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Research Article

Hybrid Rational Haar Wavelet and Block Pulse Functions Method for Solving Population Growth Model and Abel Integral Equations

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We use a computational method based on rational Haar wavelet for solving nonlinear fractional integro-differential equations. To this end, we apply the operational matrix of fractional integration for rational Haar wavelet. Also, to show the efficiency of the proposed method, we solve particularly population growth model and Abel integral equations and compare the numerical results with the exact solutions.

1. Introduction

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders (including complex orders). It is also known as generalized integral and differential calculus of arbitrary order. Fractional differential equations are generalized from classical integer-order ones, which are obtained by replacing integer-order derivatives by fractional ones. In recent years, fractional calculus and differential equations have found enormous applications in mathematics, physics, chemistry, and engineering [1–4]. A large class of dynamical systems appearing throughout the field of engineering and applied mathematics is described by fractional differential equations. For that reason, reliable and efficient techniques for the solution of fractional differential equations are indeed required. The most frequently used methods are Walsh functions [5], Laguerre polynomials [6], Fourier series [7], Laplace transform method [8], the Haar wavelets [9], Legendre wavelets [10–12], and the Chebyshev wavelets [13, 14]. Kronecker operational matrices have been developed by Kilicman and Al Zhour for some applications of fractional calculus [15]. Recently, in [16], the authors proposed a new method based on operational matrices to solve fractional Volterra integral equations.

Recently, many authors applied operational matrices of integration and derivative to reduce the original problem into an algebraic one. According to this fact that the orthogonal polynomials play an important role to solve integral and differential equations, many researchers constructed operational matrix of fractional and integer derivatives for some types of these polynomials, such as Flatlet oblique multiwavelets [17, 18], B-spline cardinal functions [19], Legendre polynomials, Chebyshev polynomials, and CAS wavelets [20]. The main aim of this paper is to use an operational matrix of fractional integration to reduce a nonlinear fractional integro-differential equation to nonlinear algebraic equations.

The rest of the paper is organized as follows: In Section 2, we introduce some basic mathematical preliminaries that we need to construct our method. Also, we recall the basic definitions from block pulse functions and fractional calculus. In Section 3, we recall definition of rational Haar wavelet. In Section 4, we apply proposed method to solve fractional population growth model and Abel integral equations. Section 5 is devoted to convergence and error analysis. Finally, in Section 6, conclusion of numerical results is presented.

2. Preliminaries

In this section, we recall some basic definitions from fractional calculus and some properties of integral calculus which we shall apply to formulate our approach.

The Riemann-Liouville fractional integral operator I^{α} of order $\alpha \geq 0$ on the usual Lebesgue space $L^1[0,b]$ is given by [21]

$$\left(I^{\alpha}u\right)(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^x (x-s)^{\alpha-1} u(s)\, ds & \alpha > 0, \\ u(x) & \alpha = 0. \end{cases} \tag{1}$$

The Riemann-Liouville fractional derivative of order $\alpha > 0$ is normally defined as

$$D^{\alpha}u(x) = \left(\frac{d}{dx}\right)^m I^{m-\alpha}u(x), \quad (m-1 < \alpha \leq m), \tag{2}$$

where m is an integer number.

The fractional derivative of order $\alpha > 0$ in the Caputo sense is given by [21]

$$D_*^{\alpha}u(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-s)^{m-\alpha-1} u^{(m)}(s)\, ds, \quad (m-1 < \alpha \leq m), \tag{3}$$

where m is an integer, $\alpha > 0$, and $u^{(m)} \in L^1[0,b]$.

The useful relation between the Riemann-Liouville operator and Caputo operator is given by the following expression:

$$I^{\alpha}D_*^{\alpha}u(x) = u(x) - \sum_{k=0}^{m-1} u^{(k)}(0^+) \frac{x^k}{k!}, \quad (m-1 < \alpha \leq m), \tag{4}$$

where m is an integer, $\alpha > 0$, and $u^{(k)} \in L^1[0,b]$.

An m-set of block pulse functions (BPFs) in the region of $[0,T)$ is defined as follows:

$$b_i(t) = \begin{cases} 1, & ih \leq t < (i+1)h, \\ 0, & \text{O.W}, \end{cases} \tag{5}$$

where $i = 1,2,\ldots,m-1$ with positive integer values for m and $h = T/m$. There are some properties for BPFs, for example, disjointness, orthogonality, and completeness.

