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Research Article

A Structural Property of Trees with an Application to Vertex-Arboricity

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We provide a structural property of trees, which is applied to show that if a plane graph G contains two edge-disjoint spanning trees, then its dual graph G^* has the vertex-arboricity at most 2. We also show that every maximal plane graph of order at least 4 contains two edge-disjoint spanning trees.

1. Introduction

All graphs considered in this paper are finite simple graphs. Given graph G, let $V(G)$, $E(G)$, $|G|$, and $\|G\|$ denote its vertex set, edge set, vertex number, and edge number, respectively. For a vertex $v \in V(G)$, let $d_G(v)$ denote the degree of v in G. Moreover, let $\Delta(G)$ and $\delta(G)$ denote the maximum degree and minimum degree of G, respectively. A *tree* is a connected graph without cycles. A *plane* graph is a particular drawing of a planar graph on the Euclidean plane.

A subgraph H of G is called a *spanning* one if $V(H) = V(G)$. A plane graph G is called *maximal* if every face of G is a triangle. A connected *Eulerian* graph is a connected one that contains no vertices of degree odd. The *dual*, denoted by G^*, of a plane graph G is a plane graph whose vertices correspond to the faces of G and edges correspond to the edges of G in this way: if e is an edge of G incident to faces f_1, f_2, then the endpoints of the dual edge $e^* \in E(G^*)$ are vertices v_1, v_2 of G^* that represent the faces of f_1, f_2 of G. The *vertex-arboricity* $a(G)$ of a graph G is the minimum number of subsets into which $V(G)$ can be partitioned so that each subset induces a forest.

The vertex-arboricity of a graph was first introduced by Chartrand et al. [1], named as *point-arboricity*. Among other things, they proved that the vertex-arboricity of planar graphs is at most 3. Chartrand and Kronk [2] showed that this bound is sharp by presenting a planar graph of the vertex-arboricity 3. More generally, Kronk [3] showed that if S is a surface with Euler genus g, then $a(S) = 3$ and if S is the sphere or the Klein bottle, then $a(S) = \lfloor (9 + \sqrt{1 + 24g})/4 \rfloor$. Hara et al. [4] extended partially Kronk's result by proving that $la(S) = 3$ if S is the projective plane or the torus, and $la(S) \leq 4$ if S is the Klein bottle. Here the vertex-arboricity of surface S is defined to be the maximum of the vertex-arboricity of all graphs embeddable into S. Other results about the vertex-arboricity of embedded graphs in the surface are referred to in [5–8].

The following theorem, due to Stein [9], characterizes completely maximal plane graphs with vertex-arboricity 2.

Theorem 1. *Let G be a maximal plane graph with* $|G| \geq 4$. *Then* $a(G) = 2$ *if and only if* G^* *is Hamiltonian.*

Hakimi and Schmeichel [10] extended Stein's theorem to the following form.

Theorem 2. *Let G be a plane graph. Then* $a(G) = 2$ *if and only if* G^* *contains a connected Eulerian spanning subgraph.*

Note that determining whether a graph to have a connected Eulerian spanning subgraph is quite difficult. Thus a natural question is as follows: which plane graphs have a connected Eulerian spanning subgraph? In this paper, we provide a sufficient condition about the problem.

2. Structural Property

Let $P = (x, y)$ denote a path from vertex x to vertex y. We call a vertex of degree 1 a *leaf* in a tree. The following lemma is of interest by itself.

Theorem 3. *Let T be a tree with $|T| \geq 2$ and $n \geq 1$ be an integer. If $S \subseteq V(T)$ with $|S| = 2n \leq |T|$, then S can be partitioned into two sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ such that T contains n edge-disjoint paths $P_i = (x_i, y_i)$, $i = 1, 2, \ldots, n$.*

Proof. The proof is proceeded by induction on the order of T. If $2 \leq |T| \leq 3$, then T is a path of length 1 or 2. Note that $n = 1$ in this case, so the theorem holds trivially. Suppose that T is a tree with $|T| \geq 4$. Let S be a subset of $V(T)$ with $|S| = 2n \leq |T|$. If T contains a leaf v which is not in S, then $|S| \leq |T| - 1$. Let $T' = T - v$. Then T' is a tree of order $|T| - 1$ and $S \subseteq V(T')$ satisfies $|S| = 2n \leq |T| - 1 = |T'|$. By the induction hypothesis, S can be partitioned into two sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ such that T' contains n edge-disjoint paths $P_i = (x_i, y_i)$, $i = 1, 2, \ldots, n$. Clearly, $S = X \cup Y$ also is a required partition of T.

