beta_j\left(\frac{\partial^2 \overline{A}_j}{\partial Y^2} - M^2\overline{A}_j\right) \\
&\quad + \gamma_j\overline{A}_j^3 = 0.
\end{aligned} \tag{45}$$

Using the traveling wave transformation, this pair of equations will be analyzed further, let

$$\overline{A}_j(T, X, Y) = F_j(\zeta), \quad \text{as } \zeta = \omega' T + K' X + M' Y, \tag{46}$$

and we have

$$\begin{aligned}
&\left[-\omega' + 2\left(KK'\alpha_j + MM'\beta_j\right)\right]\frac{dF_j}{d\zeta} = 0, \\
&\left(\alpha K'^2 + \beta M'^2\right)\frac{d^2F_j}{d\zeta^2} - \left(\alpha K^2 + \beta M^2 + \omega\right)F_j + \gamma F_j^3 \\
&\quad = 0.
\end{aligned} \tag{47}$$

From the first term of (47), we get

$$\omega' = 2\left(KK'\alpha + MM'\beta\right). \tag{48}$$

Further, according to the balance principle in trial function method, we will balance F_j'' with F_j^3. Using the solution procedure of the trial function method, we will obtain the system of algebraic equations as follows:

$$\begin{aligned}
F^3&: 2\left(\alpha_j K'^2 + \beta M'^2\right)a_4 + \gamma_j = 0, \\
F^2&: \frac{3}{2}\left(\alpha_j K'^2 + \beta_j M'^2\right)a_3 = 0, \\
F^1&: \left(\alpha_j K'^2 + \beta_j M'^2\right)a_2 - \left(\alpha_j K^2 + \beta_j M^2 + \omega_j\right) = 0, \\
F^0&: \frac{1}{2}\left(\alpha_j K'^2 + \beta_j M'^2\right)a_1 = 0.
\end{aligned} \tag{49}$$

From the equations above, we have the results of the system

$$\begin{aligned}
a_4 &= -\frac{\gamma_j}{\alpha_j K'^2 + \beta_j M'^2}, \\
a_3 &= 0, \\
a_2 &= \frac{\alpha_j K^2 + \beta_j M^2 + \omega}{\alpha_j K'^2 + \beta_j M'^2}, \\
a_1 &= 0.
\end{aligned} \tag{50}$$

Therefore, we know that F satisfied

$$\pm\left(\zeta - \zeta_0\right) = \int \frac{1}{\sqrt{a_0 + \left(\left(\alpha_j K^2 + \beta_j M^2 + \omega\right)/\left(\alpha K'^2 + \beta M'^2\right)\right)F_j^2 - \left(\gamma_j/\left(\alpha_j K'^2 + \beta_j M'^2\right)\right)F_j^4}}dF_j. \tag{51}$$

If we set $a_0 = 0$ in (51) and integrating with respect to F_j, we will obtain the following soliton solutions of (41) as follows:

$$\begin{aligned}\overline{A}_1 &= \pm\sqrt{\frac{\alpha_1 K^2+\beta_1 M^2+\omega}{\gamma_1}} \\ &\cdot \operatorname{sech}\left\{\sqrt{\frac{\alpha_1 K^2+\beta_1 M^2+\omega}{\alpha_1 K'^2+\beta_1 M'^2}}\left[K'X+M'Y\right.\right. \\ &\left.\left.-2\left(KK'\alpha_1+MM'\beta_1\right)t-\zeta_0\right]\right\}\times\exp\{i(KX \\ &+MY+\omega T)\}, \\ \overline{A}_2 &= \pm\sqrt{-\frac{\alpha_2 K^2+\beta_2 M^2+\omega}{\gamma_2}} \\ &\cdot \operatorname{csch}\left\{\sqrt{\frac{\alpha_2 K^2+\beta M^2+\omega}{\alpha_2 K'^2+\beta_2 M'^2}}\left[K'X+M'Y\right.\right. \\ &\left.\left.-2\left(KK'\alpha+MM'\beta\right)t-\zeta_0\right]\right\}\times\exp\{i(KX \\ &+MY+\omega T)\},\end{aligned} \tag{52}$$

and these solutions are the soliton solutions for CNLS equations, when $KK'\alpha_j + MM'\beta_j + \omega > 0$.

