Hindawi
Mathematical Problems in Engineering
Volume 2017, Article ID 4215685, 19 pages
https://doi.org/10.1155/2017/4215685

Research Article

Theory of Nonlinear Guided Electromagnetic Waves in a Plane Two-Layered Dielectric Waveguide

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Received 23 July 2017; Accepted 19 September 2017; Published 26 October 2017

Academic Editor: Ivan D. Rukhlenko

Propagation of transverse electric electromagnetic waves in a homogeneous plane two-layered dielectric waveguide filled with a nonlinear medium is considered. The original wave propagation problem is reduced to a nonlinear eigenvalue problem for an equation with discontinuous coefficients. The eigenvalues are propagation constants (PCs) of the guided waves that the waveguide supports. The existence of PCs that do not have linear counterparts and therefore cannot be found with any perturbation method is proven. PCs without linear counterparts correspond to a novel propagation regime that arises due to the nonlinearity. Numerical results are also presented; the comparison between linear and nonlinear cases is made.

1. Introduction

Theory of electromagnetic wave propagation in regular (planar, cylindrical, etc.) waveguides filled with linear dielectrics traditionally attracts attention [1–4]. This theory is interesting due to several reasons: first, such problems describe real physical processes that are of importance for applications; second, from the mathematical point of view, this theory is an affluent source of sophisticated and interesting mathematical problems.

Theory of electromagnetic waves in nonlinear media has also attracted attention for decades [5–12]. There are a lot of topics in this field, for example, electromagnetic wave propagation in self-focusing and self-defocusing media, higher harmonic generation (especially second and third), and Raman scattering [6, 8, 10, 13].

In the theory of nonlinear electromagnetic wave propagation, the most advanced results can be found for the case of monochromatic polarised (TE and TM) waves in planar layered dielectric waveguides. From the mathematical standpoint, similar problems for circle cylindrical waveguides are much more complicated. To the best of our knowledge, the first rigorous formulation of TE and TM wave propagation in plane and circle cylindrical waveguides with nonlinear filling had been proposed in [14], and since then these and similar problems have been studied very intensively [7, 10, 15–25]. Nevertheless, key results in the cases of TE and TM wave propagation in a layer with Kerr nonlinearity have been found only recently [23–25].

It is worth noting that the development of the wave propagation theory in a single layer is a first step towards studying stratified (or multilayered) waveguide structures. Layered and periodic waveguides are of special interest for optical guiding industry. Since such structures play an important role in a number of applications in optics, then they compel attention of researchers [26–35].

This paper focuses on the problem of monochromatic TE wave propagation in a plane two-layered dielectric waveguide Σ filled with Kerr media. The guided wave harmonically depends on one of the longitudinal coordinates and decays along the transverse coordinate. Perfectly conducted wall is located on one of the waveguide boundaries; on the opposite side, the waveguide is open and the half-space is filled with a homogeneous isotropic nonmagnetic medium having constant permittivity. We apply the approach developed in [36]. From the mathematical point of view, the problem under investigation is a nonlinear eigenvalue problem for an ordinary nonlinear autonomous differential equation of the

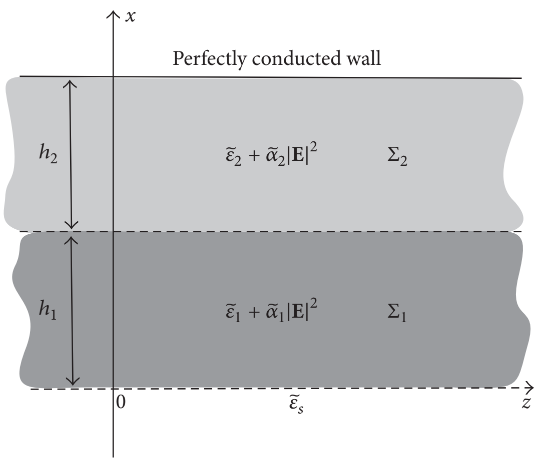


Figure 1: Geometry of the problem. Here $\tilde{\varepsilon}_i = \varepsilon_0\varepsilon_i$, $\tilde{\varepsilon}_s = \varepsilon_0\varepsilon_s$, and $\tilde{\alpha}_i = \varepsilon_0\alpha_i$.

second order with discontinuous coefficients and boundary and transmission conditions followed from electromagnetic theory. Eigenvalues of the problem are *propagation constants* (PCs) of eigenwaves of the waveguide. The PCs are solutions to the so-called *dispersion equation* (DE). We derive the DE in the general case. If one of the layers is nonlinear and the other one is linear, then the DE can be studied in detail [23, 36].

2. Statement of the Problem

We consider the propagation of a monochromatic TE wave $(\mathbf{E}, \mathbf{H})e^{-i\omega t}$, where ω is the circular frequency, in a lossless two-layered plane dielectric waveguide $\Sigma = \Sigma_1 \cup \Sigma_2$, where

$$\Sigma_1 = \left\{(x, y, z) : 0 \leqslant x < h_1,\ (y, z) \in \mathbb{R}^2\right\},$$
$$\Sigma_2 = \left\{(x, y, z) : h_1 \leqslant x \leqslant h_1 + h_2,\ (y, z) \in \mathbb{R}^2\right\}. \tag{1}$$

The TE wave is described as follows:

$$\mathbf{E} = \left(0, E_y(x)e^{i\gamma z}, 0\right)^\top,$$
$$\mathbf{H} = \left(H_x(x)e^{i\gamma z}, 0, H_z(x)e^{i\gamma z}\right)^\top, \tag{2}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the complex amplitudes [14] and γ is an unknown real propagation constant (spectral parameter).

In the half-space $x < 0$, the permittivity is constant and is equal to $\tilde{\varepsilon}_s = \varepsilon_0\varepsilon_s$, where $\varepsilon_s \geqslant 1$ and $\varepsilon_0 > 0$ is the permittivity of free space. There are no sources in the entire space. Everywhere, $\mu = \mu_0$, where μ_0 is the permeability of free space.

The waveguide Σ is characterised by the permittivity $\tilde{\varepsilon} = \varepsilon_0\varepsilon$, where

$$\varepsilon = \begin{cases} \varepsilon_1 + \alpha_1|\mathbf{E}|^2, & (x, y, z) \in \Sigma_1, \\ \varepsilon_2 + \alpha_2|\mathbf{E}|^2, & (x, y, z) \in \Sigma_2 \end{cases} \tag{3}$$

and $\alpha_1, \alpha_2 > 0$ (see Figure 1). In what follows, we assume that $\varepsilon_s < \varepsilon_1 < \varepsilon_2$ are real constants. There is a perfectly conducted wall σ at the boundary $x = h_1 + h_2$.

Complex amplitudes (2) satisfy Maxwell's equations,

$$\operatorname{rot}\mathbf{H} = -i\omega\tilde{\varepsilon}\mathbf{E},$$
$$\operatorname{rot}\mathbf{E} = i\omega\mu_0\mathbf{H}, \tag{4}$$

and decay as $O(|x|^{-1})$ when $x \to -\infty$; tangential components of the fields are continuous on the boundaries $x = 0$ and $x = h_1$; tangential component of the electric field vanishes on the boundary $x = h_1 + h_2$. It is assumed that the value $E_y|_{x=0} \neq 0$ is prescribed.

If it does not lead to misunderstanding, the explicit dependence on x and γ is omitted.

Substituting (2) into (4), one gets

$$i\gamma H_x(x) - H_z'(x) = -i\omega\varepsilon_0\varepsilon E_y(x),$$
$$-i\gamma E_y(x) = i\omega\mu_0 H_x(x), \tag{5}$$
$$E_y'(x) = i\omega\mu_0 H_z(x).$$

Let $k_0^2 = \omega^2\mu_0\varepsilon_0$. Expressing H_x and H_z from the second and third equations in (5) and substituting the results into the first equation, one obtains

$$E_y'' = \left(\gamma^2 - k_0^2\varepsilon\right)E_y. \tag{6}$$

Denoting E_y by y_1 and y_2 in the layers Σ_1 and Σ_2, respectively, one obtains the equations

$$y_1'' = -\left(\kappa_1^2 + \beta_1 y_1^2\right)y_1,$$
$$y_1 \equiv y_1(x;\gamma), \quad x \in [0, h_1], \tag{7}$$

$$y_2'' = -\left(\kappa_2^2 + \beta_2 y_2^2\right)y_2,$$
$$y_2 \equiv y_2(x;\gamma), \quad x \in [h_1, h_1 + h_2], \tag{8}$$

where $\kappa_i^2 := k_0^2\varepsilon_i - \gamma^2$ and $\beta_i := k_0^2\alpha_i$ $(i = 1, 2)$; κ_i^2 are not necessarily positive.

Equation (6) is linear in the half-space $x < 0$. Taking into account conditions at infinity, one obtains its solution in the form

$$E_y(x) = Ae^{\kappa_s x}, \tag{9}$$

where $\kappa_s^2 := \gamma^2 - k_0^2\varepsilon_s > 0$. This solution results in the condition $\gamma^2 > k_0^2\varepsilon_s$.

The continuity condition for the tangential field components results in the continuity of E_y and E_y' at $x = 0$ and $x = h_1$. Using solution (9) and the continuity of E_y and E_y', one obtains $y_1(0) = A$ and $y_1'(0) = \kappa_s A$. Since E_y corresponds to the tangential component of the electric field, then it vanishes at $x = h_1 + h_2$. In view of this, one gets the following conditions:

$$y_1'(0) - \kappa_s y_1(0) = 0,$$
$$y_2(h_1 + h_2) = 0, \tag{10}$$

$$y_1(h_1 - 0) - y_2(h_1 + 0) = 0,$$
$$y_1'(h_1 - 0) - y_2'(h_1 + 0) = 0, \tag{11}$$

where $y_1(0) = A \neq 0$ is supposed to be known (without loss of generality, $A > 0$).

Thus, the original wave propagation problem is reduced to the *problem* $P(\beta_1, \beta_2)$, which is to determine PCs $\gamma = \hat{\gamma}$, such that there exists a nontrivial function

$$y(x;\hat{\gamma}) = \begin{cases} y_1(x;\hat{\gamma}), & x \in [0, h_1]; \\ y_2(x;\hat{\gamma}), & x \in [h_1, h_1 + h_2], \end{cases} \quad (12)$$

which satisfies (7)-(8), conditions (10)-(11), and

$$y \in C^1[0, h_1 + h_2] \cap C^2[0, h_1] \cap C^2[h_1, h_1 + h_2]. \quad (13)$$

The problem $P(\beta_1, \beta_2)$ can be treated as an eigenvalue problem for a nonlinear differential equation of the second order with discontinuous coefficients on a segment with mixed boundary conditions and transmission conditions at the point $x = h_1$.

3. Linear Problem

Here we consider the case $\alpha_1 = \alpha_2 = 0$, which corresponds to the linear problem denoted by $P(0, 0)$. In this case, (7) and (8) are linear:

$$\begin{aligned} y_1''(x) &= -\kappa_1^2 y_1(x), \quad x \in [0, h_1], \\ y_2''(x) &= -\kappa_2^2 y_2(x), \quad x \in [h_1, h_1 + h_2]. \end{aligned} \quad (14)$$

Using (10), solutions to (14) are written in the following form:

(i) For $\gamma^2 > k_0^2\varepsilon_2$,

$$\begin{aligned} y_1 &= A\left(\frac{\kappa_s}{\tilde{\kappa}_1}\sinh(\tilde{\kappa}_1 x) + \cosh(\tilde{\kappa}_1 x)\right), \\ y_2 &= B\sinh(\tilde{\kappa}_2(x - h_1 - h_2)), \end{aligned} \quad (15)$$

where $\tilde{\kappa}_1^2 := -\kappa_1^2$ and $\tilde{\kappa}_2^2 := -\kappa_2^2$.

(ii) For $k_0^2\varepsilon_1 < \gamma^2 < k_0^2\varepsilon_2$,

$$\begin{aligned} y_1 &= A\left(\frac{\kappa_s}{\tilde{\kappa}_1}\sinh(\tilde{\kappa}_1 x) + \cosh(\tilde{\kappa}_1 x)\right), \\ y_2 &= B\sin(\kappa_2(x - h_1 - h_2)), \end{aligned} \quad (16)$$

where $\tilde{\kappa}_1^2 := -\kappa_1^2$.

(iii) For $k_0^2\varepsilon_s < \gamma^2 < k_0^2\varepsilon_1$,

$$\begin{aligned} y_1 &= A\left(\frac{\kappa_s}{\kappa_1}\sin(\kappa_1 x) + \cos(\kappa_1 x)\right), \\ y_2 &= B\sin(\kappa_2(x - h_1 - h_2)). \end{aligned} \quad (17)$$

Using solutions (15)–(17) and (11), the DE for the problem $P(0, 0)$ is written in the following form:

(i) For $\gamma^2 > k_0^2\varepsilon_2$,

$$\frac{\kappa_s\sinh(\tilde{\kappa}_1 h_1) + \tilde{\kappa}_1\cosh(\tilde{\kappa}_1 h_1)}{\kappa_s\cosh(\tilde{\kappa}_1 h_1) + \tilde{\kappa}_1\sinh(\tilde{\kappa}_1 h_1)} = -\frac{\tilde{\kappa}_1\sinh(\tilde{\kappa}_2 h_2)}{\tilde{\kappa}_2\cosh(\tilde{\kappa}_2 h_2)}, \quad (18)$$

where $\tilde{\kappa}_1^2 = -\kappa_1^2$ and $\tilde{\kappa}_2^2 = -\kappa_2^2$.

(ii) For $k_0^2\varepsilon_1 < \gamma^2 < k_0^2\varepsilon_2$,

$$\frac{\kappa_s\sinh(\tilde{\kappa}_1 h_1) + \tilde{\kappa}_1\cosh(\tilde{\kappa}_1 h_1)}{\kappa_s\cosh(\tilde{\kappa}_1 h_1) + \tilde{\kappa}_1\sinh(\tilde{\kappa}_1 h_1)} = -\frac{\tilde{\kappa}_1\sin(\kappa_2 h_2)}{\kappa_2\cos(\kappa_2 h_2)}, \quad (19)$$

where $\tilde{\kappa}_1^2 = -\kappa_1^2$.