The set of BPFs may be written as an m-vector as

$$B(t) = \left[b_0(t), \ldots, b_{m-1}(t)\right]^T, \tag{6}$$

where $t \in [0,1)$.

A function $f(t) \in L^2([0,1))$ may be expanded by the BPFs as

$$f(t) \simeq \sum_{i=0}^{m-1} f_i b_i(t) = F^T B(t) = B^T(t) F, \tag{7}$$

where $B(t)$ is given by (6) and F is an m-vector given by

$$F = \left[f_0, \ldots, f_{m-1}\right]^T, \tag{8}$$

and the block pulse coefficients f_i are obtained as

$$f_i = \frac{1}{h} \int_{ih}^{(i+1)h} f(t)\, dt. \tag{9}$$

The integration of vector $B(t)$ defined in (6) may be obtained as

$$\int_0^t B(\tau)\, d\tau \simeq \Upsilon B(t), \tag{10}$$

where Υ is called operational matrix of integration which can be represented by

$$\Upsilon = \frac{h}{2} \begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{11}$$

Kilicman and Al Zhour (see [15]) have given the block pulse operational matrix of fractional integration F^{α} as follows:

$$\left(I^{\alpha}B_m\right)(t) \simeq F^{\alpha}B_m(t), \tag{12}$$

where

$$F^{\alpha} = \frac{1}{m} \frac{1}{\Gamma(\alpha+2)} \begin{pmatrix} 1 & \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-2} \\ 0 & 0 & 1 & \xi_1 & \cdots & \xi_{m-3} \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \xi_1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{13}$$

and $\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$.

3. Rational Haar Wavelets

The Haar functions are an orthogonal family of switched rectangular waveforms where amplitudes can differ from one function to another. The orthogonal set of Haar wavelet functions is defined in $[0,1)$ as follows (see [22]):

$$h_i(t) = \frac{1}{\sqrt{m}} \begin{cases} 2^{j/2}, & \dfrac{k-1}{2^j} \leq t < \dfrac{k-1/2}{2^j}, \\ -2^{j/2}, & \dfrac{k-1/2}{2^j} \leq t < \dfrac{k}{2^j}, \\ 0, & \text{O.W}, \end{cases} \tag{14}$$

where $i = 0,1,2,\ldots,m-1$, $m = 2^{M+1}$ and M is a positive integer, and j and k represent the integer decomposition of the index i, that is, $i = 2^j + k - 1$, $j = 0,1,\ldots,M$, $k = 1,2,\ldots,2^j$. Also we have $h_0(t) = 1/\sqrt{m}$. This set of functions is complete, since any function $f \in L^2([0,1))$ can be expanded into Haar wavelets by

$$f(t) \simeq \sum_{i=0}^{m-1} f_i h_i(t) = F^T \Psi(t), \tag{15}$$

where

$$F = \left[f_0, f_1, f_2, \ldots, f_{m-1}\right]^T, \quad \Psi(t) = \left[h_0(t), h_1(t), h_2(t), \ldots, h_{m-1}(t)\right]^T. \tag{16}$$

Operational Matrix of Fractional Integration. Equation (7) implies that rational Haar wavelets can be also expanded into m-term BPFs as

$$h_i(t) \simeq \sum_{j=0}^{m-1} h_{ij} b_j(t), \tag{17}$$

for $i = 1, 2, \ldots, m-1$. Clearly we have

$$\Psi(t) = H_{m\times m} B(t), \tag{18}$$

where

$$H_{m\times m} = \begin{pmatrix} h_0(t_0) & h_0(t_1) & \cdots & h_0(t_{m-1}) \\ h_1(t_0) & h_1(t_1) & \cdots & h_1(t_{m-1}) \\ \vdots & \vdots & \cdots & \vdots \\ h_{m-1}(t_0) & h_{m-1}(t_1) & \cdots & h_{m-1}(t_{m-1}) \end{pmatrix}, \tag{19}$$

and $t_i = (i + 1/2)/m$, for $i = 0, 1, 2, \ldots, m-1$ (see [15, 23]).