4. Modulation Instabilities of Coupled Envelope Rossby Waves

For coupled envelope nonlinear Rossby waves, they meet the CNLS equations (39). We set the NLS equation of the $(2+1)$-dimension with constant coefficients as

$$\begin{aligned}&i\frac{\partial A_1}{\partial T}+\alpha\frac{\partial^2 A_1}{\partial X^2}+\beta\frac{\partial^2 A_1}{\partial Y^2} \\ &\quad+\left(\sigma_1 * \left|A_1\right|^2+r_{12} * \left|A_2\right|^2\right)A_1=0, \\ &i\frac{\partial A_2}{\partial T}+\alpha\frac{\partial^2 A_2}{\partial X^2}+\beta\frac{\partial^2 A_2}{\partial Y^2} \\ &\quad+\left(\sigma_1 * \left|A_2\right|^2+r_{21} * \left|A_1\right|^2\right)A_1=0.\end{aligned} \tag{53}$$

Further, introducing

$$Z = X\cos\theta + Y\sin\theta, \tag{54}$$

(53) reduces to

$$\begin{aligned}&i\frac{\partial A_1}{\partial T}+\gamma_1\frac{\partial^2 A_1}{\partial Z^2}+\left(\sigma_1 * \left|A_1\right|^2+r_{12} * \left|A_2\right|^2\right)A_1=0, \\ &i\frac{\partial A_2}{\partial T}+\gamma_2\frac{\partial^2 A_2}{\partial Z^2}+\left(\sigma_1 * \left|A_2\right|^2+r_{21} * \left|A_1\right|^2\right)A_2=0,\end{aligned} \tag{55}$$

where $\gamma_1 = \gamma_2 = \alpha^2\cos^2\theta + \beta^2\sin^2\theta$.

From Section 3, we know the exact periodic wave solutions, taking the simple form as follows:

$$\begin{aligned}A_1 &= a\exp i\left(m_1 Z-\Omega_1 T\right), \\ A_2 &= b\exp i\left(m_2 Z-\Omega_2 T\right),\end{aligned} \tag{56}$$

where m_1, Ω_1, m_2, and Ω_2 satisfy

$$\begin{aligned}\Omega_1 &= \gamma_1 m_1^2-\left(\sigma_1 a^2+\gamma_{12}b^2\right), \\ \Omega_2 &= \gamma_2 m_2^2-\left(\sigma_2 b^2+\gamma_{21}a^2\right).\end{aligned} \tag{57}$$

We assume that a, b are real numbers and m_1, m_2 represent wave number. Equation (54) shows that each nonlinear Rossby wave dispersion not only contains itself wave number and amplitude, but also contains another wave amplitude, which is characteristic of the interaction between wave and wave.

Next, we will analyze the stability of waves solution below. Assume the solutions for the disturbance as follows:

$$\begin{aligned}A_1 &= \exp i\left(m_1 Z-\Omega_1 T\right)\left(a+\epsilon\phi_1(Z,T)\right), \\ A_2 &= \exp i\left(m_2 Z-\Omega_2 T\right)\left(b+\epsilon\phi_2(Z,T)\right).\end{aligned} \tag{58}$$

Substituting (58) into (53), we can get the linear equations as follows:

$$\begin{aligned}&i\frac{\partial\phi_1}{\partial T}+\gamma_1\left(2m_1 i\frac{\partial\phi_1}{\partial Z}+\frac{\partial^2\phi_1}{\partial Z^2}\right)+\sigma_1 a^2\left(\phi_1+\phi_1^*\right) \\ &\quad+\gamma_{12}ab\left(\phi_2+\phi_2^*\right)=0, \\ &i\frac{\partial\phi_2}{\partial T}+\gamma_2\left(2m_1 i\frac{\partial\phi_2}{\partial Z}+\frac{\partial^2\phi_2}{\partial Z^2}\right)+\sigma_2 b^2\left(\phi_2+\phi_2^*\right) \\ &\quad+\gamma_{21}ab\left(\phi_1+\phi_1^*\right)=0.\end{aligned} \tag{59}$$