So assume that S contains all the leaves of T. If there exist two leaves x and y adjacent to a common vertex u, we let $T' = T - \{x, y\}$ and $S' = S \setminus \{x, y\}$. Then T' is a tree of order $|T| - 2$ and $S' \subseteq V(T')$ satisfies $|S'| = |S| - 2 = 2n - 2 \leq |T| - 2 = |T'|$. By the induction hypothesis, S' can be partitioned into two sets $X' = \{x_1, x_2, \ldots, x_{n-1}\}$ and $Y' = \{y_1, y_2, \ldots, y_{n-1}\}$ such that T' contains $n - 1$ edge-disjoint paths $P_i' = (x_i, y_i)$, $i = 1, 2, \ldots, n-1$. Defining $P_n = xuy$, $P_i = P_i'$ for $i = 1, 2, \ldots, n-1$, $X = X' \cup \{x\}$, and $Y = Y' \cup \{y\}$, we obtain a desired partition $S = X \cup Y$ of T.

Now suppose that every vertex of T is adjacent to at most one leaf. Let $Q = v_1 v_2 v_3 \cdots v_k$ be a longest path in T. Then it is easy to see that $k \geq 4$ and v_1, v_k are leaves. If v_2 is adjacent to some vertex, u_1, different from v_1 and v_3, then u_1 is not a leaf by the assumption. There exists a vertex u_2, other than v_2, adjacent to u_1. However, $Q' = u_2 u_1 v_2 v_3 \cdots v_k$ is a path whose length is greater than that of Q, contradicting the choice of Q. Hence it follows that $d_T(v_2) = 2$. Similarly, we derive that $d_T(v_{k-1}) = 2$.

If $v_2 \in S$, we define $T' = T - \{v_1, v_2\}$ and $S' = S \setminus \{v_1, v_2\}$. Then $S' \subseteq V(T')$ satisfies $|S'| = |S| - 2 = 2n - 2 \leq |T| - 2 = |T'|$. By the induction hypothesis, S' admits a required partition $X' \cup Y'$ with $|X'| = |Y'| = n - 1$ so that T' contains $n - 1$ edge-disjoint paths $P_1, P_2, \ldots, P_{n-1}$. In T, we define $P_n = v_1 v_2$, $X = X' \cup \{v_1\}$, and $Y = Y' \cup \{v_2\}$.

If $v_2 \notin S$, then $|S| \leq |T| - 1$. Let $T' = T - v_2 + v_1 v_3$ and $S' = S$. Then T' is a tree of order $|T| - 1$ and $S' \subseteq V(T')$ with $|S'| = 2n \leq |T| - 1 = |T'|$. By the induction hypothesis, S' can be partitioned into two sets $X' = \{x_1, x_2, \ldots, x_n\}$ and $Y' = \{y_1, y_2, \ldots, y_n\}$ such that T' contains n edge-disjoint paths $P_i' = (x_i, y_i)$, $i = 1, 2, \ldots, n$. Since v_1 is a leaf of T', v_1 is an end of some path P_{i^0}'. Furthermore, $v_1 v_3 \in E(P_{i^0}')$. Without loss of generality, suppose that $i^0 = n$ and $P_n = v_1 v_3 z_1 z_2 \cdots z_m$. In T, we let $X = X'$, $Y = Y'$, $P_n = v_1 v_2 v_3 z_1 z_2 \cdots z_m$, and $P_i = P_i'$ for $i = 1, 2, \ldots, n-1$. A desired partition of S in T is established. This proves the theorem. □

By means of Theorem 3, we can give a simple proof for the following result.

Theorem 4. *Let G be a graph with $|G| \geq 4$. If G contains two edge-disjoint spanning trees, then G contains a connected Eulerian spanning subgraph.*

Proof. Suppose that T_1 and T_2 are two edge-disjoint spanning trees of G. Then $V(T_1) = V(T_2) = V(G)$. Let S^* denote the subset of vertices of degree odd in T_1. Then $|S^*|$ is even. Since T_1 contains at least two leaves, $2 \leq |S^*| = 2n \leq |T_1| = |T_2|$. Since $S^* \subseteq V(T_2)$, by Theorem 3, S^* can be partitioned into two sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ such that T_2 contains n edge-disjoint paths $P_i = (x_i, y_i)$, $i = 1, 2, \ldots, n$.