For example, for $M = 2$, the Haar operational matrix into BPFs can be expressed as

$$H_{8\times 8} = \begin{pmatrix} 0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 \\ 0.3536 & 0.3536 & 0.3536 & 0.3536 & -0.3536 & -0.3536 & -0.3536 & -0.3536 \\ 0.5000 & 0.5000 & -0.5000 & -0.5000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5000 & 0.5000 & -0.5000 & -0.5000 \\ 0.7071 & -0.7071 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7071 & -0.7071 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.7071 & -0.7071 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.7071 & -0.7071 \end{pmatrix}. \tag{20}$$

Let

$$(I^\alpha \Psi)(t) \simeq P^\alpha_{m\times m} \Psi(t), \tag{21}$$

where matrix $P^\alpha_{m\times m}$ is called the Haar wavelet operational matrix of fractional integration. Using (18) and (12), we have

$$\begin{aligned} (I^\alpha \Psi)(t) &\simeq (I^\alpha H_{m\times m} B_m)(t) = H_{m\times m} (I^\alpha B_m)(t) \\ &\simeq H_{m\times m} F^\alpha B_m(t). \end{aligned} \tag{22}$$

By (21) and (22), we get

$$P^\alpha_{m\times m} \Psi(t) \simeq H_{m\times m} F^\alpha B_m(t); \tag{23}$$

therefore the Haar wavelet operational matrix of fractional integration $P^\alpha_{m\times m}$ is as follows:

$$P^\alpha_{m\times m} = H_{m\times m} F^\alpha H^T_{m\times m}. \tag{24}$$

For example, with $M = 2$, the Haar operational matrix into BPFs can be expressed as

$$P^{1/2}_{8\times 8} = \begin{pmatrix} 0.0940 & 0.1719 & 0.2226 & 0.2637 & 0.2991 & 0.3307 & 0.3595 & 0.3862 \\ 0.0940 & 0.1719 & 0.2226 & 0.2637 & 0.1110 & -0.0132 & -0.0858 & -0.1411 \\ 0.1330 & 0.2431 & 0.0489 & -0.1134 & -0.0738 & -0.0349 & -0.0226 & -0.0162 \\ 0 & 0 & 0 & 0 & 0.1330 & 0.2431 & 0.0489 & -0.1134 \\ 0.1881 & -0.0323 & -0.0544 & -0.0194 & -0.0112 & -0.0076 & -0.0056 & -0.0043 \\ 0 & 0 & 0.1881 & -0.0323 & -0.0544 & -0.0194 & -0.0112 & -0.0076 \\ 0 & 0 & 0 & 0 & 0.1881 & -0.0323 & -0.0544 & -0.0194 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.1881 & -0.0323 \end{pmatrix}. \tag{25}$$

Table 1: Comparison of exact value of $u_{\max}$ with the proposed method (RHM), ADM and HPM.

		$\alpha = 1$			$\alpha = 3/4$	
κ	Exact. $u_{\max}$	RHM	ADM	HPM	RHM	HPM
0.02	0.9234271700	0.9234473344	0.9234270	0.922942037	0.9186276936	2.92000000
0.04	0.8737199832	0.8737197006	0.8612401	0.873725344	0.8590308516	2.01600000
0.1	0.7697414491	0.7697404269	0.7651130	0.765113089	0.7517051345	1.23200000
0.2	0.6590503816	0.6590500286	0.6579123	0.659050432	0.6362736501	0.81130000
0.5	0.4851902914	0.4851895440	0.4852823	0.485190290	0.4475550726	0.44320000

4. Implementation of the Method

In this section, we present a computational method for solving the nondimensional fractional population growth model and Abel integral equations.

4.1. Population Growth Model. The Volterra model for nondimensional fractional population growth model is as follows:

$$\kappa D_*^\alpha u(t) - u(t) + u^2(t) + u(t)\int_0^t u(\tau)\,d\tau = 0, \qquad u(0) = u_0,\ 0 < \alpha \le 1. \tag{26}$$

The analytical solution (26) for $\alpha = 1$ is (see [24])

$$u(t) = u_0 \exp\left(\frac{1}{\kappa}\int_0^t \left(1 - u(\tau) - \int_0^\tau u(s)\,ds\right)d\tau\right). \tag{27}$$

The exact values of $u_{\max}$ were evaluated by using

$$u_{\max} = 1 + \kappa \ln\left(\frac{\kappa}{1+\kappa-u_0}\right). \tag{28}$$

For solving (26), we first approximate $D_*^\alpha u(t)$ as

$$D_*^\alpha u(t) \simeq U^T \Psi(t), \tag{29}$$

where U is an unknown vector which should be found and $\Psi(t)$ is the vector which is defined in (16). By using initial condition, $u(0) = u_0$, and (4), we have

$$u(t) \simeq U^T P^\alpha \Psi(t) + u_0. \tag{30}$$

By using (18) and (30), we conclude that

$$u(t) \simeq U^T P^\alpha H_{m'\times m'} B_{m'}(t) + u_0 [1, 1, \ldots, 1] B_{m'}(t). \tag{31}$$