Further, assume

$$\begin{aligned}\phi_1 &= p_1\exp\left(i\lambda Z+\sigma T\right)+q_1\exp\left(-i\lambda Z+\sigma^* T\right), \\ \phi_2 &= p_2\exp\left(i\lambda Z+\sigma T\right)+q_2\exp\left(-i\lambda Z+\sigma^* T\right),\end{aligned} \tag{60}$$

and, substituting (60) into (59), we can get

$$\begin{aligned}&\left[\left(\sigma_1 a^2-\gamma_1\lambda^2\right)+\left(i\sigma-2\gamma_1 m_1\lambda\right)\right]p_1+\gamma_{12}abp_2 \\ &\quad+\sigma_1 a^2 q_1^*+\gamma_{12}abq_2^*=0, \\ &\sigma_1 a^2 p_1^*+\gamma_{12}abp_2^* \\ &\quad+\left[\left(\sigma_1 a^2-\gamma_1\lambda^2\right)-\left(i\sigma^*+2\gamma_1 m_1\lambda\right)\right]q_1 \\ &\quad+\gamma_{12}abq_2=0, \\ &\gamma_{21}abp_1+\left[\left(\sigma_2 b^2-\gamma_2\lambda^2\right)+\left(i\sigma-2\gamma_2 m_2\lambda\right)\right]p_2 \\ &\quad+\gamma_{21}abq_1^*+\sigma_2 b^2 q_2^*=0, \\ &\gamma_{21}abp_1^*-\sigma_2 b^2 p_2^*+\gamma_{21}abq_1 \\ &\quad+\left[\left(\sigma_2 b^2-\gamma_2\lambda^2\right)+\left(i\sigma-2\gamma_2 m_2\lambda\right)\right]q_2=0.\end{aligned} \tag{61}$$

The above equations are linear homogeneous equations for p_1, p_2, q_1, and q_2. If there are nonzero solutions, the coefficients determinant must be zero. So we can get the next type:

$$\begin{aligned}&\left[\left(\gamma_1^2\lambda^4 - 2\gamma_1\sigma_1\lambda^2a^2\right) - \left(i\sigma - 2\gamma m_1\lambda\right)^2\right]\\&\quad\cdot\left[\left(\gamma_2^2\lambda^4 - 2\gamma_2\sigma_2\lambda^2b^2\right) - \left(i\sigma - 2\gamma_2 m_2\lambda\right)^2\right]\\&\quad= 4\gamma_1\gamma_2\gamma_{12}\gamma_{21}a^2b^2\lambda^4,\end{aligned} \tag{62}$$

and these are the four algebraic equations of $i\sigma$. When the parameters are certain, the value of λ makes $\mathrm{Re}\sigma > 0$. But, for the influence of the interaction between wave and wave on the stability of wave, we will give a special case for discussion. Assume the number of waves satisfies

$$\gamma_1 m_1 = \gamma_2 m_2, \tag{63}$$

and, moreover, set

$$\begin{aligned}\Delta_1 &= \gamma_1^2\lambda^2\left(\lambda^2 - \frac{2\sigma_1}{\gamma_1}a^2\right),\\ \Delta_2 &= \gamma_2^2\lambda^2\left(\lambda^2 - \frac{2\sigma_2}{\gamma_2}b^2\right),\\ S &= 4\gamma_1\gamma_2\gamma_{12}\gamma_{21}a^2b^2\lambda^4.\end{aligned} \tag{64}$$

Clearly, when $\Delta_1 \geq 0$ the first wave is stable and there is no interaction and the contrary occurs when $\Delta_1 < 0$ is instable. Similarly, when $\Delta_2 \geq 0$ the second wave is stable and there is no interaction, and the contrary occurs when $\Delta_2 < 0$ is instable. From (62), we get

$$\sigma = \mathrm{Im}\left\{\gamma_1 m_1\lambda \pm \left[\frac{(\Delta_1 + \Delta_2) \pm \sqrt{(\Delta_1 - \Delta_2)^2 + 4S}}{2}\right]^{1/2}\right\}, \tag{65}$$

where σ represents the gain for the frequency shift, which has been described in Figure 1. Equation (65) shows that when $\Delta_1 < 0$ and $\Delta_2 < 0$, that is, $\Delta_1 + \Delta_2 < 0$, no matter what value S takes, at least there is one which satisfies $\mathrm{Re}\sigma > 0$; therefore, the waves have instability. This conclusion shows that when $\gamma_1 m_1 = \gamma_2 m_2$, two modulated unstable waves, the interaction of the two waves is still unstable.