(iii) For $k_0^2\varepsilon_s < \gamma^2 < k_0^2\varepsilon_1$,

$$\frac{\kappa_s\sin(\kappa_1 h_1) + \kappa_1\cos(\kappa_1 h_1)}{\kappa_s\cos(\kappa_1 h_1) - \kappa_1\sin(\kappa_1 h_1)} = -\frac{\kappa_1\sin(\kappa_2 h_2)}{\kappa_2\cos(\kappa_2 h_2)}. \quad (20)$$

It is clear that (18) has no solutions. Indeed, the left-hand side is positive and right-hand side is negative. Thus, solutions to the linear problem satisfy the condition

$$k_0^2\varepsilon_s < \gamma^2 < k_0^2\varepsilon_2. \quad (21)$$

Let us consider (19). Introduce the function

$$\begin{aligned} F(\gamma^2) &= \kappa_2\cos(\kappa_2 h_2)(\kappa_s\sinh(\tilde{\kappa}_1 h_1) + \tilde{\kappa}_1\cosh(\tilde{\kappa}_1 h_1)) \\ &\quad + \tilde{\kappa}_1\sin(\kappa_2 h_2)(k_s\cosh(\tilde{\kappa}_1 h_1) + \tilde{\kappa}_1\sinh(\tilde{\kappa}_1 h_1)). \end{aligned} \quad (22)$$

Let $\sin(\kappa_2 h_2) = 0$; therefore $\gamma^2 = k_0^2\varepsilon_2 - (\pi n/h_2)^2 = \gamma_*^2$, where $n \geqslant 1$ is an integer, such that $k_0^2\varepsilon_2 - (\pi n/h_2)^2 > k_0^2\varepsilon_1$ [see formula (19)]. Thus, one obtains

$$F(\gamma_*^2) = \frac{\pi n}{h_2}(-1)^n(\kappa_s\sinh(\tilde{\kappa}_1 h_1) + \tilde{\kappa}_1\cosh(\tilde{\kappa}_1 h_1)). \quad (23)$$

Choosing the lowest possible $n = 1$, one gets $F(\gamma_*^2) < 0$. The inequality $\varepsilon_2 - \pi^2/k_0^2h_2^2 > \varepsilon_1$ must be fulfilled.

Now let $\cos(\kappa_2 h_2) = 0$; therefore $\gamma^2 = k_0^2\varepsilon_2 - ((\pi + 2\pi m)/2h_2)^2 = \gamma^{*2}$, where $m \geqslant 0$ is an integer, such that $k_0^2\varepsilon_2 - ((\pi + 2\pi m)/2h_2)^2 > k_0^2\varepsilon_1$ [see formula (19)]. Thus, one obtains

$$F(\gamma^{*2}) = \tilde{\kappa}_1(-1)^m(\kappa_s\cosh(\tilde{\kappa}_1 h_1) + \tilde{\kappa}_1\sinh(\tilde{\kappa}_1 h_1)). \quad (24)$$

Choosing the lowest possible $m = 0$, one gets $F(\gamma^{*2}) > 0$. The inequality $\varepsilon_2 - \pi^2/4k_0^2h_2^2 > \varepsilon_1$ must be fulfilled.

Since F is continuous for $\gamma \in (\gamma_*, \gamma^*)$ and $F(\gamma_*^2)F(\gamma^{*2}) < 0$, then there is $\tilde{\gamma} \in (\gamma_*, \gamma^*)$ such that $F(\tilde{\gamma}) = 0$. The inequality

$$h_2 > \frac{\pi}{k_0\sqrt{\varepsilon_2 - \varepsilon_1}} \quad (25)$$

is a sufficient condition of existence of solutions to (19).

Let us pass to (20). Introduce the function

$$\begin{aligned} F(\gamma^2) &= \kappa_2\cos(\kappa_2 h_2)(\kappa_s\sin(\kappa_1 h_1) + \kappa_1\cos(\kappa_1 h_1)) \\ &\quad + \kappa_1\sin(\kappa_2 h_2)(\kappa_s\cos(\kappa_1 h_1) - \kappa_1\sin(\kappa_1 h_1)). \end{aligned} \quad (26)$$

Let $\sin(\kappa_2 h_2) = 0$; therefore $\gamma^2 = k_0^2\varepsilon_2 - (\pi n/h_2)^2 = \gamma_*^2$, where $n \geqslant 1$ is an integer, such that $k_0^2\varepsilon_s < k_0^2\varepsilon_2 - (\pi n/h_2)^2 < k_0^2\varepsilon_1$ [see formula (20)]. Thus, one obtains

$$F(\gamma_*^2) = \frac{\pi n}{h_2}(-1)^n(\kappa_s \sin(\kappa_1 h_1) + \kappa_1 \cos(\kappa_1 h_1)) = \frac{\pi n}{h_2}(-1)^n\sqrt{\kappa_1^2+\kappa_s^2}\sin(\kappa_1 h_1 + \phi), \quad (27)$$

where $\sin\phi = \kappa_1/\sqrt{\kappa_1^2+\kappa_s^2}$ and $\cos\phi = \kappa_s/\sqrt{\kappa_1^2+\kappa_s^2}$. Choosing the lowest possible $n = 1$, one gets $F(\gamma_*^2) < 0$ subject to $0 < \kappa_1 h_1 + \phi < \pi$. Since the inequality $\varepsilon_s < \varepsilon_2 - (\pi/k_0 h_2)^2 < \varepsilon_1$ must be fulfilled, then $\pi/k_0\sqrt{\varepsilon_2-\varepsilon_s} < h_2 < \pi/k_0\sqrt{\varepsilon_2-\varepsilon_1}$.

Now let $\cos(\kappa_2 h_2) = 0$; therefore $\gamma^2 = k_0^2\varepsilon_2 - ((\pi + 2\pi m)/2h_2)^2 = \gamma^{*2}$, where $m \geqslant 0$ is an integer, such that $k_0^2\varepsilon_s < k_0^2\varepsilon_2 - ((\pi+2\pi m)/2h_2)^2 < k_0^2\varepsilon_1$ [see formula (20)]. Thus, one obtains

$$F(\gamma^{*2}) = \kappa_1(-1)^m(\kappa_s \cos(\kappa_1 h_1) - \kappa_1 \sin(\kappa_1 h_1)) = \kappa_1(-1)^m\sqrt{\kappa_1^2+\kappa_s^2}\cos(\kappa_1 h_1 + \phi). \quad (28)$$

Choosing the lowest possible $m = 0$, one gets $F(\gamma^{*2}) > 0$ subject to $\kappa_1 h_1 + \phi < \pi/2$. Since the inequality $\varepsilon_s < \varepsilon_2 - (\pi/2k_0 h_2)^2 < \varepsilon_1$ must be fulfilled, then $\pi/2k_0\sqrt{\varepsilon_2-\varepsilon_s} < h_2 < \pi/2k_0\sqrt{\varepsilon_2-\varepsilon_1}$.

It follows from the above that the inequalities

$$\frac{\pi}{k_0\sqrt{\varepsilon_2-\varepsilon_s}} < h_2 < \frac{\pi}{2k_0\sqrt{\varepsilon_2-\varepsilon_1}}, \qquad h_1 < \frac{1}{\kappa_1}\left(\frac{\pi}{2}-\phi\right) \quad (29)$$

give a sufficient condition of existence of solutions to (20).

Thus, we formulate the following.

Statement 1. The problem $P(0,0)$ does not have more than a finite number of PCs $\widetilde{\gamma}_i$, $i = \overline{1,l}$. For any $i = \overline{1,l}$, it is true that $k_0^2\varepsilon_s < \widetilde{\gamma}_i^2 < k_0^2\varepsilon_2$.

Proof. The existence of solutions to (19) and (20) subject to conditions (25) and (29), respectively, results from the analysis given above. These conditions are only sufficient. It is also clear how conditions (25) and (29) change when the number of solutions increases.

The functions F depend analytically on γ. Since γ belongs to a finite interval [see formula (21)], then each of (19) and (20) does not have more than a finite number of isolated solutions inside the interval. □

4. Nonlinear Problem: Dispersion Equations and Theorem of Equivalence

In the following, we need auxiliary results given below by Statements 2 and 3.

Statement 2. The Cauchy problem for (7) with initial data

$$y_1(0) = A, \qquad y_1'(0) = A\kappa_s, \quad (30)$$

where $A > 0$ and $\kappa_s = \sqrt{\gamma^2 - k_0^2\varepsilon_s} > 0$ are constants, has a unique continuous solution $y_1 \equiv y_1(x;\gamma)$ defined globally on $[0, x^*]$, where $x^* > 0$ is an arbitrary real point. This solution depends continuously on γ for all $\gamma^2 > k_0^2\varepsilon_s$.

Proof. First integral of (7) takes the form

$$y_1'^2 + \kappa_1^2 y_1^2 + \frac{1}{2}\beta_1 y_1^4 = C_1, \quad (31)$$

where, using conditions (30), one finds

$$C_1 = \left(k_0^2\varepsilon_1 - k_0^2\varepsilon_s + \frac{1}{2}\beta_1 A^2\right)A^2. \quad (32)$$

It is clear that C_1 does not depend on γ, and $C_1 > 0$ if $\beta_1 \geqslant 0$.

Introduce new variables:

$$\tau_1(x) = y_1^2(x), \qquad \eta_1(x) = \frac{y_1'(x)}{y_1(x)}. \quad (33)$$

Equation (7) can be rewritten as a system:

$$\tau_1' = 2\tau_1\eta_1, \qquad \eta_1' = -\left(\eta_1^2 + \kappa_1^2 + \beta_1\tau_1\right). \quad (34)$$

First integral (31) takes the form

$$\frac{1}{2}\beta_1\tau_1^2 + \left(\eta_1^2 + \kappa_1^2\right)\tau_1 = C_1. \quad (35)$$

Solving (35) with respect to τ_1, taking into account the fact that $\tau_1 \geqslant 0$, and substituting the result into the right-hand side of the second equation in (34), one obtains

$$\eta_1' = -w_1(\eta_1;\gamma), \quad (36)$$

where $w_1(\eta_1;\gamma) = \sqrt{(\eta_1^2+\kappa_1^2)^2 + 2\beta_1 C_1}$ and the radicand is positive for all real η_1 and γ.

Using conditions (30), one finds

$$\eta_1(0) = \kappa_s > 0. \quad (37)$$

Since $\eta_1' < 0$, η_1 monotonically decreases for $x > 0$. In the general case, $y_1(x)$ can have zeros at some points on the interval $(0, x^*)$. Suppose that $y_1(x)$ has n_1 zeros $x_1, \ldots, x_{n_1} \in (0, x^*)$. Then η_1 has n_1 break points $x_1, \ldots, x_{n_1} \in (0, x^*)$. If $n_1 = 0$, then $y_1(x)$ does not become zero for any $x \in [0, x^*]$ and, therefore, η_1 is continuous for $x \in [0, x^*]$. It is clear that $y_1'(x_i) \neq 0$ for all $i = \overline{1,n_1}$. Formula (36) implies that

$$\eta_1(x_i - 0) = -\infty, \qquad \eta_1(x_i + 0) = +\infty, \qquad i = \overline{1,n_1}. \quad (38)$$

Thereby, solutions to (36) are sought on each of the intervals $[0, x_1)$, $(x_1, x_2), \ldots, (x_{n_1}, x^*)$:

$$\begin{aligned} &\int_{\eta(x)}^{\eta(x_1-0)} \frac{ds}{w_1(s;\gamma)} = x + c_0, \quad 0 \leqslant x < x_1; \\ &-\int_{\eta(x_i+0)}^{\eta(x)} \frac{ds}{w_1(s;\gamma)} = x + c_i, \\ &\qquad x_i < x < x_{i+1},\ i = \overline{0, n_1 - 1}; \\ &-\int_{\eta(x_{n_1}+0)}^{\eta(x)} \frac{ds}{w_1(s;\gamma)} = x + c_{n_1}, \quad x_{n_1} < x < x^*. \end{aligned} \tag{39}$$

Substituting $x = 0$, $x = x_{i+1} - 0$, and $x = x^*$ into the first, second, and third equations, respectively, in (39), one determines

$$\begin{aligned} c_0 &= \int_{\eta(0)}^{\eta(x_1-0)} \frac{ds}{w_1(s;\gamma)}; \\ c_i &= -\int_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} \frac{ds}{w_1(s;\gamma)} - x_{i+1}, \quad i = \overline{1, n_1 - 1}; \\ c_{n_1} &= -\int_{\eta(x_{n_1}+0)}^{\eta(x^*)} \frac{ds}{w_1(s;\gamma)} - x^*. \end{aligned} \tag{40}$$

Using the found c_i, one can rewrite (39) in the following form:

$$\begin{aligned} &\int_{\eta(x)}^{\eta(x_1-0)} \frac{ds}{w_1(s;\gamma)} = x + \int_{\eta(0)}^{\eta(x_1-0)} \frac{ds}{w_1(s;\gamma)}, \\ &\qquad 0 \leqslant x < x_1; \\ &-\int_{\eta(x_i+0)}^{\eta(x)} \frac{ds}{w_1(s;\gamma)} = x - \int_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} \frac{ds}{w_1(s;\gamma)} - x_{i+1}, \\ &\qquad x_i < x < x_{i+1},\ i = \overline{1, n_1 - 1}; \\ &-\int_{\eta(x_{n_1}+0)}^{\eta(x)} \frac{ds}{w_1(s;\gamma)} = x - \int_{\eta(x_{n_1}+0)}^{\eta(x^*)} \frac{ds}{w_1(s;\gamma)} - x^*, \\ &\qquad x_{n_1} < x < x^*. \end{aligned} \tag{41}$$

By substituting $x = x_1 - 0$, $x = x_i + 0$, and $x = x_n + 0$ into the first, second, and third equations, respectively, of previous equation, one obtains

$$\begin{aligned} 0 &= x_1 + \int_{\eta(0)}^{\eta(x_1-0)} \frac{ds}{w_1(s;\gamma)}, \\ 0 &= x_i - \int_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} \frac{ds}{w_1(s;\gamma)} - x_{i+1}, \quad i = \overline{1, n_1 - 1}; \\ 0 &= x_{n_1} - \int_{\eta(x_{n_1}+0)}^{\eta(x^*)} \frac{ds}{w_1(s;\gamma)} - x^*. \end{aligned} \tag{42}$$

Taking into account (37) and (38), one finds, from (42),

$$\begin{aligned} 0 < x_1 &= \int_{-\infty}^{\kappa_s} \frac{ds}{w_1(s;\gamma)}, \\ 0 < x_{i+1} - x_i &= \int_{-\infty}^{+\infty} \frac{ds}{w_1(s;\gamma)}, \quad i = \overline{1, n_1 - 1}; \\ 0 < h_1 - x_{n_1} &= \int_{\eta(x^*)}^{+\infty} \frac{ds}{w_1(s;\gamma)}. \end{aligned} \tag{43}$$

Formulas (43) give explicit expressions for distances between zeros of y_1. Moreover, since the left-hand sides in (43) are finite, the right-hand sides are also finite. Therefore, the improper integrals on the right-hand sides converge.