Let

$$\begin{aligned} A^T &= [a_1, a_2, \ldots, a_{m'}] \\ &= U^T P^\alpha H_{m'\times m'} + u_0 [1, 1, \ldots, 1]. \end{aligned} \tag{32}$$

By using (31) and (32), we have $u(t) \simeq A^T B_{m'}(t)$. From (5) we have

$$u^2(t) \simeq \left[a_1^2, a_2^2, \ldots, a_{m'}^2\right] B_{m'}(t) = \widetilde{A}^T B_{m'}(t). \tag{33}$$

Also, from (10) we have

$$\begin{aligned} \int_0^t u(\tau)\,d\tau &\simeq \int_0^t A^T B_{m'}(\tau)\,d\tau \simeq A^T \Upsilon B_{m'}(t) \\ &= C^T B_{m'}(t), \end{aligned} \tag{34}$$

where $C^T = A^T \Upsilon$. By using (5), (31), and (34), we have

$$u(t)\int_0^t u(\tau)\,d\tau \simeq \widetilde{U}^T B_{m'}(t), \tag{35}$$

where

$$\widetilde{U}^T = [a_1c_1, a_2c_2, \ldots, a_{m'}c_{m'}]. \tag{36}$$

Now by substituting (29), (31), (33), and (35) into (26), we obtain

$$\left(\kappa U^T H_{m'\times m'} - A^T + \widetilde{A}^T + \widetilde{U}^T\right) B_{m'}(t) \simeq 0. \tag{37}$$

By replacing $\simeq$ by $=$, we obtain the following system of nonlinear algebraic equations:

$$\kappa U^T H_{m'\times m'} - A^T + \widetilde{A}^T + \widetilde{U}^T = 0. \tag{38}$$

Finally, by solving this system, we obtain the approximate solution of the problem as $u(t) \simeq A^T B_{m'}(t)$. As a numerical example, we consider the nonlinear fractional integro-differential equation (26) with the initial condition $u(0) = 0.1$; for more details, see Table 1 and Figures 1 and 2.

4.2. Abel Equations. Consider the generalized linear Abel integral equations of the first and second kinds, respectively, as [25]

$$\begin{aligned} f(x) &= \int_0^x \frac{y(t)}{(x-t)^{1-\alpha}}dt, \quad 0 < \alpha < 1, \\ y(x) &= f(x) + \int_0^x \frac{y(t)}{(x-t)^{1-\alpha}}dt, \quad 0 < \alpha < 1, \end{aligned} \tag{39}$$

where $f(x)$ and $y(x)$ are differentiable functions. Here, we apply fractional integration operational matrix of rational Haar wavelet to solve Abel integral equations as fractional integral equations.

Let $y(x) = U^T \Psi(x)$. Clearly, we can write (1) as follows:

$$\int_0^x \frac{y(t)}{(x-t)^{1-\alpha}}dt = \Gamma(\alpha) I^\alpha y(x). \tag{40}$$

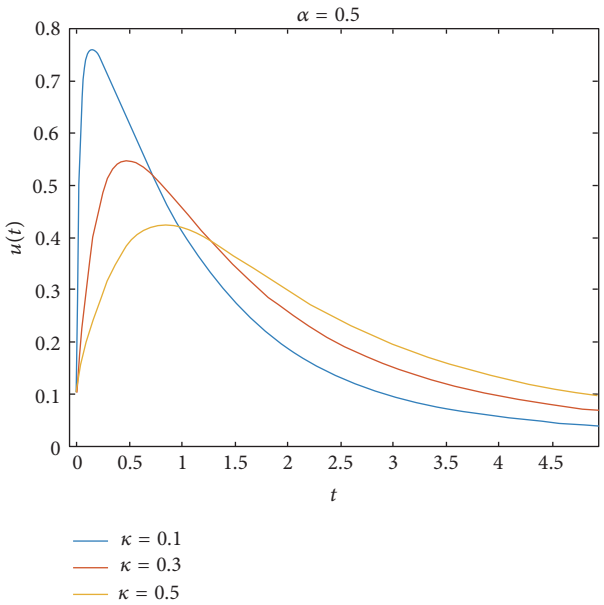

FIGURE 1: Numerical solutions of the fractional population growth model for different values of κ.