When $\Delta_1 > 0, \Delta_2 > 0$, that is, $\Delta_1 + \Delta_2 > 0$, corresponding to no interaction, the two waves are stable. If $S > 0$, when $S \leq \Delta_1\Delta_2$, two waves are stable after interaction. When $S > \Delta_1\Delta_2$, two waves are unstable after interaction. If $S < 0$, when $-(1/4)(\Delta_1 - \Delta_2)^2 \leq S \leq \Delta_1\Delta_2$, the two waves are stable. When $S < -(1/4)(\Delta_1 - \Delta_2)^2$ or $S > \Delta_1\Delta_2$ two waves are unstable after interaction. From the above analysis, we can find that when $\Delta_1 > 0$ and $\Delta_2 > 0$, even with two stable nonlinear waves through interaction, the stable feature is decided by the value of S.

When $\Delta_1 < 0$ and $\Delta_2 > 0$, corresponding to no interaction, the first wave is stable, but the second wave is unstable, while when $\Delta_1 > 0$, $\Delta_2 < 0$, the condition is opposite.

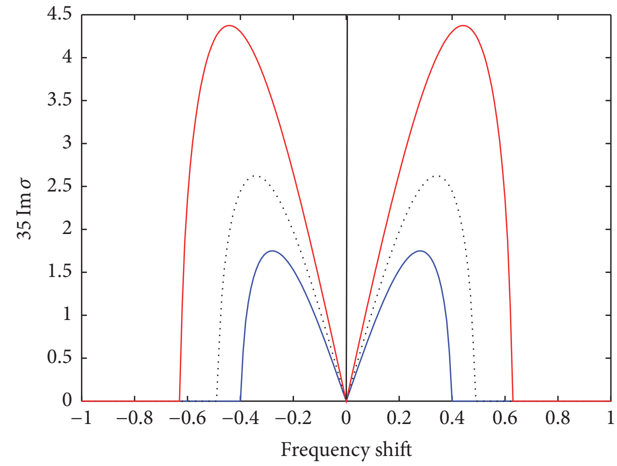

FIGURE 1: Gain spectra for frequency shift.

5. Chirp Effect

With summary of previous studies on Rossby waves, it is not hard to find that nonlinearity and dispersion are important factors affecting the propagation of Rossby waves. In this section, we use the concept of chirp in the field of optical soliton communication to study the chirp effect caused by nonlinearity and dispersion in the propagation of Rossby waves.

when $\alpha = \beta$, the NLS equation (41) for describing the characteristics of Rossby wave propagation transforms to

$$i\frac{\partial A}{\partial T} + \alpha\left(\frac{\partial^2 A}{\partial X^2} + \frac{\partial^2 A}{\partial Y^2}\right) + \gamma|A|^2 A = 0, \tag{66}$$

where α is the coefficient of dispersion and γ is the nonlinear coefficient. Here, based on the soliton solution of the NLS equation, we take the initial wave form of $(2+1)$-dimensional Rossby solitary waves as follows, setting $K = M = K' = M' = 1$:

$$A = \sqrt{\frac{2\alpha + \omega}{\gamma}}\,\mathrm{sech}\left[\sqrt{\frac{2\alpha + \omega}{2\alpha}}(X + Y)\right]. \tag{67}$$

5.1. Chirp Effect Caused by Dispersion. Let us consider the dispersion effect of chirp, and (66) becomes

$$A_T = -i\alpha\left(A_{XX} + A_{YY}\right). \tag{68}$$

Reviewing the time T from $0 \rightarrow \Delta T$, where ΔT is an infinitesimal variable, and introducing (67) into (68), we can get the approximate solution of (68) as follows:

$$A(\Delta T, X, Y) = \sqrt{\frac{2\alpha+\omega}{\gamma}} \operatorname{sech}\left[\sqrt{\frac{2\alpha+\omega}{2\alpha}}(X+Y)\right] \cdot \exp\left\{-i(2\alpha+\omega)\Delta T \times \operatorname{sech}^2\left[\sqrt{\frac{2\alpha+\omega}{2\alpha}}(X+Y)\right]\right\}, \quad (69)$$

so that the phase of the wave meets

$$\varphi_D = -i(2\alpha+\omega)\Delta T \operatorname{sech}^2\left[\sqrt{\frac{2\alpha+\omega}{2\alpha}}(X+Y)\right], \quad (70)$$

and, based on (70), we can obtain the chirp effect caused by the dispersion

$$\Delta\nu_D = -\nabla\varphi_D = \sqrt{2}(2\alpha+\omega)^{3/2} \cdot \Delta T\left\{\operatorname{sech}^2\left[\sqrt{\frac{2\alpha+\omega}{2\alpha}}(X+Y)\right] \cdot \tanh\left[\sqrt{\frac{2\alpha+\omega}{2\alpha}}(X+Y)\right]\right\}. \quad (71)$$