Summing up all the terms in (43), one gets

$$\begin{aligned} &x_1 + x_2 - x_1 + x_3 - x_2 + \cdots + x_{n_1-1} - x_{n_1-2} + x_{n_1} \\ &\quad - x_{n_1-1} + x^* - x_{n_1} \\ &\quad = \int_{-\infty}^{\kappa_s} w_1 ds + (n_1 - 1) \int_{-\infty}^{+\infty} w_1 ds + \int_{\eta(x^*)}^{+\infty} w_1 ds. \end{aligned} \tag{44}$$

From the last formula, one finds

$$\int_{\eta(x^*)}^{\kappa_s} w_1 ds + n_1 \int_{-\infty}^{+\infty} w_1 ds = x^*. \tag{45}$$

Formula (45) shows that the solution of the Cauchy problem to (7) with initial conditions (30) exists and is defined globally at any segment $[0, x^*]$. The uniqueness of this solution and its continuity with respect to γ follows from smoothness of the right-hand side of (7) with respect to y_1 and γ [37]. □

Let us consider the function $p(\gamma) \coloneqq y_1'(h_1;\gamma)/y_1(h_1;\gamma)$. Since the right-hand side of (7) depends on γ analytically, the solution $y_1 \equiv y_1(x;\gamma)$ of the considered Cauchy problem depends on γ analytically as well [38] and therefore y_1 and y_1' depend analytically on γ. Since y_1 and y_1' do not vanish simultaneously, $p \equiv p(\gamma)$ is an analytical function that can have only poles of the first order.

Passing to the limit $x^* \to h_1 - 0$ in (45), one gets

$$\Phi_1(\gamma; n_1, p) \equiv \int_p^{\kappa_s} w_1 ds + n_1 \int_{-\infty}^{+\infty} w_1 ds = h_1, \tag{46}$$

where $n_1 = 0, 1, 2, \ldots$. We notice that if $p > \kappa_s$, then necessarily $n_1 \geqslant 1$.

Statement 3. The Cauchy problem for (8) with initial data

$$\begin{aligned} y_2(h_1 + h_2) &= 0, \\ y_2'(h_1 + h_2) &= B, \end{aligned} \tag{47}$$

where B is a real constant, has a unique continuous solution $y_2 \equiv y_2(x;\gamma)$ defined globally on $[x^*, h_1 + h_2]$, where $x^* < h_1 + h_2$ is an arbitrary real point. This solution depends continuously on γ for all $\gamma^2 > k_0^2 \varepsilon_s$.

Proof. Now let us consider (8). First integral of (8) has the form

$$y_2'^2 + \kappa_2^2 y_2^2 + \frac{1}{2}\beta_2 y_2^4 = C_2, \quad (48)$$

where, using condition (47), one calculates

$$C_2 = B^2 > 0, \quad (49)$$

where B depends on γ. The positivity of C_2 is essential.

Introduce new variables:

$$\tau_2(x) := y_2^2(x),$$
$$\eta_2(x) := \frac{y_2'(x)}{y_2(x)}. \quad (50)$$

Equation (8) can be rewritten as a system:

$$\tau_2' = 2\tau_2\eta_2,$$
$$\eta_2' = -\left(\eta_2^2 + \kappa_2^2 + \beta_2\tau_2\right). \quad (51)$$

First integral (48) takes the form

$$\frac{1}{2}\beta_2\tau_2^2 + \left(\eta_2^2 + \kappa_2^2\right)\tau_2 = C_1. \quad (52)$$

Solving (52) with respect to τ_2, taking into account the fact that $\tau_2 \geqslant 0$, and substituting the result into the right-hand side of the second equation in (51), one obtains

$$\eta_2' = -w_2(\eta_2;\gamma), \quad (53)$$

where $w(\eta_2;\gamma) = \sqrt{(\eta_2^2+\kappa_2^2)^2 + 2\beta_2 C_2}$ and the radicand is positive for all real η_2 and γ.

Using condition (47) and the fact that $\eta_2' < 0$, one finds

$$\eta_2(h_1 + h_2 - 0) = -\infty. \quad (54)$$

Since $\eta_2' < 0$, η_2 monotonically decreases for $x \in (x^*, h_1 + h_2)$. However, η_2 is continuous if and only if $y_2(x)$ does not vanish for all $x \in (x^*, h_1 + h_2)$. In the general case, $y_2(x)$ can have zeros at some points on the interval $(h_1, h_1 + h_2)$. Suppose that $y_2(x)$ has n_2 zeros $x_1, \ldots, x_{n_2} \in (x^*, h_1 + h_2)$. Then η_2 has n_2 break points $x_1, \ldots, x_{n_2} \in (x^*, h_1 + h_2)$. It is clear that $y_2'(x_i) \neq 0$ for all $i = \overline{1, n_2}$. Formula (53) implies that

$$\eta_2(x_i - 0) = -\infty,$$
$$\eta_2(x_i + 0) = +\infty, \quad (55)$$
$$i = \overline{1, n_2}.$$

Thereby, solutions to (53) are sought on each of the intervals $(x^*, x_1), (x_1, x_2), \ldots, (x_{n_2}, h_1 + h_2)$.

Using the same reasoning as in the proof of Statement 2, one obtains the expression

$$\int_{-\infty}^{\eta(x^*)} \frac{ds}{w_2(s;\gamma)} + n_2 \int_{-\infty}^{+\infty} \frac{ds}{w_2(s;\gamma)} = h_2. \quad (56)$$

Formula (56) shows that the solution to the Cauchy problem for (8) with initial conditions (47) exists and is defined globally at any segment $[x^*, h_1 + h_2]$. The uniqueness of this solution and its continuity with respect to γ follows from smoothness of the right-hand side of (8) with respect to y_2 and γ [37]. □

Passing to the limit $x^* \to h_1 + 0$ in (56), one gets

$$\Phi_2(\gamma; n_2, p) \equiv \int_{-\infty}^{p} \frac{ds}{w_2(s;\gamma)} + n_2 \int_{-\infty}^{+\infty} \frac{ds}{w_2(s;\gamma)} = h_2, \quad (57)$$

where $n_2 = 0, 1, 2, \ldots$.

The following theorem takes place.

Theorem 1 (of equivalence). *The value $\hat{\gamma}$ is a PC of the problem $P(\beta_1, \beta_2)$ if and only if there are integers $n_1 = \hat{n}_1 \geqslant 0$ and $n_2 = \hat{n}_2 \geqslant 0$ such that $\gamma = \hat{\gamma}$ is a solution to the DE:*

$$\Phi_1(\gamma; n_1, p) = h_1,$$
$$\Phi_2(\gamma; n_2, p) = h_2 \quad (58)$$

with certain $p = \hat{p}$.

Proof. It follows from the derivation of expressions (46) and (57) that if $\gamma = \hat{\gamma}$ is an eigenvalue of the problem $P(\beta_1, \beta_2)$, then it is a solution to system (58) with $n_1 = \hat{n}_1$, $n_2 = \hat{n}_2$, and $p = \hat{p}$. Let us prove that each solution $\gamma = \hat{\gamma}$ to system (58) is an eigenvalue.

Let system (58) have a solution $\gamma = \hat{\gamma}$ with $n_1 = \hat{n}_1$, $n_2 = \hat{n}_2$, $p = \hat{p}$, and $y_1(0) = A$.

Consider the Cauchy problem for (7) with initial data (30), where $\kappa_s = \sqrt{\hat{\gamma}^2 - k_0^2\varepsilon_s}$. In accordance with Statement 2, its solution $y_1 \equiv y_1(x, \hat{\gamma})$ exists, is unique, and is defined for $x \in [0, h_1]$. At this step, we do not claim that $y_1'(h_1)/y_1(h_1) = \hat{p}$. Using the found solution y_1 and formula (33), one determines the functions τ_1 and η_1. It is clear that $\tau_1(0) = A^2$ and $\eta_1(0) = \kappa_s$.

Assuming that $\eta_1(h_1) = p_1 < \hat{p}$ and using the found τ_1 and η_1, one obtains the expression

$$\int_{p_1}^{\kappa_s} \frac{ds}{w_1(s;\hat{\gamma})} + \hat{n}_1 \int_{-\infty}^{+\infty} \frac{ds}{w_1(s;\hat{\gamma})} = h_1, \quad (59)$$

which corresponds to the first line in (58). We rewrite it in the form

$$\int_{p_1}^{\hat{p}} \frac{ds}{w_1(s;\hat{\gamma})} + \int_{\hat{p}}^{\kappa_s} \frac{ds}{w_1(s;\hat{\gamma})} + \hat{n}_1 \int_{-\infty}^{+\infty} \frac{ds}{w_1(s;\hat{\gamma})} = h_1. \quad (60)$$

Since $\gamma = \hat{\gamma}$ satisfies the first line in (58) with $p = \hat{p}$, then in the first line in (58) and in (60) the integrands coincide. Subtracting one from another, one obtains

$$\int_{p_1}^{\hat{p}} \frac{ds}{w_1(s;\hat{\gamma})} = 0. \quad (61)$$

Due to the obvious estimates $\int_{-\infty}^{+\infty}(ds/w_1(s;\hat{\gamma})) > \int_{p_1}^{\hat{p}}(ds/w_1(s;\hat{\gamma})) > 0$, one obtains that (61) is fulfilled only if $p_1 = \hat{p}$. Therefore, the condition $\eta(h_1) = p_1 < \hat{p}$ is false. In the same way, it can be shown that the condition $\eta(h_1) = p_1 > \hat{p}$ is also false. Thus, $p_1 = \hat{p}$.

Now let us pass to the Cauchy problem for (8) with initial data (47), where $B \equiv B(\hat{\gamma})$. In accordance with Statement 3, its solution $y_2 \equiv y_2(x, \hat{\gamma})$ exists, is unique, and is defined globally for $x \in [h_1, h_1 + h_2]$. In this case, using the continuity of E_y and E_y', first integral (48), and (49), one finds that the quantity B is determined from the equation

$$B^2 = y_1'^2 + \left(k_0^2\varepsilon_2 - \hat{\gamma}^2\right) y_1^2 + \frac{1}{2}\beta_2 y_1^4 > 0, \qquad (62)$$

where y_1 and y_1' are calculated at $x = h_1$ with $\gamma = \hat{\gamma}$ (y_1 and y_1' are already found).

At this step, we do not claim that $y_2'(h_1)/y_2(h_1) = \hat{p}$. Using the found solution y_2 and formula (50), one determines the functions τ_2 and η_2.

We assume that $\eta_2(h_1) = p_2 > \hat{p}$. Using the found τ_2 and η_2, one obtains the expression

$$\int_{-\infty}^{p_2} \frac{ds}{w_2(s;\hat{\gamma})} + \hat{n}_2 \int_{-\infty}^{+\infty} \frac{ds}{w_2(s;\hat{\gamma})} = h_2, \qquad (63)$$

which corresponds to the second line in (58). We rewrite it in the following form:

$$\int_{-\infty}^{\hat{p}} \frac{ds}{w_2(s;\hat{\gamma})} + \int_{\hat{p}}^{p_2} \frac{ds}{w_2(s;\hat{\gamma})} + m_2 \int_{-\infty}^{+\infty} \frac{ds}{w_2(s;\hat{\gamma})} = h_2. \qquad (64)$$

Since $\gamma = \hat{\gamma}$ satisfies the second line in (58) with $p = \hat{p}$, then in the second line in (58) and in (64) the integrands coincide. Subtracting one from another, one obtains

$$\int_{\hat{p}}^{p_2} \frac{ds}{w_2(s;\hat{\gamma})} = 0. \qquad (65)$$

Due to the obvious estimates $\int_{-\infty}^{+\infty}(ds/w_2(s;\hat{\gamma})) > \int_{\hat{p}}^{p_2}(ds/w_2(s;\hat{\gamma})) > 0$, one obtains that (65) is fulfilled only if $p_2 = \hat{p}$. Thus, the condition $\eta_2(h_1) = p_2 > \hat{p}$ is false. In the same way, it can be shown that the condition $\eta(h_1) = p_2 < \hat{p}$ is also false. Thus, $p_2 = \hat{p}$.

In other words, we have shown that the functions y_1 and y_2 satisfy (7) and (8) and conditions (10) and (11), and, therefore, the function

$$y(x;\hat{\gamma}) = \begin{cases} y_1(x;\hat{\gamma}), & x \in [0, h_1]; \\ y_2(x;\hat{\gamma}), & x \in [h_1, h_1 + h_2] \end{cases} \qquad (66)$$

is an eigenfunction of the problem $P(\beta_1, \beta_2)$ corresponding to the eigenvalue $\gamma = \hat{\gamma}$. □

5. Solvability of the Nonlinear Problem

Theorem 1 is derived for the general case, that is, $\beta_1, \beta_2 > 0$. The DE (58) can be studied theoretically and numerically. However, the existence of infinitely many PCs for the general case can be proven only if a special restriction is imposed on $B \equiv B(\gamma)$. This restriction establishes the behaviour of B for big γ. To be more precise, $B(\gamma)$ must not decrease "too" rapid. As a matter of fact, such a restriction does not result from the physical formulation of the problem. For this reason below we study two simplified problems, where either β_1 or β_2 vanishes.

In this section, we use the following notation for the eigenvalues $\hat{\gamma}$ of problems $P(0, \beta_2)$ and $P(\beta_1, 0)$: $\hat{\gamma}_i$ means that all the eigenvalues are arranged in the ascending order; $\hat{\gamma}(m)$ means that this eigenvalue is a solution of (57) with $n_2 = m$ [for the problem $P(0, \beta_2)$] and (46) with $n_1 = m$ [for the problem $P(\beta_1, 0)$].