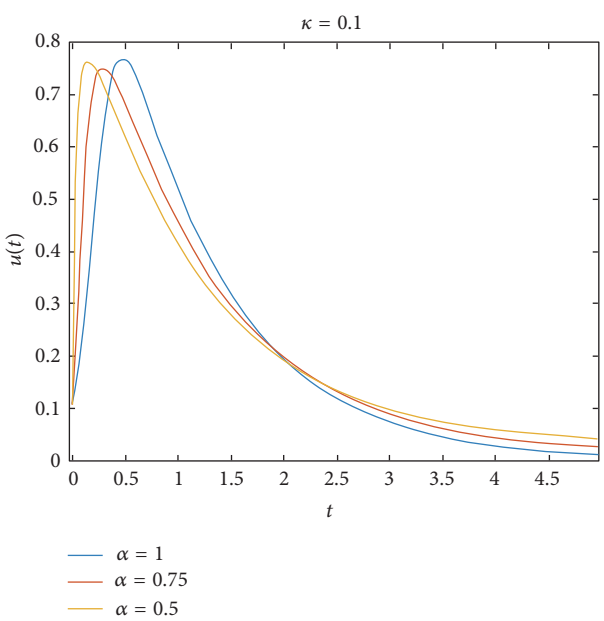

FIGURE 2: Numerical solutions of the fractional population growth model with $\kappa = 0.1$ for different values of α.

Now, by using (40), we obtain fractional form of Abel integral equations of the first and second kind, respectively, as the following form:

$$\begin{aligned} f(x) &= \Gamma(\alpha) U^T P^\alpha \Psi(x), \\ f(x) &= \left(U^T - \Gamma(\alpha) U^T P^\alpha\right) \Psi(x). \end{aligned} \tag{41}$$

By collocating (41) in $t_i = (i + 1/2)/m$, for $i = 0, 1, 2, \ldots, m-1$, we obtain the following systems of algebraic equations:

$$\begin{aligned} f(x_i) &= \Gamma(\alpha) U^T P^\alpha \Psi(x_i), \\ f(x_i) &= \left(U^T - \Gamma(\alpha) U^T P^\alpha\right) \Psi(x_i). \end{aligned} \tag{42}$$

Finally, by solving this system and determining U, we obtain the approximate solution of (41) as $y(x) = U^T \Psi(x)$.

Example 1. Consider the second kind Abel integral equation of the form

$$y(x) = x^2 + \frac{16}{15} x^{5/2} - \int_0^x \frac{y(t)}{(x-t)^{1/2}} dt. \tag{43}$$

The exact solution is $y(x) = x^2$.

Let $y(x) = U^T \Psi(x)$ and $X = (l - .5)/m'$, for $l = 1, 2, \ldots, m'$.

From (1), (18), and (21), we have

$$\begin{aligned} \int_0^x \frac{y(t)}{(x-t)^{1/2}} dt &= \Gamma\left(\frac{1}{2}\right) I^\alpha (y(x)) \\ &= \Gamma\left(\frac{1}{2}\right) I^\alpha \left(U^T \Psi(x)\right) \\ &= \Gamma\left(\frac{1}{2}\right) U^T P^\alpha H_{m' \times m'} B(x). \end{aligned} \tag{44}$$

So, we get the algebraic equations as follows:

$$\begin{aligned} U^T H_{m' \times m'} - \left(X^2\right)^T - \frac{16}{15} \left(X^{5/2}\right)^T \\ + \Gamma\left(\frac{1}{2}\right) U^T P^\alpha H_{m' \times m'} = 0. \end{aligned} \tag{45}$$

By solving this system, we obtain the approximate solution of the problem as $y(x) = U^T \Psi(x)$. Figure 3 shows the plot of error for presented method and the exact solution of this example.

Example 2. Consider Abel integral equation of the first kind

$$x = \int_0^x \frac{y(t)}{(x-t)^{4/5}} dt. \tag{46}$$

The exact solution is $y(x) = (5/4)(\sin(\pi/5)/\pi) x^{4/5}$.

To compare the numerical results and the exact solution, one can refer to Figure 4.