5.2. Chirp Effect Caused by Nonlinearity. Separately considering the nonlinear effect of chirp, the NLS equation (66) becomes

$$A_T = -i\gamma|A|^2 A, \quad (72)$$

and, investigating the condition of time T from $0 \rightarrow \Delta T$, where ΔT is an infinitesimal variable, and introducing (67) into (72), we can get the approximate solution of (72) as follows:

$$A(\Delta T, X, Y) = \sqrt{\frac{2\alpha+\omega}{\gamma}} \operatorname{sech}\left[\sqrt{\frac{2\alpha+\omega}{2\alpha}}(X+Y)\right] \cdot \exp\left\{-i(2\alpha+\omega)\Delta T \times \operatorname{sech}^2\left[\sqrt{\frac{2\alpha+\omega}{2\alpha}}(X+Y)\right]\right\}, \quad (73)$$

and the phase of the wave meets

$$\varphi_N = -(2\alpha+\omega)\Delta T \times \operatorname{sech}^2\left[\sqrt{\frac{2\alpha+\omega}{2\alpha}}(X+Y)\right], \quad (74)$$

and, based on (74), we can get the chirp effect caused by the nonlinear

$$\Delta\nu_N = -\nabla\varphi_N = -4(2\alpha+\omega)^{3/2} \cdot \sqrt{2\alpha}\Delta T\left\{\operatorname{sech}^2\left[\sqrt{\frac{2\alpha+\omega}{2\alpha}}(X+Y)\right] \cdot \tanh\left[\sqrt{\frac{2\alpha+\omega}{2\alpha}}(X+Y)\right]\right\}. \quad (75)$$

5.3. Discussion of Total Chirp. According to (71) and (75), the total chirp is

$$\Delta\nu_s = \Delta\nu_D + \Delta\nu_N = \left(\sqrt{2} - 4\sqrt{2\alpha}\right)(2\alpha+\omega)^{3/2} \cdot \Delta T\left\{\operatorname{sech}^2\left[\sqrt{\frac{2\alpha+\omega}{2\alpha}}(X+Y)\right] \cdot \tanh\left[\sqrt{\frac{2\alpha+\omega}{2\alpha}}(X+Y)\right]\right\}. \quad (76)$$

(1) When the dispersion and nonlinear effects cancel each other, as follows:

$$\Delta\nu_s = \Delta\nu_D + \Delta\nu_N = 0, \quad (77)$$

we can get $\alpha = 1/16$.

(2) When the dispersion effect is greater than the nonlinear effect, as follows:

$$|\Delta\nu_D| > |\Delta\nu_N|, \quad (78)$$

we get $\alpha < 1/16$.

(3) When the dispersion is less than the nonlinear effect, as follows:

$$|\Delta\nu_D| < |\Delta\nu_N|, \quad (79)$$

we get $\alpha > 1/16$. As we know, the total chirp effect is related to the sea conditions in the propagation region and the amplitude of the initial wave. If the propagation area and the sea condition parameter is determined, the total chirp effect is only related to the initial amplitude. The chirp effect caused by dispersion and nonlinearity is described in Figures 2 and 3. The above calculation shows that the magnitude of the initial amplitude is determined by the parameter α. Combining Figure 2, Figure 3, and calculation, we can get the following conclusions.

(1) When $\alpha = 1/16$, the dispersion and nonlinear effects cancel each other, and solitary waves propagate over long distances and keep waveforms constant.

(2) When $\alpha > 1/16$, the nonlinear effect is greater than the dispersion effect, so that the total chirp is not zero. In this case, the isolated waves present nonlinear characteristics, and the amplitude of isolated waves is changed periodically.

(3) When $\alpha > 1/16$, the dispersion effect is greater than the nonlinear effect, so that the total chirp is not zero. In this case, the isolated waves present dispersion property, and the amplitude of isolated waves is changed periodically.