5.1. Case $\beta_1 = 0$ and $\beta_2 \neq 0$. If $\beta_1 = 0$ (or $\alpha_1 = 0$), solutions to (7) are found elementarily. This allows one to explicitly compute the quantity p, which is

$$p = \kappa_1 \frac{\kappa_s \cos(\kappa_1 h_1) - \kappa_1 \sin(\kappa_1 h_1)}{\kappa_s \sin(\kappa_1 h_1) + \kappa_1 \cos(\kappa_1 h_1)} = \kappa_1 \frac{\kappa_s - \kappa_1 \tan(\kappa_1 h_1)}{\kappa_1 + \kappa_s \tan(\kappa_1 h_1)}. \qquad (67)$$

This expression (due to its analytical dependence on γ) can be used for $\kappa_1^2 \geqslant 0$ as well as for $\kappa_1^2 \leqslant 0$. Note that p is a real value for all $\gamma^2 > k_0^2\varepsilon_s$. Indeed, for $\gamma^2 > k_0^2\varepsilon_1$, one has $\kappa_1 = i\tilde{\kappa}_1$, where $\tilde{\kappa}_1^2 = \gamma^2 - k_0^2\varepsilon_1$, and then $\cos(ix) = \cosh(x)$ and $\sin(ix) = i\sinh(x)$; then

$$p = \tilde{\kappa}_1 \frac{\kappa_s \cosh(\tilde{\kappa}_1 h_1) + \tilde{\kappa}_1 \sinh(\tilde{\kappa}_1 h_1)}{\kappa_s \sinh(\tilde{\kappa}_1 h_1) + \tilde{\kappa}_1 \cosh(\tilde{\kappa}_1 h_1)} = \kappa_1 \frac{\kappa_s + \tilde{\kappa}_1 \tanh(\tilde{\kappa}_1 h_1)}{\tilde{\kappa}_1 + \kappa_s \tanh(\tilde{\kappa}_1 h_1)}. \qquad (68)$$

It can be checked that the substitution of the explicit expression for p into the first equation of system (58) leads to the identity. Thus, in this case, system (58) reduces to the only equation

$$\Phi_2(\gamma; n_2, p) = h_2, \qquad (69)$$

where Φ_2 and p are given by (57) and (67), respectively; $n_2 = 0, 1, 2, \ldots$.

The solvability of the problem $P(0, \beta_2)$ is established by the following.

Theorem 2. *For $\beta_1 = 0$, any $\beta_2 > 0$, and any fixed $A \neq 0$, the problem $P(0, \beta_2)$ has an infinite number of PCs $\hat{\gamma}_i$ $(i = 1, 2, \ldots)$ with the following properties:*

(1) *If $\hat{\gamma}_i$ is the solution to $P(0, \beta_2)$, then $\hat{\gamma}_i^2 \in (k_0^2\varepsilon_s, +\infty)$ and $\lim_{i\to\infty}\hat{\gamma}_i^2 = +\infty$.*

(2) *If the linear problem $P(0, 0)$ has q solutions $\tilde{\gamma}_1 < \tilde{\gamma}_2 < \cdots < \tilde{\gamma}_q$, then there exists a constant $\beta_0 > 0$ such that,*

for any $\beta_2 = \beta_2' < \beta_0$, *it is true that* $\hat{\gamma}_i^2 \in (k_0^2\varepsilon_s, k_0^2\varepsilon_2)$ *and* $\lim_{\beta_1' \to 0} \hat{\gamma}_i = \gamma_i$ $(i = \overline{1,q})$, *where* $\hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_q$ *are first* q *solutions to* $P(0, \beta_2')$.

(3) *If* $\hat{\gamma}_i \to \infty$, *then* $\max_{x\in(h_1,h_1+h_2)} |y_2(x, \hat{\gamma}_i)| \to \infty$.

(4) *For big* $\hat{\gamma}$ *and arbitrary small* $\Delta > 0$, *the asymptotic two-sided inequality*

$$\begin{aligned} (1-\Delta)\frac{1}{h_1}\ln\frac{m\pi}{h_2 A\sqrt{\beta_2}} &\leqslant \hat{\gamma}(m) \\ &\leqslant (1+\Delta)\frac{1}{h_1}\ln\frac{\sqrt{2}(m+1)\pi}{h_2 A\sqrt{\beta_2}} \end{aligned} \tag{70}$$

is valid.

Proof. Now let us study the behaviour of $C_2 \equiv C_2(\gamma)$ with respect to γ. Explicit solution to (7) with initial data (30) has the form $y_1(x) = A((\kappa_s/\kappa_1)\sin(\kappa_1 x) + \cos(\kappa_1 x))$. At the point $x = h_1$, one gets

$$\begin{aligned} y_1(h_1) &= A\left(\frac{\kappa_s}{\kappa_1}\sin(\kappa_1 h_1) + \cos(\kappa_1 h_1)\right), \\ y_1'(h_1) &= A\kappa_1\left(\frac{\kappa_s}{\kappa_1}\cos(\kappa_1 h_1) - \sin(\kappa_1 h_1)\right). \end{aligned} \tag{71}$$

Substituting (71) into (48), one obtains

$$\begin{aligned} C_2 = {} & \kappa_1^2\left(\frac{\kappa_s}{\kappa_1}\cos(\kappa_1 h_1) - \sin(\kappa_1 h_1)\right)^2 A^2 \\ & + \kappa_2^2\left(\frac{\kappa_s}{\kappa_1}\sin(\kappa_1 h_1) + \cos(\kappa_1 h_1)\right)^2 A^2 \\ & + \frac{1}{2}\beta_2\left(\frac{\kappa_s}{\kappa_1}\sin(\kappa_1 h_1) + \cos(\kappa_1 h_1)\right)^4 A^4. \end{aligned} \tag{72}$$

If $k_0^2\varepsilon_s < \gamma^2 \leqslant k_0^2\varepsilon_1$, then $\kappa_1^2 \geqslant 0$ and $\kappa_2^2 \geqslant 0$ and, therefore, $C_2 > 0$.

If $k_0^2\varepsilon_1 < \gamma^2 \leqslant k_0^2\varepsilon_2$, then $\kappa_1^2 < 0$ and $\kappa_2^2 \geqslant 0$. As before, we denote $\kappa_1 = i\tilde{\kappa}_1$, where $\tilde{\kappa}_1^2 = \gamma^2 - k_0^2\varepsilon_1 > 0$. Then

$$\begin{aligned} C_2 = {} & \tilde{\kappa}_1^2\left(\frac{\kappa_s}{\tilde{\kappa}_1}\cosh(\tilde{\kappa}_1 h_1) + \sinh(\tilde{\kappa}_1 h_1)\right)^2 A^2 \\ & + \kappa_2^2\left(\frac{\kappa_s}{\tilde{\kappa}_1}\sinh(\tilde{\kappa}_1 h_1) + \cosh(\tilde{\kappa}_1 h_1)\right)^2 A^2 \\ & + \frac{1}{2}\beta_2\left(\frac{\kappa_s}{\tilde{\kappa}_1}\sinh(\kappa_1 h_1) + \cosh(\tilde{\kappa}_1 h_1)\right)^4 A^4 \\ & > 0. \end{aligned} \tag{73}$$

If $\gamma^2 > k_0^2\varepsilon_2$, then $\kappa_1^2 < 0$ and $\kappa_2^2 < 0$. As before, we denote $\kappa_1 = i\tilde{\kappa}_1$ and $\kappa_2 = i\tilde{\kappa}_2$, where $\tilde{\kappa}_1^2 = \gamma^2 - k_0^2\varepsilon_1 > 0$ and $\tilde{\kappa}_2^2 = \gamma^2 - k_0^2\varepsilon_2 > 0$. Then

$$\begin{aligned} C_2 = {} & \tilde{\kappa}_1^2\left(\frac{\kappa_s}{\tilde{\kappa}_1}\cosh(\tilde{\kappa}_1 h_1) + \sinh(\tilde{\kappa}_1 h_1)\right)^2 A^2 \\ & - \tilde{\kappa}_2^2\left(\frac{\kappa_s}{\tilde{\kappa}_1}\sinh(\tilde{\kappa}_1 h_1) + \cosh(\tilde{\kappa}_1 h_1)\right)^2 A^2 \\ & + \frac{1}{2}\beta_2\left(\frac{\kappa_s}{\tilde{\kappa}_1}\sinh(\kappa_1 h_1) + \cosh(\tilde{\kappa}_1 h_1)\right)^4 A^4. \end{aligned} \tag{74}$$

Grouping terms in the last expression for C_2, one gets

$$\begin{aligned} & \frac{\kappa_s}{\tilde{\kappa}_1}\cosh(\tilde{\kappa}_1 h_1) + \sinh(\tilde{\kappa}_1 h_1) \\ & \quad = \frac{1}{2}\left(\frac{\kappa_s}{\tilde{\kappa}_1}\left(e^{\tilde{\kappa}_1 h_1} + e^{-\tilde{\kappa}_1 h_1}\right) + \left(e^{\tilde{\kappa}_1 h_1} - e^{-\tilde{\kappa}_1 h_1}\right)\right) \\ & \quad = \frac{1}{2}\left(a_+ e^{\tilde{\kappa}_1 h_1} + a_- e^{-\tilde{\kappa}_1 h_1}\right), \\ & \frac{\kappa_s}{\tilde{\kappa}_1}\sinh(\tilde{\kappa}_1 h_1) + \cosh(\tilde{\kappa}_1 h_1) \\ & \quad = \frac{1}{2}\left(\frac{\kappa_s}{\tilde{\kappa}_1}\left(e^{\tilde{\kappa}_1 h_1} - e^{-\tilde{\kappa}_1 h_1}\right) + \left(e^{\tilde{\kappa}_1 h_1} + e^{-\tilde{\kappa}_1 h_1}\right)\right) \\ & \quad = \frac{1}{2}\left(a_+ e^{\tilde{\kappa}_1 h_1} - a_- e^{-\tilde{\kappa}_1 h_1}\right), \end{aligned} \tag{75}$$

where $a_+ := \kappa_s/\tilde{\kappa}_1 + 1$ and $a_- := \kappa_s/\tilde{\kappa}_1 - 1$.

Then

$$\begin{aligned} C_2 = {} & \frac{1}{32}\beta_2 A^4 a_+^4 e^{4\tilde{\kappa}_1 h_1} \\ & + \frac{1}{4}A^2\left(k_0^2\varepsilon_2 - k_0^2\varepsilon_1 - \frac{1}{2}\beta_2 a_+ a_- A^2\right) a_+^2 e^{2\tilde{\kappa}_1 h_1} \\ & + \frac{1}{2}A^2\left(\tilde{\kappa}_1^2 + \tilde{\kappa}_2^2 + \frac{3}{8}\beta_2 a_+ a_- A^2\right) a_+ a_- \\ & + \frac{1}{4}A^2\left(k_0^2\varepsilon_2 - k_0^2\varepsilon_1 - \frac{1}{2}\beta_2 a_+ a_- A^2\right) a_-^2 e^{-2\tilde{\kappa}_1 h_1} \\ & + \frac{1}{32}\beta_2 A^4 a_-^4 e^{-4\tilde{\kappa}_1 h_1}. \end{aligned} \tag{76}$$

Since $a_+ > 1$ and $a_- > 0$ are bounded above, then it is clear that for sufficiently small $\beta_2 > 0$ the constant $C_2 > 0$ for all $\gamma^2 > k_0^2\varepsilon_2$. It is also clear that for any fixed $\beta_2 > 0$ there exists γ_* such that $C_2 > 0$ for all $\gamma > \gamma_*$.

Using the expansion $(1 + x)^\alpha = 1 + O(x)$, where $|x| < 1$ and $-1 < \alpha < 1$, for sufficiently large γ, one gets the asymptotics

$$C_2 = \beta_2 A^4\left(\frac{1}{2} + O\left(\gamma^{-2}\right)\right) e^{4h_1\gamma}. \tag{77}$$

Since both integrands in (69) are positive, the estimate

$$n_2 T \leqslant \Phi_2(\gamma; n_2, p) \leqslant (n_2 + 1) T \tag{78}$$

takes place, where $n_2 \geqslant 0$ and $T = \int_{-\infty}^{+\infty}(ds/w_2)$. Thus we need to evaluate T. For further analysis, we use the easy checked inequalities $1/(a+b) \leqslant 1/\sqrt{a^2+b^2} \leqslant \sqrt{2}/(a+b)$, where $a \geqslant 0$ and $b > 0$. These inequalities give

$$T^* \leqslant T \leqslant \sqrt{2}T^*, \tag{79}$$

where $T^* = \int_{-\infty}^{+\infty}(ds/(|s^2+\kappa_2^2| + \sqrt{2\beta_2 C_2}))$. Integral T^* is calculated explicitly.

For $\gamma^2 < k_0^2\varepsilon_2$, the value $\eta^2 + \kappa_2^2 > 0$ and, therefore, one has

$$\begin{aligned} T^* &= 2\int_0^{+\infty} \frac{ds}{|s^2+\kappa_2^2| + \sqrt{2\beta_2 C_2}} \\ &= 2\int_0^{+\infty} \frac{ds}{s^2+\kappa_2^2 + \sqrt{2\beta_2 C_2}} = \frac{\pi}{\sqrt{|\kappa_2^2| + \sqrt{2\beta_2 C_2}}}. \end{aligned} \tag{80}$$

Let $\gamma^2 \geqslant k_0^2\varepsilon_2$; then

$$\begin{aligned} T^* &= 2\int_0^{+\infty} \frac{ds}{|s^2+\kappa_2^2| + \sqrt{2\beta_2 C_2}} \\ &= 2\int_0^{|\kappa_2|} \frac{ds}{-s^2-\kappa_2^2 + \sqrt{2\beta_2 C_2}} \\ &\quad + 2\int_{|\kappa_2|}^{+\infty} \frac{ds}{s^2+\kappa_2^2 + \sqrt{2\beta_2 C_2}} \\ &= 2\int_0^{|\kappa_2|} \frac{ds}{|\kappa_2^2| + \sqrt{2\beta_2 C_2} - s^2} \\ &\quad + 2\int_{|\kappa_2|}^{+\infty} \frac{ds}{s^2 + \sqrt{2\beta_2 C_2} - |\kappa_2^2|} = 2I_1 + 2I_2. \end{aligned} \tag{81}$$

Obviously, the denominator in I_1 is always positive.