5. Error Analysis

In this section, we assume that $f(t)$ is a differentiable function and also $f'(t)$ is bounded on the interval $[0, 1]$; that is,

$$\exists K > 0; \quad \forall t \in [0, 1] : \left| f'(t) \right| \leqslant K. \tag{47}$$

If $f_m(t)$ is the approximation of $f(t)$ as

$$f(t) \simeq \sum_{i=0}^{m-1} f_i h_i(t), \tag{48}$$

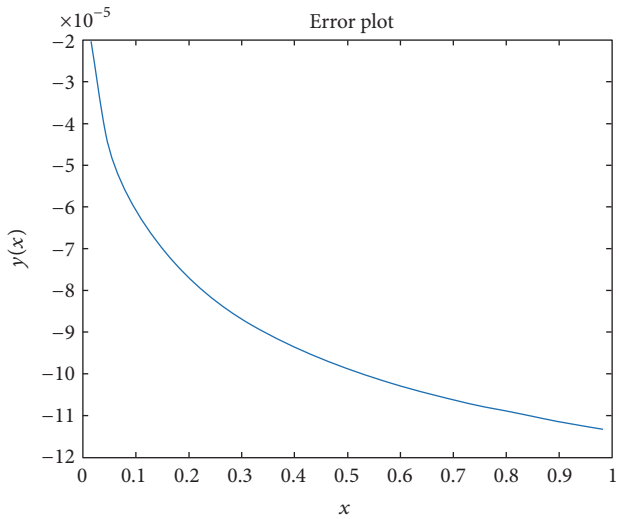

Figure 3: Error plot for Example 1 with $m' = 32$.

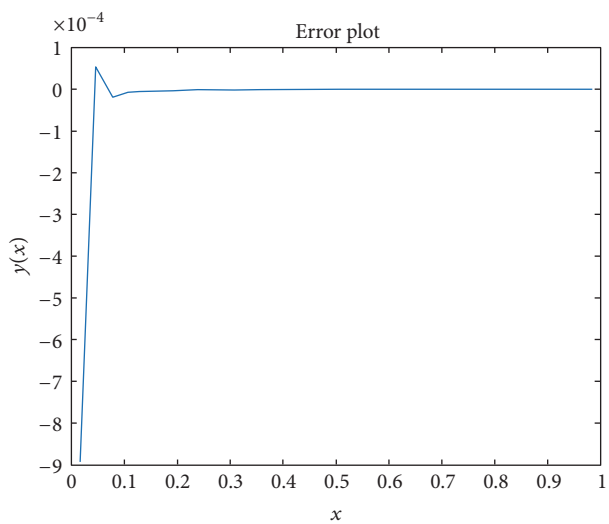

Figure 4: Error plot for Example 2 with $m' = 32$.

where $m = 2^{M+1}$ and M is a positive integer, the corresponding error function is denoted by $e_m(x) = f(x) - f_m(x)$.

Theorem 3. *Suppose that $f(x) \in L^2[0,1]$ with bounded first derivative, $|f'(t)| \leqslant K$, and $f(t) \simeq \sum_{i=0}^{m-1} f_i h_i(t)$. Then we have the error bound as follows:*

$$\|e_m(x)\|_E = \|f(x) - f_m(x)\|_E \leqslant \frac{K}{\sqrt{3}m}, \tag{49}$$

where $\|e_m(x)\|_E = \int_0^1 (e_m(x))^2 dx$.

Proof. See [26]. □

In other words, by increasing m, the error function, $e_m(t)$, approaches to zero. If $e_m(t) \to 0$ when m is sufficiently large enough, then the error decreases.

6. Conclusion

In this paper, we presented a numerical scheme for solving fractional population growth model and Abel integral equations of the first and second kinds. The method which is employed is based on the rational Haar wavelet. In Figures 1 and 2, the solutions of fractional population growth model for different values of κ and α are shown. Table 1 represents the exact value of $u_{\max}$ and comparison of our used rational Haar method (RHM) with ADM (Adomian Decomposition Method [27]) and HPM (Homotopy-Padé Approximation Method [28]). By considering Abel integral equations of the first and second kinds as a fractional integral equation, we use fractional calculus properties for solving these singular integral equations. The fractional integration is described in the Riemann-Liouville sense. This matrix is used to approximate the numerical solution of the generalized Abel integral equations of the first and second kinds. Presented approach was based on the collocation method. Figures 3 and 4 show the plot of the error of presented method and the exact solution of Abel equations (Examples 1 and 2, resp.). The obtained results show that the used technique can be a suitable method to solve the fractional integro-differential equations.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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