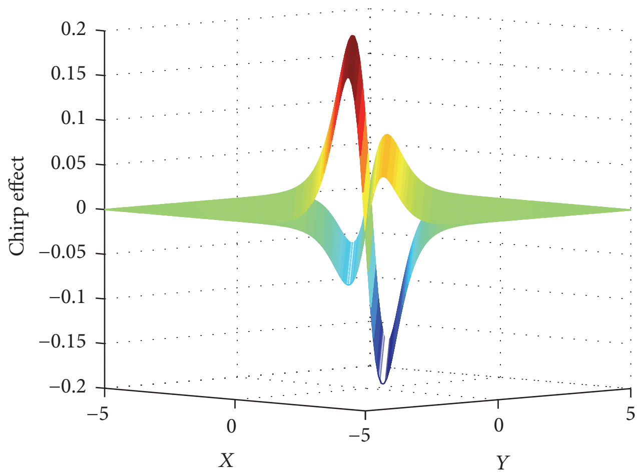

FIGURE 2: Three-dimensional waveform with dispersion and nonlinear effects.

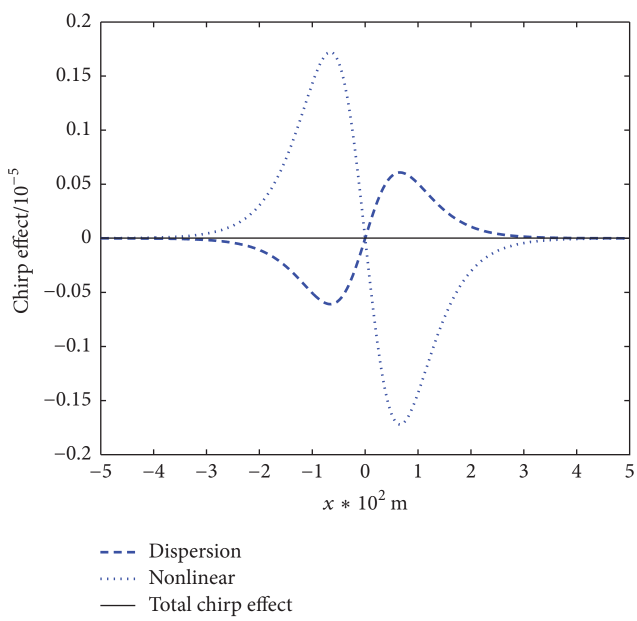

FIGURE 3: The variation of the chirp effect under the dispersion and nonlinearity.

6. Conclusions

In this paper, we obtained the (2 + 1)-dimensional coupled NLS (CNLS) equations and discussed the solutions of the single nonlinear Schrödinger equation. In addition, we know that, for two nonlinear Rossby waves, they can be described by CNLS equations. The equations show that no matter the two waves' interaction process, their respective energy and momentum are conserved. Furthermore, modulational instability of coupled envelope Rossby waves in (2 + 1)-dimensional condition is also discussed. we can find that the stable feature of coupled envelope Rossby waves is decided by the value of S, which implies that the instability condition depends not only on the prescribed perturbation wave numbers p, q, but also on the amplitude of the Rossby waves. Finally, introducing the concept of chirp in the optical soliton communication field we study the chirp effect caused by nonlinearity and dispersion in the propagation of Rossby waves.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

The project is supported by the Scientific and Technological Innovation Project Financially Supported by Qingdao National Laboratory for Marine Science and Technology (no. 2016ASKJ12), China Postdoctoral Science Foundation Funded Project (2017M610436), Open Fund of the Key Laboratory of Meteorological Disaster of Ministry of Education (Nanjing University of Information Science and Technology) (no. KLME1507), and Open Fund of the Key Laboratory of Ocean Circulation and Waves, Chinese Academy of Sciences (no. KLOCAW1401).

References

[1] R. R. Long, "Solitary waves in one and two-fluids system," *Journal of the Atmospheric Sciences*, vol. 21, pp. 197–200, 1964.

[2] D. J. Benny, "Long non-linear waves in fluid flows," *Journal of Mathematical Physics*, vol. 45, pp. 52–63, 1966.

[3] L. G. Redekopp, "On the theory of solitary Rossby waves," *Journal of Fluid Mechanics*, vol. 82, no. 4, pp. 725–745, 1977.