Calculating I_1, one gets

$$\begin{aligned} I_1 &= \int_0^{|\kappa_2|} \frac{ds}{|\kappa_2^2| + \sqrt{2\beta_2 C_2} - s^2} \\ &= \frac{-1}{2\sqrt{|\kappa_2^2| + \sqrt{2\beta_2 C_2}}} \ln \frac{\sqrt{2\beta_2 C_2}}{\left(|\kappa_2| + \sqrt{|\kappa_2^2| + \sqrt{2\beta_2 C_2}}\right)^2}. \end{aligned} \tag{82}$$

Two cases are possible for I_2. If $|\kappa_2^2| - \sqrt{2\beta_2 C_2} \geqslant 0$, that is, $\gamma^2 \geqslant k_0^2\varepsilon_2 + \sqrt{2\beta_2 C_2}$, then

$$\begin{aligned} I_2 &= \int_{|\kappa_2|}^{+\infty} \frac{ds}{s^2 - \left(|\kappa_2^2| - \sqrt{2\beta_2 C_2}\right)} \\ &= -\frac{1}{2\sqrt{|\kappa_2^2| - \sqrt{2\beta_2 C_2}}} \ln \frac{\sqrt{2\beta_2 C_2}}{\left(|\kappa_2| + \sqrt{|\kappa_2^2| - \sqrt{2\beta_2 C_2}}\right)^2}. \end{aligned} \tag{83}$$

If $-|\kappa_2^2| + \sqrt{2\beta_2 C_2} > 0$, that is, $k_0^2\varepsilon_2 \leqslant \gamma^2 < k_0^2\varepsilon_2 + \sqrt{2\beta_2 C_2}$, then

$$\begin{aligned} I_2 &= \int_{|\kappa_2|}^{+\infty} \frac{ds}{s^2 - |\kappa_2^2| + \sqrt{2\beta_2 C_2}} \\ &= \frac{1}{\sqrt{-|\kappa_2^2| + \sqrt{2\beta_2 C_2}}} \left(\frac{\pi}{2} \right. \\ &\quad \left. - \arctan \frac{|\kappa_2|}{\sqrt{-|\kappa_2^2| + \sqrt{2\beta_2 C_2}}}\right). \end{aligned} \tag{84}$$

Combining the found results, one arrives at the formula

$$T^* = \begin{cases} \dfrac{\pi}{\theta}, & \gamma^2 < k_0^2\varepsilon_2, \\ -\dfrac{1}{\theta} \ln \dfrac{\sqrt{2\beta_2 C_2}}{\left(|\kappa_2| + \theta\right)^2} + \dfrac{2}{\theta_1}\left(\dfrac{\pi}{2} - \arctan \dfrac{|\kappa_2|}{\theta_1}\right), & k_0^2\varepsilon_2 \leqslant \gamma^2 \leqslant k_0^2\varepsilon_2 + \sqrt{2\beta_2 C_2}, \\ -\dfrac{1}{\theta} \ln \dfrac{\sqrt{2\beta_2 C_2}}{\left(|\kappa_2| + \theta\right)^2} - \dfrac{1}{\theta_2} \ln \dfrac{\sqrt{2\beta_2 C_2}}{\left(|\kappa_2| + \theta_2\right)^2}, & \gamma^2 \geqslant k_0^2\varepsilon_2 + \sqrt{2\beta_2 C_2}, \end{cases} \tag{85}$$

where $\theta = \sqrt{|\kappa_2^2| + \sqrt{2\beta_2 C_2}}$, $\theta_1 = \sqrt{-|\kappa_2^2| + \sqrt{2\beta_2 C_2}}$, and $\theta_2 = \sqrt{|\kappa_2^2| - \sqrt{2\beta_2 C_2}}$.

Taking into account formula (77), it is clear that the inequality $\gamma^2 \geqslant k_0^2\varepsilon_2 + \sqrt{2\beta_2 C_2}$ with fixed β_2 holds only if γ belongs to a finite interval.

Taking into account the fact that for large γ the value C_2 is asymptotically equal to $(1/2)\beta_2 A^4 e^{4h_1\gamma}$, one finds that the second formula in (85) must be used for sufficiently big γ.

Further, we need the following expansions: $\ln(1+x) = x + O(x^2)$ and $\arctan(x) = x + O(x^3)$, which are valid for $|x| < 1$. Using these expansions for the second line in (85), one finds

$$\begin{aligned} T^* &= -\frac{1}{\sqrt[4]{2\beta_2 C_2}}\left(1 + \frac{|\kappa_2^2|}{\sqrt{2\beta_2 C_2}}\right)^{-1/2} \\ &\quad \cdot \left(\ln\sqrt{2\beta_2 C_2} - 2\ln\left(|\kappa_2| + \theta\right)\right) \\ &\quad + \frac{2}{\sqrt[4]{2\beta_2 C_2}}\left(1 - \frac{|\kappa_2^2|}{\sqrt{2\beta_2 C_2}}\right)^{-1/2} \end{aligned}$$

$$\cdot\left(\frac{\pi}{2}-\frac{|\kappa_2|}{\theta_1}+O\left(\left(\frac{|\kappa_2|}{\theta_1}\right)^2\right)\right)$$
$$=\frac{1}{\sqrt[4]{2\beta_2 C_2}}\left(\pi+O\left(\frac{|\kappa_2^2|}{\sqrt[4]{2\beta_2 C_2}}\right)\right). \tag{86}$$

Taking into account the fact that

$$(2\beta_2 C_2)^{-1/4}=\left(2\beta_2\cdot\frac{1}{2}\beta_2 A^4\left(1+O\left(\gamma^{-2}\right)\right)e^{4h_1\gamma}\right)^{-1/4}$$
$$=\frac{e^{-h_1\gamma}}{A\sqrt{\beta_2}}\left(1+O\left(\gamma^{-2}\right)\right), \tag{87}$$

one finally obtains

$$T^*=\frac{\pi e^{-h_1\gamma}}{A\sqrt{\beta_2}}\left(1+O\left(\gamma^{-2}\right)\right) \tag{88}$$

for sufficiently large γ.

It follows from (88) that $\lim_{\gamma^2\to\infty}T^*=0$. This means that for any $h_1,h_2>0$ there is an integer $n^*\geqslant 0$ such that, for any integer $n_2\geqslant n^*$, the DE (69) has at least one solution. This implies the existence of infinitely many $\widehat{\gamma}_i$ with an accumulation point at infinity.

Passing to the limit $\beta_2\to 0$ in the first line of (85), one obtains either (20) or (19) depending on $k_0^2\varepsilon_s<\gamma^2<k_0^2\varepsilon_1$ or $k_0^2\varepsilon_1<\gamma^2<k_0^2\varepsilon_2$. This implies item (2) of the theorem.

Multiplying (8) by y_2 and integrating over $[h_1,h_1+h_2]$, one gets the following for $\gamma^2>k_0^2\varepsilon_2$:

$$\int_{h_1}^{h_1+h_2}y_2y_2''dx=-\kappa_2^2\int_{h_1}^{h_1+h_2}y_2^2dx-\beta_2\int_{h_1}^{h_1+h_2}y_2^4dx. \tag{89}$$

This gives

$$y_2y_2'\Big|_{h_1}^{h_1+h_2}-\int_{h_1}^{h_1+h_2}y_2'^2dx$$
$$=-\kappa_2^2\int_{h_1}^{h_1+h_2}y_2^2dx-\beta_2\int_{h_1}^{h_1+h_2}y_2^4dx. \tag{90}$$

Taking into account the fact that

$$y_2(h_1)\,y_2'(h_1)$$
$$=\frac{A^2}{\widetilde{\kappa}_1}\left(\kappa_s\sinh\left(\widetilde{\kappa}_1h_1\right)+\widetilde{\kappa}_1\cosh\left(\widetilde{\kappa}_1h_1\right)\right) \tag{91}$$
$$\cdot\left(\kappa_s\cosh\left(\widetilde{\kappa}_1h_1\right)+\widetilde{\kappa}_1\sinh\left(\widetilde{\kappa}_1h_1\right)\right)>0,$$

where $\widetilde{\kappa}_1h_1>0$ and $y_2(h_1+h_2)=0$, formula (90) results in

$$\beta_2\int_{h_1}^{h_1+h_2}y_2^4dx$$
$$=\frac{A^2}{\widetilde{\kappa}_1}\left(\kappa_s\sinh\left(\widetilde{\kappa}_1h_1\right)+\widetilde{\kappa}_1\cosh\left(\widetilde{\kappa}_1h_1\right)\right)$$
$$\cdot\left(\kappa_s\cosh\left(\widetilde{\kappa}_1h_1\right)+\widetilde{\kappa}_1\sinh\left(\widetilde{\kappa}_1h_1\right)\right)+\int_{h_1}^{h_1+h_2}y_2'^2dx \tag{92}$$
$$+\left(\gamma^2-k_0^2\varepsilon_2\right)\int_{h_1}^{h_1+h_2}y_2^2dx.$$

If $\gamma\to\infty$, then, obviously, the right-hand side of this equality tends to ∞ and so does the left-hand side. This implies that $\lim_{\gamma^2\to\infty}\max_{x\in[h_1,h_1+h_2]}|y_2(x;\gamma)|=\infty$.

For $\max_{x\in[h_1,h_1+h_2]}|y_2(x;\gamma)|$ using (48), one can derive an exact formula provided that $y_2(x;\gamma)$ has at least two zeros inside the segment.

In order to derive the asymptotic estimates for $\widehat{\gamma}_i$, let us get back to formula (88). It follows from (78) and (79) that

$$n_2T^*\leqslant n_2T\leqslant\Phi_2\left(\gamma;n_2,p\right)\leqslant\left(n_2+1\right)T$$
$$\leqslant\sqrt{2}\left(n_2+1\right)T^*. \tag{93}$$

Since n_2T^* bounds h_2 from below and $\sqrt{2}(n_2+1)T^*$ bounds h_2 from above, then, solving the equations $h_2=n_2(\pi e^{-h_1\gamma}/A\sqrt{\beta_2})$ and $h_2=\sqrt{2}(n_2+1)(\pi e^{-h_1\gamma}/A\sqrt{\beta_2})$, with respect to γ, one finds (70). □

5.2. Case $\beta_1\neq 0$ and $\beta_2=0$. If $\beta_2=0$ (or $\alpha_2=0$), solutions to (8) are found elementarily. This allows one to explicitly compute the quantity p, which is

$$p=\kappa_2\cot\left(\kappa_2h_2\right). \tag{94}$$

This expression (due to its analytical dependence on γ) can be used for $\kappa_2^2\geqslant 0$ as well as for $\kappa_2^2\leqslant 0$. Note that p is a real value for all $\gamma^2>k_0^2\varepsilon_s$. Indeed, for $\gamma^2>k_0^2\varepsilon_1$, one has $\kappa_2=i\widetilde{\kappa}_2$, where $\widetilde{\kappa}_2^2=\gamma^2-k_0^2\varepsilon_2$ and then $\cos(ix)=\cosh x$ and $\sin(ix)=i\sinh x$; then

$$p=\kappa_2\frac{\cos\left(\kappa_2h_2\right)}{\sin\left(\kappa_2h_2\right)}=i\widetilde{\kappa}_2\frac{\cosh\left(\widetilde{\kappa}_2h_2\right)}{i\sinh\left(\widetilde{\kappa}_2h_2\right)}$$
$$=\widetilde{\kappa}_2\frac{\cosh\left(\widetilde{\kappa}_2h_2\right)}{\sinh\left(\widetilde{\kappa}_2h_2\right)}. \tag{95}$$

It is clear that the substitution of the explicit expression for p into the second equation of system (58) leads to the identity. Thus, in this case, system (58) reduces to the only equation

$$\Phi_1\left(\gamma;n_1,p\right)=h_1, \tag{96}$$

where Φ_1 and p are given by (46) and (94), respectively; $n_1=0,1,2,\ldots$.

The solvability of the problem $P(\beta_1,0)$ is established by the following.

Theorem 3. *For $\beta_2 = 0$, any $\beta_1 > 0$, and any fixed $A \neq 0$, the problem $P(\beta_1, 0)$ has an infinite number of PCs $\widehat{\gamma}_i$ $(i = 1, 2, \ldots)$ with the following properties:*

(1) *If $\widehat{\gamma}_i$ is the solution to $P(\beta_1, 0)$, then $\widehat{\gamma}_i^2 \in (k_0^2\varepsilon_s, +\infty)$ and $\lim_{i\to\infty}\widehat{\gamma}_i^2 = \infty$.*

(2) *If the linear problem $P(0, 0)$ has q solutions $\widetilde{\gamma}_1 < \widetilde{\gamma}_2 < \cdots < \widetilde{\gamma}_q$, then there exists a constant $\beta_0 > 0$ such that, for any $\beta_1 = \beta_1' < \beta_0$, it is true that $\widehat{\gamma}_i^2 \in (k_0^2\varepsilon_s, k_0^2\varepsilon_2)$ and $\lim_{\beta_1'\to 0}\widehat{\gamma}_i = \widetilde{\gamma}_i$ $(i = \overline{1, q})$, where $\widehat{\gamma}_1, \widehat{\gamma}_2, \ldots, \widehat{\gamma}_q$ are first q solutions to $P(\beta_1', 0)$.*

(3) *If $s > q$, then $\lim_{\beta_1'\to +0}\widehat{\gamma}_s = +\infty$.*

(4) *If $\widehat{\gamma}_i \to \infty$, then $\max_{x\in(0,h_1)}|y_2(x, \widehat{\gamma}_i)| \to \infty$.*

(5) *For big $\widehat{\gamma}$ and arbitrary small $\Delta > 0$, the asymptotic two-sided inequality*

$$(1 - \Delta)\,\gamma_\bullet(m) \leqslant \widehat{\gamma}(m) \leqslant (1 + \Delta)\,\gamma_\bullet\left(\sqrt{2}\,(m + 1)\right) \quad (97)$$

is valid, where $\gamma_\bullet^2(m) = \varepsilon_1 + [f^{-1}(h/4m)]^2$ and f^{-1} is the inversion of $f(t) = t^{-1}\ln t$.

Proof. The constant C_1 in (96) is defined explicitly by (32).

Using (16) at the point $x = h_1$, one obtains

$$y_2(h_1) = -B\sin(\kappa_2 h_2),$$
$$y_2'(h_1) = B\kappa_2\cos(\kappa_2 h_2). \quad (98)$$

Substituting these values into (31) and using (32), one obtains the biquadratic equation

$$B^4 + 2\frac{\kappa_1^2\sin^2(\kappa_2 h_2) + \kappa_2^2\cos^2(\kappa_2 h_2)}{\beta_1\sin^4(\kappa_2 h_2)}B^2 - \frac{2k_0^2(\varepsilon_1 - \varepsilon_s) + \beta_1 A^2}{\beta_1\sin^4(\kappa_2 h_2)}A^2 = 0. \quad (99)$$

Using the same estimations as in the proof of Theorem 2 and evaluating the integral contained in (96), one finds

$$n_1 T^* \leqslant n_1 T \leqslant \Phi_1(\gamma; n_1, p) \leqslant (n_1 + 1)\,T \leqslant \sqrt{2}\,(n_1 + 1)\,T^*, \quad (100)$$

where $T^* = \int_{-\infty}^{+\infty}(d\eta/(|\eta^2 + \kappa_1^2| + \sqrt{2\beta_1 C_1}))$.