Let M denote the subgraph of G induced by the edge set $E(P_1) \cup \cdots \cup E(P_n)$. Let $H = T_1 \cup M$. T_1 is connected and spanning, so is H. Let $v \in V(H) = V(G)$. It is easy to observe that $d_H(v) = d_{T_1}(v) + d_M(v)$. If $v \notin S^*$, then both $d_{T_1}(v)$ and $d_M(v)$ are even, and hence $d_H(v)$ is even. If $v \in S^*$, then both $d_{T_1}(v)$ and $d_M(v)$ are odd, and so $d_H(v)$ is even. It follows that H is Eulerian. Thus H is a connected Eulerian spanning subgraph of G. The proof of the theorem is complete. □

Combining Theorems 2 and 4, we obtain the main result in this paper.

Theorem 5. *Let G be a plane graph. If G^* contains two edge-disjoint spanning trees, then $a(G) = 2$.*

3. Spanning Trees in Maximal Plane Graphs

Let G be a maximal plane graph. Then the following properties (a)–(d) hold:

(a) $\|G\| = 3|G| - 6$;

(b) $3 \leq \delta(G) \leq 5$ if $|G| \geq 4$;

(c) Each of the faces in G is of degree 3;

(d) If $|G| = 3$, then G is isomorphic to K_3; if $|G| = 4$, then G is isomorphic to K_4; if $|G| = 5$, then G is isomorphic to $K_5 - e$, where e is any edge of K_5. In particular, when $|G| \geq 5$, we have that $\Delta(G) \geq 4$.

Theorem 6. *Every maximal plane graph G with $\Delta(G) \geq 3$ contains two edge-disjoint spanning trees.*

Proof. We prove the theorem by induction on the vertex number $|G|$. Since $\Delta(G) \geq 3$, it follows that $|G| \geq 4$. If $|G| = 4$, then G is isomorphic to K_4, and K_4 can be easily edge-partitioned into two edge-disjoint spanning trees. Hence the basis step of induction is established. Let G be a maximal plane graph with $|G| \geq 5$. So $\Delta(G) \geq 4$. By the above property (b), G contains a vertex v with $3 \leq d_G(v) \leq 5$. Set $k = d_G(v)$. Let $v_0, v_1, \ldots, v_{k-1}$ denote the neighbors of v in G in clockwise order and $f_0, f_1, \ldots, f_{k-1}$ denote the incident faces of v in G in clockwise order with vv_i, vv_{i+1} as two boundary edges of f_i for $i = 0, 1, \ldots, k-1$. Here all the indices are taken modulo k. Note that $f_0, f_1, \ldots, f_{k-1}$ are all 3 faces.

The proof is split into the following three cases, depending on the size of k.

Case 1 ($k = 3$). Let $G' = G - v$. Then G' is a maximal plane graph with $|G'| < |G|$ and $\Delta(G') \geq \Delta(G) - 1 \geq 3$. By the induction hypothesis, G' contains two edge-disjoint spanning trees T_1' and T_2'. Let $T_1 = T_1' + vv_1$ and $T_2 = T_2' + vv_2$. Obviously, T_i is a spanning tree of G for $i = 1, 2$, and $E(T_1) \cap E(T_2) = \emptyset$. Thus, the theorem holds in this situation.

Case 2 ($k = 4$). In view of the planarity of G, at least one of v_0v_2 and v_1v_3 does not belong to $E(G)$. Without loss of generality, assume that $v_0v_2 \notin E(G)$. Let $G' = G - v + v_0v_2$. Then G' is a maximal plane graph with $|G'| < |G|$ and $\Delta(G') \geq \Delta(G) - 1 \geq 3$. By the induction hypothesis, G' contains two edge-disjoint spanning trees T_1' and T_2'. If $v_0v_2 \notin E(T_1') \cup E(T_2')$, then we define $T_i = T_i' + vv_i$ for $i = 1, 2$. If $v_0v_2 \in E(T_1')$, then we define $T_1 = (T_1' - v_0v_2) + \{vv_0, vv_2\}$ and $T_2 = T_2' + vv_1$. If $v_0v_2 \in E(T_2')$, then we define $T_1 = T_1' + vv_1$ and $T_2 = (T_2' - v_0v_2) + \{vv_0, vv_2\}$. It is easy to inspect that T_1 and T_2 are two edge-disjoint spanning trees of G in each of the above three cases.