[4] H. Yang, Q. Zhao, B. Yin, and H. Dong, "A new integro-differential equation for rossby solitary waves with topography effect in deep rotational fluids," *Abstract and Applied Analysis*, vol. 2013, Article ID 597807, 2013.

[5] H. Yang, B. Yin, Y. Shi, and Q. Wang, "Forced ILW-Burgers equation as a model for Rossby solitary waves generated by topography in finite depth fluids," *Journal of Applied Mathematics*, vol. 2012, Article ID 491343, 2012.

[6] D. H. Luo, "The nonlinear schr÷dinger equation rotating barotropic atmosphere and atmospheric blocking," *Acta. Meteor. Sinica*, vol. 48, p. 265, 1990.

[7] X.-X. Xu, "A deformed reduced semi-discrete Kaup-Newell equation, the related integrable family and Darboux transformation," *Applied Mathematics and Computation*, vol. 251, pp. 275–283, 2015.

[8] H. Li, Y. Li, and Y. Chen, "Bi-Hamiltonian structure of multi-component Yajima-Oikawa hierarchy," *Zeitschrift fur Naturforschung - Section A Journal of Physical Sciences*, vol. 70, no. 11, pp. 929–934, 2015.

[9] Q. L. Zhao and X. Z. Wang, "The integrable coupling system of a 3 × 3 discrete matrix spectral problem," *Applied Mathematics and Computation*, vol. 216, no. 3, pp. 730–743, 2010.

[10] Q.-L. Zhao and X.-Y. Li, "A Bargmann system and the involutive solutions associated with a new 4-order lattice hierarchy," *Analysis and Mathematical Physics*, vol. 6, no. 3, pp. 237–254, 2016.

[11] X.-X. Xu, "An integrable coupling hierarchy of the Mkdv-Integrable systems, its hamiltonian structure and corresponding nonisospectral integrable hierarchy," *Applied Mathematics and Computation*, vol. 216, no. 1, pp. 344–353, 2010.

[12] W.-X. Ma, "Variational identities and applications to Hamiltonian structures of soliton equations," *Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal*, vol. 71, no. 12, pp. e1716–e1726, 2009.

[13] L. G. Redekopp and P. D. Weidman, "Solitary rossby waves in zonal shear flows and their interactions," *Journal of the Atmospheric Sciences*, vol. 35, no. 5, pp. 790–804, 1978.

[14] D. J. Benney, "The perturbed plane-wave solutions of the cubic Schrödinger equation," *Studies in Applied Mathematics*, vol. 60, p. 1, 1979.

[15] T. Yamagata, "he stability, modulation and long wave resonance of a planetary wave in a rotating, two-layer fluid on a channel β-plane," *Journal of the Meteorological Society of Japan*, vol. 58, no. 3, pp. 160–171, 1980.

[16] D. Luo, "Envelope solitary ROSsby waves and modulational instabilities of uniform ROSsby wave trains in two space dimensions," *Wave Motion. An International Journal Reporting Research on Wave Phenomena*, vol. 24, no. 3, pp. 315–325, 1996.

[17] Y. Shi, B. Yin, H. Yang, D. Yang, and Z. Xu, "Dissipative nonlinear Schrödinger Equation for envelope solitary Rossby waves with dissipation effect in stratified fluids and its solution," *Abstract and Applied Analysis*, Article ID 643652, Art. ID 643652, 9 pages, 2014.

[18] S. Y. Song, J. Wang, J. M. Meng, J. B. Wang, and P. X. Hu, "Nonlinear Schrödinger equation for internal waves in deep sea," *Acta Physica Sinica*, vol. 59, no. 2, pp. 1123–1129, 2010.

[19] H. W. Yang, Z. H. Xu, D. Z. Yang, X. R. Feng, B. S. Yin, and H. H. Dong, "ZK-Burgers equation for three-dimensional Rossby solitary waves and its solutions as well as chirp effect," *Advances in Difference Equations*, 2016:167, 22 pages, 2016.

[20] J. Wang and Y. M. Shi, "Study of chirps induced by the higher-order nonlinear effects in the photonic crystal fiber," *Acta Physica Sinica*, vol. 55, no. 6, pp. 1000–3290, 2006.