Integral T^* is computed in the same way as in the proof of Theorem 2; the result has the form

$$T^* = \begin{cases} \dfrac{\pi}{\theta}, & \gamma^2 < k_0^2\varepsilon_1, \\ -\dfrac{1}{\theta}\ln\dfrac{\sqrt{2\beta_1 C_1}}{(|\kappa_1| + \theta)^2} + \dfrac{2}{\theta_1}\left(\dfrac{\pi}{2} - \arctan\dfrac{|\kappa_1|}{\theta_1}\right), & k_0^2\varepsilon_1 \leqslant \gamma^2 \leqslant k_0^2\varepsilon_1 + \sqrt{2\beta_1 C_1}, \\ -\dfrac{1}{\theta}\ln\dfrac{\sqrt{2\beta_1 C_1}}{(|\kappa_1| + \theta)^2} - \dfrac{1}{\theta_2}\ln\dfrac{\sqrt{2\beta_1 C_1}}{(|\kappa_1| + \theta_2)^2}, & \gamma^2 \geqslant k_0^2\varepsilon_1 + \sqrt{2\beta_1 C_1}, \end{cases} \quad (101)$$

where $\theta = \sqrt{|\kappa_1^2| + \sqrt{2\beta_1 C_1}}$, $\theta_1 = \sqrt{-|\kappa_1^2| + \sqrt{2\beta_1 C_1}}$, and $\theta_2 = \sqrt{|\kappa_1^2| - \sqrt{2\beta_1 C_1}}$.

In order to derive the asymptotic estimates for the eigenvalues $\widehat{\gamma}_i$, let us get back to formula (101). Since C_1 is a (fixed) constant [it does not depend on γ; see formula (32)], the third line in (101) corresponds to sufficiently big γ. Thus, for sufficiently big γ, the third line of (101) gives the asymptotics

$$T^* = 4\frac{\ln\gamma}{\gamma} + O\left(\gamma^{-1}\right). \quad (102)$$

It follows from (102) that $\lim_{\gamma^2\to\infty}T^* = 0$. This implies that for any prescribed $h_1, h_2 > 0$ there is an integer $n^* \geqslant 0$ such that (96) has at least one solution for any integer $n_1 \geqslant n^*$. Thus, there exists an infinite number of positive eigenvalues $\widehat{\gamma}_i$ with accumulation point at infinity.

Passing to the limit $\beta_1 \to +0$ in the first line of (101) gives either (20) or (19) depending on $k_0^2\varepsilon_s < \gamma^2 < k_0^2\varepsilon_1$ or $k_0^2\varepsilon_1 < \gamma^2 < k_0^2\varepsilon_2$. This implies item (2) of the theorem.

Item (3) of the theorem results from the behaviour of the second and third lines in (101) as $\beta_1 \to +0$.

Multiplying (7) by y_1 and integrating over $[0, h_1]$, one obtains

$$y_2(h_1)\,y_2'(h_1) - \kappa_s A^2 + \int_0^{h_1} y_1'^2 dx = \kappa_1^2\int_0^{h_1} y_1^2 dx + \beta_1\int_0^{h_1} y_1^4 dx. \quad (103)$$

It follows from (103) that $\int_0^{h_1} y_1^4 dx \to \infty$ as $\gamma \to \infty$. This implies item (4) of the theorem.

The asymptotic estimates in the theorem result from formula (102). □

6. Numerical Results

Figures 2–13 show results of numerical experiments. The following parameters are used: $A = 1$ [**E**] and $\varepsilon_s = 1$; $\varepsilon_1 = 2$ and $\varepsilon_2 = 2.5$ (Figures 2–9); $\varepsilon_1 = 5$ and $\varepsilon_2 = 10$ (Figures 10 and 11); $\varepsilon_1 = 5$ and $\varepsilon_2 = 5.01$ (Figures 12 and 13); other parameters are specified in the captions; $\omega = 2\pi f/c$, where c is the speed of light.

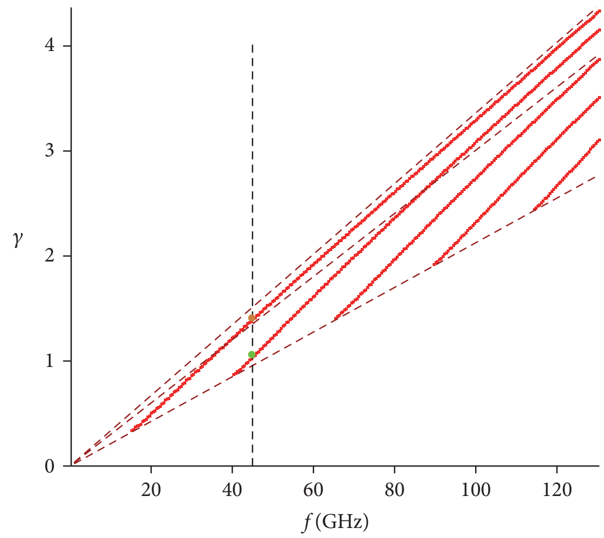


FIGURE 2: DCs for the linear case: $h_1 = 1$ and $h_2 = 4$ [mm]; vertical dashed line $f = 45$ [GHz]; $\widetilde{\gamma}_1 \approx 1.0170$ (green dot) and $\widetilde{\gamma}_2 \approx 1.3709$ (brown dot).

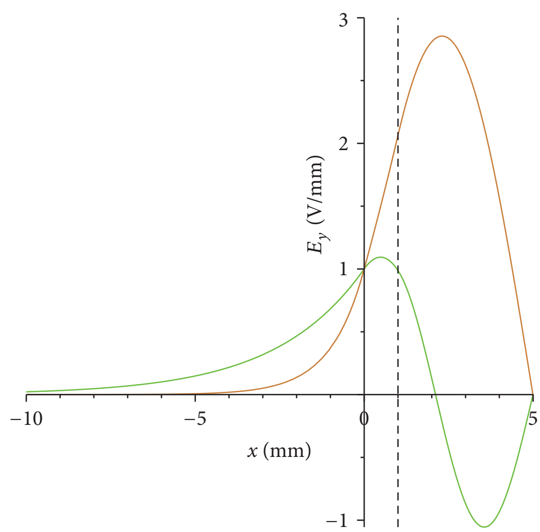


FIGURE 3: Eigenmodes $E_y(x)$ for the PCs marked in Figure 2: $\widetilde{\gamma}_1 \approx 1.0170$ (green) and $\widetilde{\gamma}_2 \approx 1.3709$ (brown).

In Figures 2, 4, 6, 8, 10, and 12(a), the dispersion curves (DCs) are plotted (γ versus f). The DCs for the linear and nonlinear cases are given in red and blue colours, respectively. Dashed lines in Figure 2 correspond to the boundaries for γ. Points of intersections of the vertical dashed line with DCs are PCs.

Dashed lines in Figures 3, 5, 7, and 9 correspond to the boundary between the layers.

Figures 2 and 3 correspond to the linear problem ($\alpha_1 = \alpha_2 = 0$). For chosen frequency, there are 2 PCs; eigenmodes for them are shown in Figure 3.

The dependence γ versus A, where $A = E_y(0)$, is shown in Figures 2, 4, 6, 8, 10, and 12(b). The linear and nonlinear cases are given in red and blue colours, respectively. Points of intersections of the vertical dashed line (corresponding to $A = 1$) with the curves are PCs. Obviously, in the linear case, the PCs do not depend on A and therefore linear solutions are horizontal (red) lines.

There are two types of PCs for the nonlinear case. The first type corresponds to the PCs, which are close to the corresponding linear solutions (in the linear limit, they reduce to the linear solutions; see item (2) in Theorems 2 and 3). The second type corresponds to the PCs, which present a novel guiding regime (they do not reduce to any linear solutions in the linear limit; see item (3) in Theorem 3). In the latter case, we call them "purely nonlinear" PCs; see also [23–25]. In Figures 4, 6, and 8, the PC $\widehat{\gamma}_1$ marked with a blue dot corresponds to the first type; the PCs $\widehat{\gamma}_2$ and $\widehat{\gamma}_3$ marked with green and brown dots, respectively, correspond to the second type.

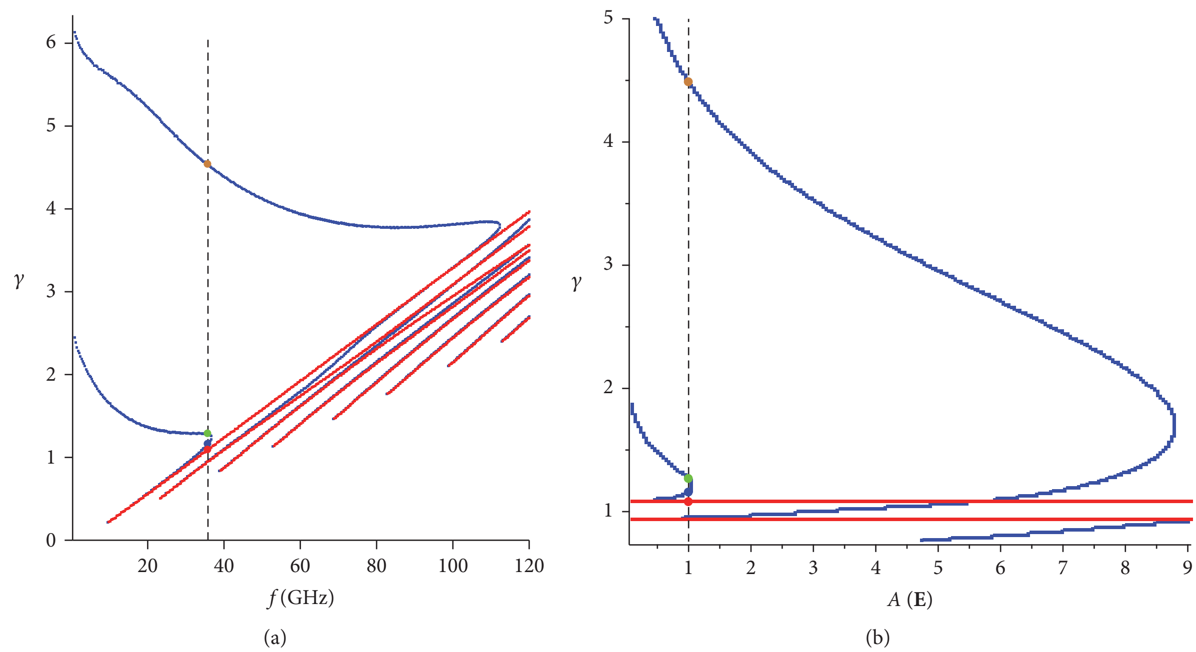


FIGURE 4: (a) DCs for nonlinear (blue) and linear (red) cases: $\alpha_1 = 0.02$ [$\mathbf{E}^{-2}$], $\alpha_2 = 0$, $h_1 = 5$, $h_2 = 4$ [mm], and $E_y(0) = 1$ [**E**]; vertical dashed line $f = 36.1$ [GHz]; nonlinear PCs: $\widehat{\gamma}_1 \approx 1.1509$ (blue dot), $\widehat{\gamma}_2 \approx 1.2701$ (green dot), and $\widehat{\gamma}_3 \approx 4.5003$ (brown dot), respectively; linear PC: $\widetilde{\gamma}_2 \approx 1.0860$ (red dot). (b) The dependence γ versus A for nonlinear (blue) and linear (red) cases: $\alpha_1 = 0.02$ [$\mathbf{E}^{-2}$], $\alpha_2 = 0$, $h_1 = 5$, $h_2 = 4$ [mm], and $f = 36.1$ [GHz]; vertical dashed line $E_y(0) = 1$ [**E**]; nonlinear PCs: $\widehat{\gamma}_1 \approx 1.1509$ (blue dot), $\widehat{\gamma}_2 \approx 1.2701$ (green dot), and $\widehat{\gamma}_3 \approx 4.5003$ (brown dot), respectively; linear PC: $\widetilde{\gamma}_2 \approx 1.0860$ (red dot).

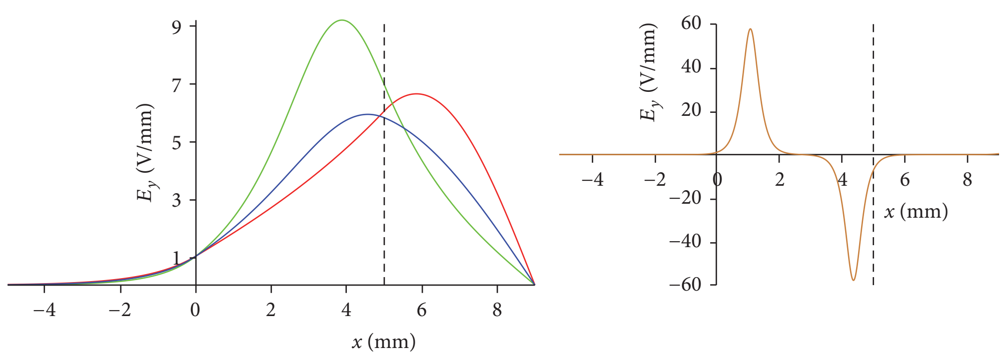


FIGURE 5: Eigenmodes $E_y(x)$ for the PCs marked in Figure 4: $\widetilde{\gamma}_2 \approx 1.0860$ (red, linear case), $\widehat{\gamma}_1 \approx 1.1509$ (blue), $\widehat{\gamma}_2 \approx 1.2701$ (green), and $\widehat{\gamma}_3 \approx 4.5003$ (brown).

In Figures 6 and 8, for any frequency, there are infinitely many PCs in the nonlinear cases (few of them are shown and marked in the figures) and only 2 PCs in the linear case.