Case 3 ($k = 5$). Again, by the planarity of G, there exists a vertex v_i, say $i = 0$, such that $v_0v_2, v_0v_3 \notin E(G)$. Let $G' = G - v + \{v_0v_2, v_0v_3\}$. Then G' is a maximal plane graph with $|G'| < |G|$ and $\Delta(G') \geq \Delta(G) - 1 \geq 3$. By the induction hypothesis, G' contains two edge-disjoint spanning trees T_1' and T_2'. We have to consider the following subcases.

Case 3.1. At least one of v_0v_2 and v_0v_3 does not belong to $E(T_1') \cup E(T_2')$, say $v_0v_3 \notin E(T_1') \cup E(T_2')$.

If $v_0v_2 \notin E(T_1') \cup E(T_2')$, then we set $T_i = T_i' + vv_i$ for $i = 1, 2$. If $v_0v_2 \in E(T_1')$, then we set $T_1 = (T_1' - v_0v_2) + \{vv_0, vv_2\}$ and $T_2 = T_2' + vv_3$. If $v_0v_2 \in E(T_2')$, then we set $T_1 = T_1' + vv_1$ and $T_2 = (T_2' - v_0v_2) + \{vv_0, vv_2\}$.

Case 3.2 ($v_0v_2, v_0v_3 \in E(T_1') \cup E(T_2')$). We need to deal with the following two subcases by symmetry.

Case 3.2.1 ($v_0v_2, v_0v_3 \in E(T_i')$ for some $i \in \{1, 2\}$). Without loss of generality, assume that $i = 1$. It is enough to set $T_1 = (T_1' - \{v_0v_2, v_0v_3\}) + \{vv_0, vv_2, vv_3\}$ and $T_2 = T_2' + vv_1$.

Case 3.2.2 ($v_0v_2 \in E(T_1')$ and $v_0v_3 \in E(T_2')$). Let A_1 and B_1 denote the two components of $T_1' - v_0v_2$ with $v_0 \in V(A_1)$ and $v_2 \in V(B_1)$, and A_2 and B_2 denote the two components of $T_2' - v_0v_3$ with $v_0 \in V(A_2)$ and $v_3 \in V(B_2)$. Since v_0v_2 is a cut edge of T_1', it follows that v_1 belongs to exactly one of $V(A_1)$ and $V(B_1)$. Similarly, v_4 belongs to exactly one of $V(A_2)$ and $V(B_2)$.

If $v_1 \in V(A_1)$, then we define $T_1 = (T_1' - v_0v_2) + \{vv_1, vv_2\}$ and $T_2 = (T_2' - v_0v_3) + \{vv_0, vv_3\}$.

If $v_4 \in V(A_2)$, then we define $T_1 = (T_1' - v_0v_2) + \{vv_0, vv_2\}$ and $T_2 = (T_2' - v_0v_3) + \{vv_3, vv_4\}$.

Now assume that $v_1 \in V(B_1)$ and $v_4 \in V(B_2)$. This implies that $v_0v_1 \notin E(T_1')$ and $v_0v_4 \notin E(T_2')$. If $v_0v_1 \notin E(T_2')$, then we define $T_1 = (T_1' - v_0v_2) + \{v_0v_1, vv_1\}$ and $T_2 = (T_2' - v_0v_3) + \{vv_0, vv_3\}$. If $v_0v_4 \notin E(T_1')$, then we define $T_1 = (T_1' - v_0v_2) + \{vv_0, vv_2\}$ and $T_2 = (T_2' - v_0v_3) + \{v_0v_4, vv_4\}$. Otherwise, $v_0v_1 \in E(T_2')$ and $v_0v_4 \in E(T_1')$. It suffices to define $T_1 = (T_1' - v_0v_2) + \{vv_2, vv_4\}$ and $T_2 = (T_2' - v_0v_3) + \{vv_1, vv_3\}$.

It is easy to inspect that T_1 and T_2 are two edge-disjoint spanning trees of G in every possible case above. This completes the proof of the theorem. □

The following consequence follows immediately from Theorems 4 and 6.

Corollary 7. *Every maximal planar graph G with $|G| \geq 4$ contains a connected Eulerian spanning subgraph.*

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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