[21] L. G. Yang, C. J. Da, J. Song, H. Zhang, H. Yang, and Y. Hou, "Rossby waves with linear topography in barotropic fluids," *Chinese Journal of Oceanology and Limnology*, vol. 26, no. 3, pp. 334–338, 2008.

[22] T. Xu and Y. Chen, "Localized waves in three-component coupled nonlinear Schrödinger equation," *Chinese Physics B*, vol. 25, no. 9, Article ID 090201, 2016.

[23] Y. Li, W.-X. Ma, and J. E. Zhang, "Darboux transformations of classical Boussinesq system and its new solutions," *Physics Letters. A*, vol. 275, no. 1-2, pp. 60–66, 2000.

[24] Q.-L. Zhao, X.-Y. Li, and F.-S. Liu, "Two integrable lattice hierarchies and their respective Darboux transformations," *Applied Mathematics and Computation*, vol. 219, no. 10, pp. 5693–5705, 2013.

[25] D.-Y. Chen, D.-J. Zhang, and S.-F. Deng, "The novel multi-soliton solutions of the MKdV-sine Gordon equations," *Journal of the Physical Society of Japan*, vol. 71, no. 2, pp. 658-659, 2002.

[26] X. Guo, "On bilinear representations and infinite conservation laws of a nonlinear variable-coefficient equation," *Applied Mathematics and Computation*, vol. 248, pp. 531–535, 2014.

[27] E. G. Fan and H. Q. Zhang, "Homogeneous balance method for nonlinear soliton equations," *Chinese Journal of Physics*, vol. 47, no. 3, pp. 353–363, 1998.

[28] S. D. Liu, Z. T. Fu, S. K. Liu, and Q. Zhao, "Jacobi elliptic function envelope periodic solutions for nonlinear wave equations," *Chinese Physics*, vol. 51, no. 4, pp. 718–723, 2002.

[29] W. X. Ma and R. G. Zhang, "Adjont symmetry constraints of muliti-component AKNS equations," *Chinese Annals of Mathematics*, vol. 349, pp. 153–163, 2006.

[30] L. Huang and Y. Chen, "Nonlocal symmetry and similarity reductions for the Drinfeld-Sokolov-SATsuma-Hirota system," *Applied Mathematics Letters. An International Journal of Rapid Publication*, vol. 64, pp. 177–184, 2017.

[31] Y. Zhang, H. Dong, X. Zhang, and H. Yang, "Rational solutions and lump solutions to the generalized $(3+1)$-dimensional Shallow Water-like equation," *Journal of Computers & Mathematics with Applications*, vol. 73, no. 2, pp. 246–252, 2017.

[32] Y. Zhang and W.-X. Ma, "Rational solutions to a KdV-like equation," *Applied Mathematics and Computation*, vol. 256, pp. 252–256, 2015.

[33] H. Dong, K. Zhao, H. Yang, and Y. Li, "Generalised (2+1)-dimensional super MKdV hierarchy for integrable systems in soliton theory," *East Asian Journal on Applied Mathematics*, vol. 5, no. 3, pp. 256–272, 2015.

[34] X.-Y. Li, Y.-X. Li, and H.-X. Yang, "Two families of Liouville integrable lattice equations," *Applied Mathematics and Computation*, vol. 217, no. 21, pp. 8671–8682, 2011.

[35] H. H. Dong, B. Y. Guo, and B. S. Yin, "Generalized fractional supertrace identity for Hamiltonian structure of NLS–MKdV hierarchy with self-consistent sources," *Analysis and Mathematical Physics*, vol. 6, no. 2, pp. 199–209, 2016.

[36] X. Li and Q. Zhao, "A new integrable symplectic map by the binary nonlinearization to the super AKNS system," *Journal of Geometry and Physics*, vol. 121, pp. 123–137, 2017.

[37] B. Tan and J. P. Boyd, "Coupled-mode envelope solitary waves in a pair of cubic Schrödinger equations with cross modulation: Analytical solution and collisions with applicaion to Rossby waves," *Chaos, Solitons & Fractals*, vol. 11, no. 7, pp. 1113–1129, 2000.

[38] R. Hirota, *The Direct Method in Soliton Theory*, Cambridge University Press, Cambridge, UK, 2004.

[39] R. Xiang, L. M. Ling, and X. Lü, "Some novel solutions for the two-coupled nonlinear Schröinger equations," *Applied Mathematics Letters*, vol. 68, pp. 163–170, 2017.