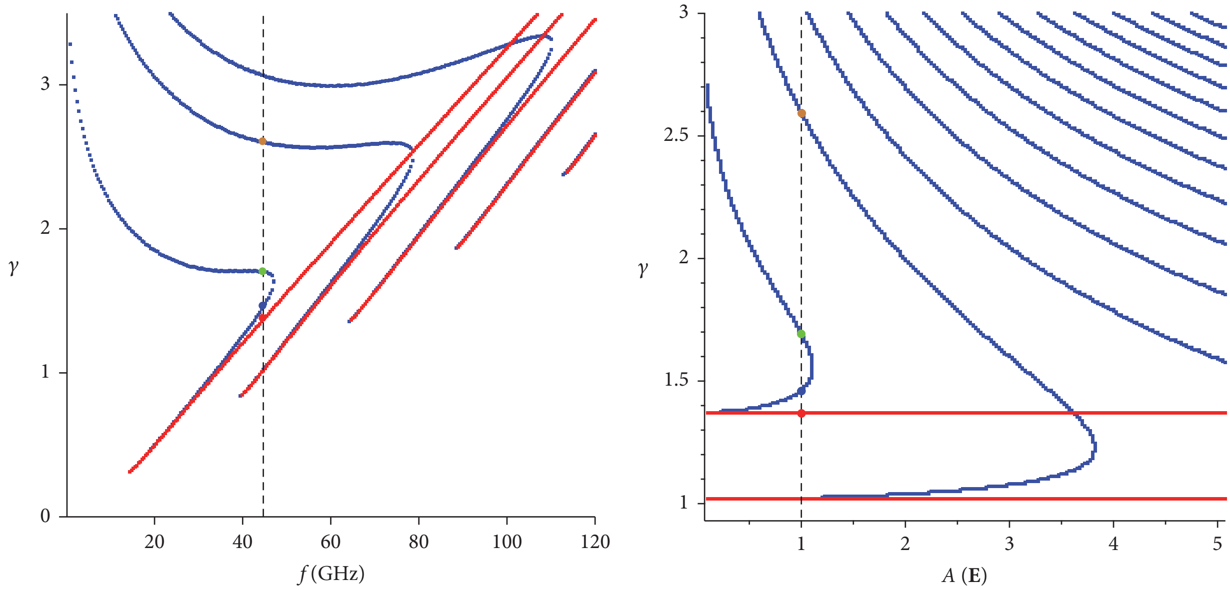
DCs for the linear cases (red DCs) presented in Figures 6 and 8 coincide with the DCs for the linear case presented in Figure 2.

Figures 5, 7, and 9 allow one to compare linear and nonlinear modes. It can be seen from these pictures that, for a "linear" PC and for a "nonlinear" one (which reduces to the "linear" one in the linear limit), eigenmodes are also close. Obviously, for such a nonlinear PC, perturbation methods can be applied. However, two other nonlinear eigenmodes (shown in green and brown colours) are not close to any linear solution and they do not reduce to linear solutions in the linear limit; these very eigenmodes we call purely nonlinear.

In Figures 8–13, the nonlinear case for different sets of α_1, α_2 is shown, where both coefficients α_1 and α_2 are nonzero. In this case, we did not prove the existence of infinitely many PCs; however, these figures are similar to corresponding figures, where α_1 or α_2 equals zero.

7. Conclusion

The paper focuses on the problem of wave propagation in a plane layered dielectric waveguide filled with Kerr medium. The existence of guided modes that have linear counterparts



(a) (b)

Figure 6: (a) DCs for nonlinear (blue) and linear (red) cases: $\alpha_1 = 0$, $\alpha_2 = 0.03\,[\mathbf{E}^{-2}]$, $h_1 = 1\,[\text{mm}]$, $h_2 = 4\,[\text{mm}]$, and $E_y(0) = 1\,[\mathbf{E}]$; vertical dashed line $f = 45\,[\text{GHz}]$; nonlinear PCs: $\hat{\gamma}_1 \approx 1.4625$ (blue dot), $\hat{\gamma}_2 \approx 1.7012$ (green dot), and $\hat{\gamma}_3 \approx 2.6038$ (brown dot), respectively. The linear case coincides with one presented in Figure 2. (b) The dependence γ versus A for nonlinear (blue) and linear (red) cases: $\alpha_1 = 0$, $\alpha_2 = 0.03\,[\mathbf{E}^{-2}]$, $h_1 = 1\,[\text{mm}]$, $h_2 = 4\,[\text{mm}]$, and $f = 45\,[\text{GHz}]$; vertical dashed line $E_y(0) = 1\,[\mathbf{E}]$; nonlinear PCs: $\hat{\gamma}_1 \approx 1.4625$ (blue dot), $\hat{\gamma}_2 \approx 1.7012$ (green dot), and $\hat{\gamma}_3 \approx 2.6038$ (brown dot), respectively; linear PC: $\tilde{\gamma}_2 \approx 1.3709$ (red dot).

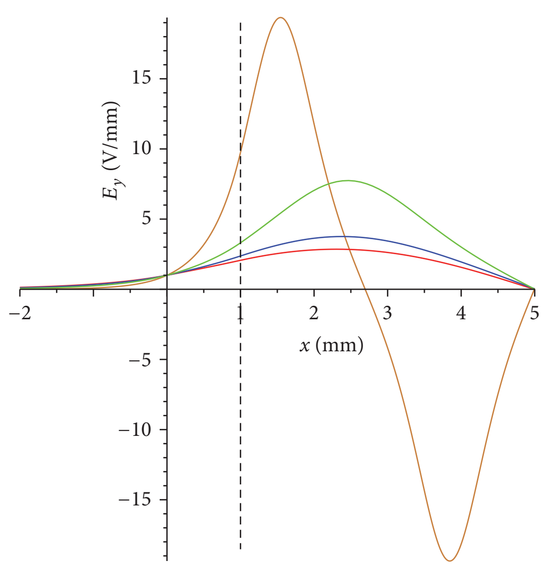


Figure 7: Eigenmodes $E_y(x)$: $\tilde{\gamma}_2 \approx 1.3709$ (red, linear case), $\hat{\gamma}_1 \approx 1.4625$ (blue), $\hat{\gamma}_2 \approx 1.7012$ (green), and $\hat{\gamma}_3 \approx 2.6038$ (brown).

and guided modes that do not have linear counterparts is proven. The latter waves correspond to a novel guided regime. Since the Kerr nonlinearity is widely studied in nonlinear optics (see, e.g., [7–14]), the results found here can be interesting and important from both theoretical and applied points of view.

It is worth noting that similar results have been found for some other cases. Indeed, it was proven lately that, even in a simpler case of a one-layer waveguide, the Kerr nonlinearity results in the existence of novel guided regimes as well [23–25]. Moreover, similar results have recently been found in the case of polynomial nonlinearity [39]. Thus, the existence of infinitely many nonperturbative PCs is a general feature of polynomial (nonlinear) permittivities with positive terms.

One of the most known peculiarities of nonlinear guided waves is power-dependent PCs; this is clearly shown in Figures 2, 4, 6, 8, 10, and 12(b). It is interesting that varying the value A, one strongly affects the corresponding PC and

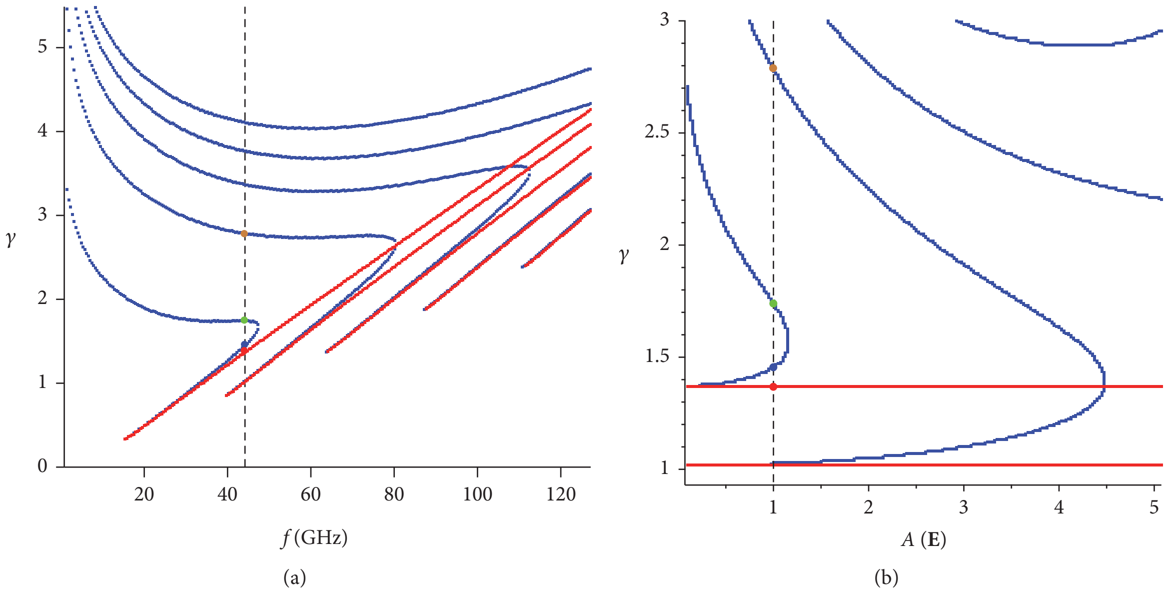


FIGURE 8: (a) DCs for nonlinear (blue) and linear (red) cases: $\alpha_1 = 0.02$, $\alpha_2 = 0.03\,[\mathbf{E}^{-2}]$, $h_1 = 1$, $h_2 = 4\,[\text{mm}]$, and $E_y(0) = 1\,[\mathbf{E}]$; black dashed line $f = 45\,[\text{GHz}]$; nonlinear PCs: $\widehat{\gamma}_1 \approx 1.4548$ (blue dot), $\widehat{\gamma}_2 \approx 1.7476$ (green dot), and $\widehat{\gamma}_3 \approx 2.7896$ (brown dot), respectively. The linear case coincides with one presented in Figure 2. (b) The dependence γ versus A for nonlinear (blue) and linear (red) cases: $\alpha_1 = 0.02$, $\alpha_2 = 0.03\,[\mathbf{E}^{-2}]$, $h_1 = 1$, $h_2 = 4\,[\text{mm}]$, and $f = 45\,[\text{GHz}]$; black dashed line $E_y(0) = 1\,[\mathbf{E}]$; nonlinear PCs: $\widehat{\gamma}_1 \approx 1.4548$ (blue dot), $\widehat{\gamma}_2 \approx 1.7476$ (green dot), and $\widehat{\gamma}_3 \approx 2.7896$ (brown dot), respectively; linear PC: $\widetilde{\gamma}_2 \approx 1.3709$ (red dot).

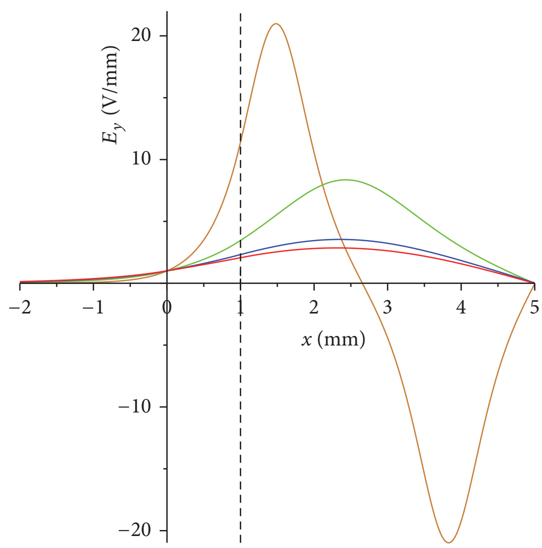


FIGURE 9: Eigenmodes $E_y(x)$ for the PCs marked in Figure 8: $\widetilde{\gamma}_2 \approx 1.3709$ (red, linear case), $\widehat{\gamma}_1 \approx 1.4548$ (blue), $\widehat{\gamma}_2 \approx 1.7476$ (green), and $\widehat{\gamma}_3 \approx 2.7896$ (brown).

therefore the eigenmode. The fact that the power-dependent PCs can have some potential for optical signal processing is pointed out in many papers (see, e.g., [28–34]). For additional description of applications of nonlinear guided waves and Kerr effect, see [32, 34], where many useful references are also given.

The result given in the paper clearly shows that nonlinear problems can have solutions that cannot be considered as perturbations of solutions of corresponding linear problems. Thus, it is necessary to be careful when one linearizes a nonlinear problem and considers the linearized problem without proving that there are no other solutions.

Of course, it is impossible to expect the existence of purely nonlinear waves for infinitely many PCs. However, it is possible that purely nonlinear waves can be observed in an experiment for some first purely nonlinear PCs. Indeed, as is seen from Figures 5, 7, and 9, max values of few first purely nonlinear eigenmodes are not too big (in comparison with the linear eigenmode). This probably gives an opportunity to observe such waves in an experiment. For the rest of the PCs,

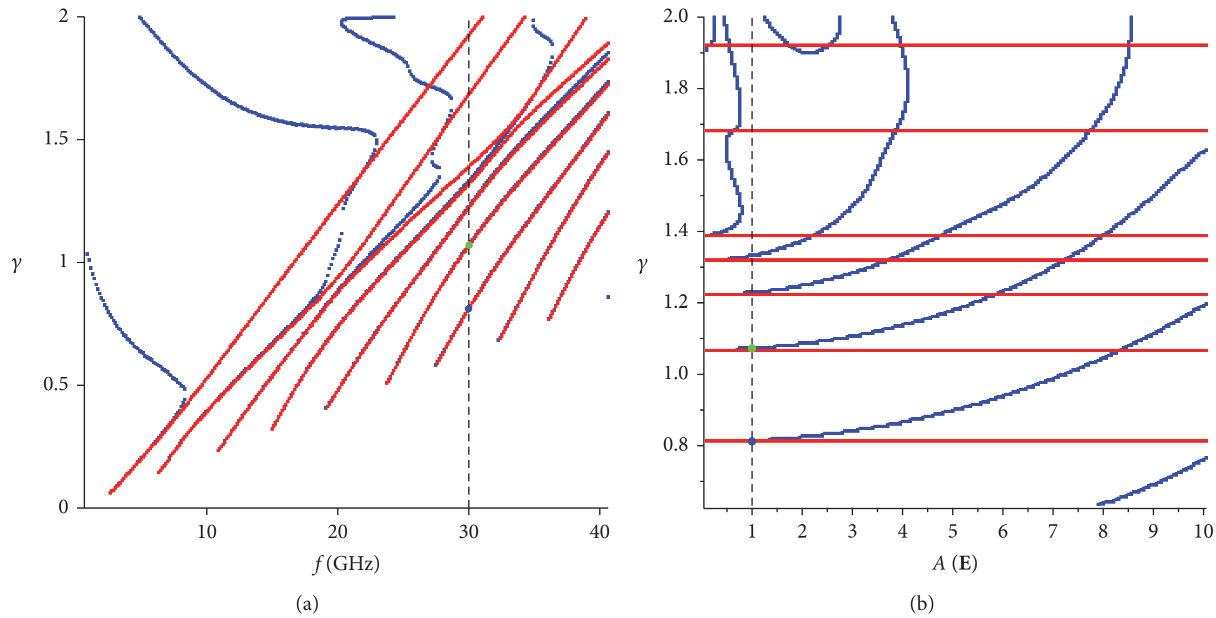


Figure 10: (a) DCs for nonlinear (blue) and linear (red) cases: $\alpha_1 = 0.02$, $\alpha_2 = 0.01\,[\mathbf{E}^{-2}]$, $h_1 = 10$, $h_2 = 5\,[\text{mm}]$, and $E_y(0) = 1\,[\mathbf{E}]$; black dashed line $f = 30.2\,[\text{GHz}]$; nonlinear PCs: $\hat{\gamma}_1 \approx 0.8145$ (blue dot) and $\hat{\gamma}_2 \approx 1.0723$ (green dot), respectively. (b) The dependence γ versus A for nonlinear (blue) and linear (red) cases: $\alpha_1 = 0.02$, $\alpha_2 = 0.01\,[\mathbf{E}^{-2}]$, $h_1 = 10$, $h_2 = 5\,[\text{mm}]$, and $f = 30.2\,[\text{GHz}]$; black dashed line $E_y(0) = 1\,[\mathbf{E}]$; nonlinear PCs: $\hat{\gamma}_1 \approx 0.8145$ (blue dot) and $\hat{\gamma}_2 \approx 1.0723$ (green dot), respectively.

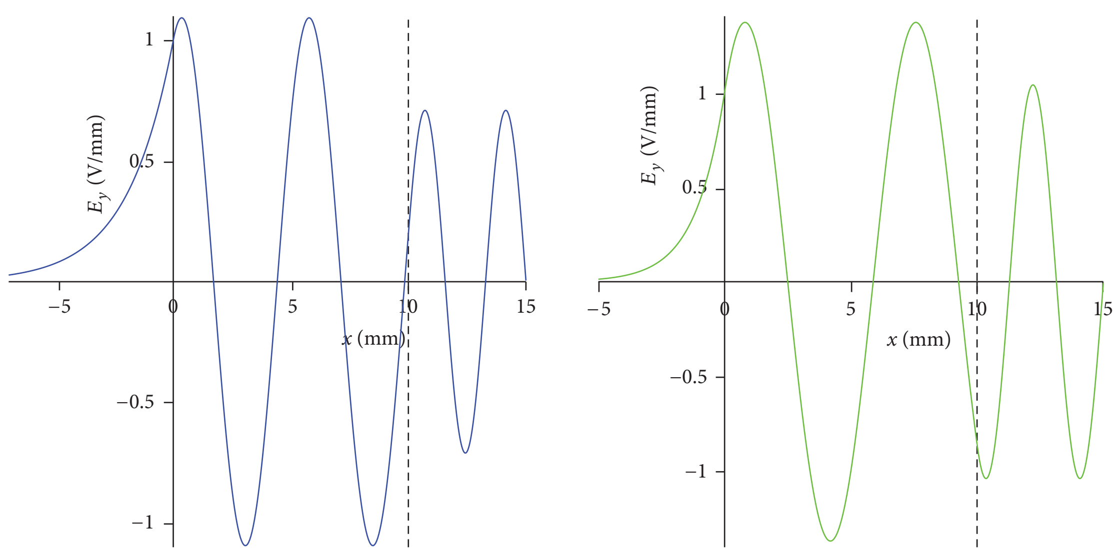


Figure 11: Eigenmodes $E_y(x)$ for the PCs marked in Figure 10: $\hat{\gamma}_1 \approx 0.8145$ (blue) and $\hat{\gamma}_2 \approx 1.0723$ (green).

the value $\max_{x\in[0,h_1+h_2]}|E_y(x;\hat{\gamma}_i)|$ is so high that the Kerr law is no longer valid.

As a matter of fact, different types of nonlinearities admit nonlinear solutions that become linear ones in the linear limit. For example, for a wide range of saturated nonlinearities, there exist only a finite number of PCs [40, 41]. Thus, there is a qualitative difference between saturated and unbounded (Kerr, qubic-quintic-septic, and, more generally, power and polynomial) nonlinearities. It seems that this difference can be used in order to understand what kinds of nonlinear permittivities are closer to real situations.

If the purely nonlinear guided modes are confirmed by experiment, the theory of nonlinear guided wave propagation will definitely advance. If they are not observed in experiments, then well-known and widespread formulas for nonlinear permittivities must be changed so that mathematical analysis of these models can give results that better satisfy reality.

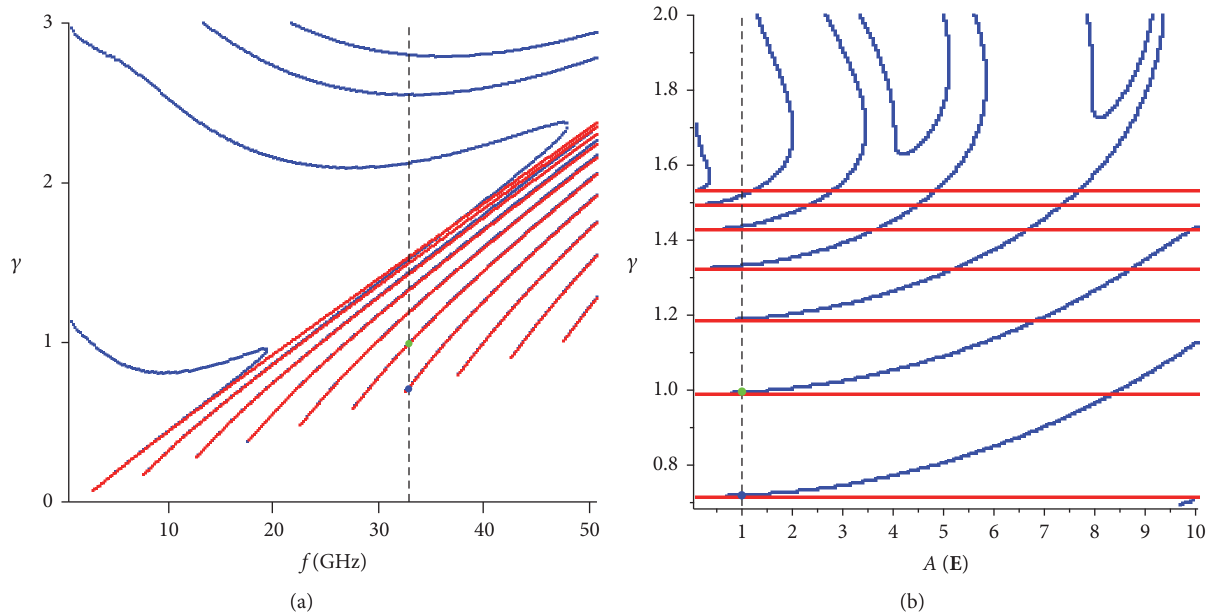


FIGURE 12: (a) DCs for nonlinear (blue) and linear (red) cases: $\alpha_1 = 0.02$, $\alpha_2 = 0.01$ [$\mathbf{E}^{-2}$], $h_1 = 10$, $h_2 = 5$ [mm], and $E_y(0) = 1$ [**E**]; black dashed line $f = 33$ [GHz]; nonlinear PCs: $\hat{\gamma}_1 \approx 0.7189$ (blue dot) and $\hat{\gamma}_2 \approx 0.9937$ (green dot), respectively. (b) The dependence γ versus A for nonlinear (blue) and linear (red) cases: $\alpha_1 = 0.02$, $\alpha_2 = 0.01$ [$\mathbf{E}^{-2}$], $h_1 = 10$, $h_2 = 5$ [mm], and $E_y(0) = 1$ [**E**]; black dashed line $f = 33$ [GHz]; nonlinear PCs: $\hat{\gamma}_1 \approx 0.7189$ (blue dot) and $\hat{\gamma}_2 \approx 0.9937$ (green dot), respectively.

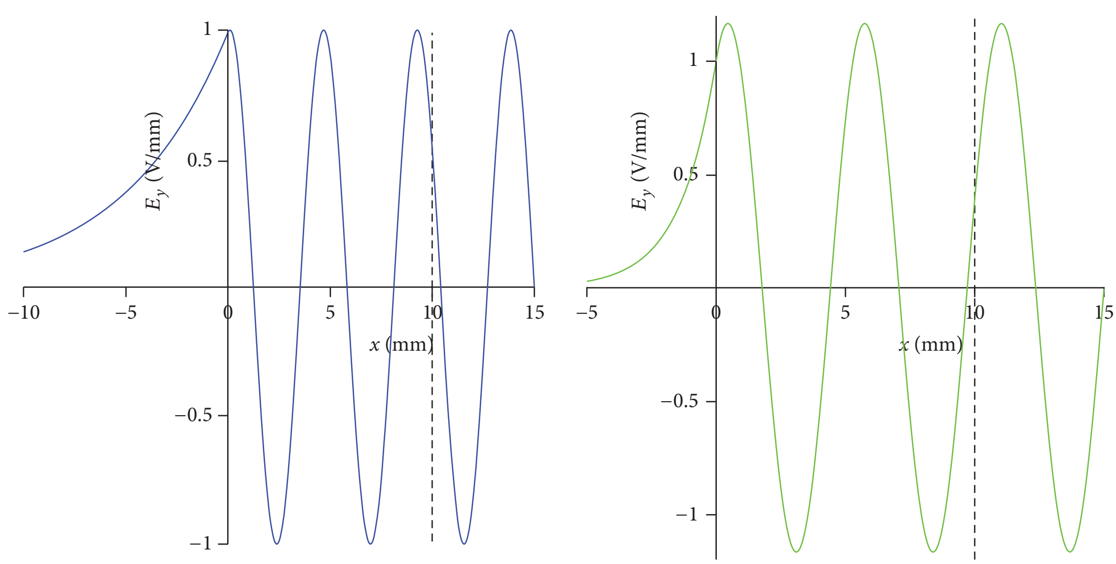


FIGURE 13: Eigenmodes $E_y(x)$ for the PCs marked in Figure 12: $\hat{\gamma}_1 \approx 0.7189$ (blue) and $\hat{\gamma}_2 \approx 0.9937$ (green).

In spite of the fact that in some papers the nonlinear eigenmodes are searched in an explicit form (see, e.g., [7, 16, 17, 20–22, 34]), this way is not an appropriate one. Indeed, on one hand, explicit solutions to nonlinear equations are often complicated special functions (if it is possible to find them at all) and, therefore, it is almost impossible to study such solutions. On the other hand, many properties/characteristics of eigenwaves can be calculated from original differential equations and boundary conditions, like it is done in this paper. In order to check calculations, one solves numerically the DE (58) with respect to γ in a prescribed segment; then, for each found PC, one solves the Cauchy problem for (7) and (8) with initial data at $x = 0$ and transmission conditions (11) at $x = h_1$. In this case, the second condition (10) is fulfilled indispensably (obviously, in numerical calculations, $y_2(h_1 + h_2)$ cannot be exactly zero). Thus, Theorems 1, 2, and 3 together with computations allow one to find all nonlinear eigenwaves for which the PCs are calculated.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This study is supported by the Russian Foundation for Basic Research (Grant no. 15-01-00206), the Russian Federation President Grant (Project no. MK-4684.2016.1), and the Ministry of Education and Science of the Russian Federation (Agreement no. 1.894.2017/4.6).

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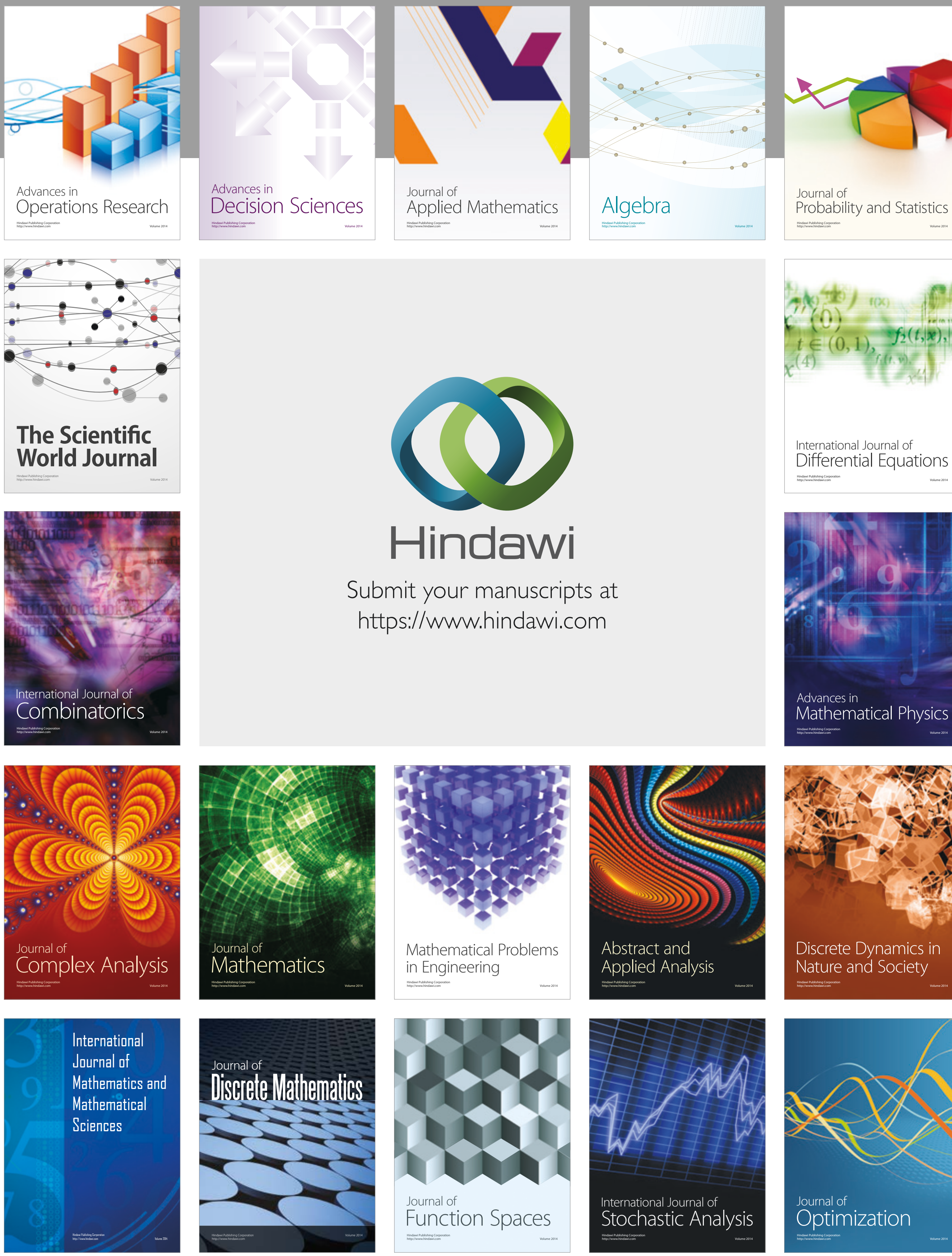
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