Hindawi
Mathematical Problems in Engineering
Volume 2017, Article ID 4391970, 9 pages
https://doi.org/10.1155/2017/4391970

Research Article

Mathematical Model of (R, Q) Inventory Policy under Limited Storage Space for Continuous and Periodic Review Policies with Backlog and Lost Sales

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Received 31 May 2017; Accepted 25 October 2017; Published 10 December 2017

Academic Editor: Thomas Hanne

This paper involves developing new mathematical expressions to find reorder point and order quantity for inventory management policies that explicitly consider storage space capacity. Both continuous and periodic reviews, as well as backlogged and lost demand during stockout, are considered. With storage space capacity, when on-hand inventory exceeds the capacity, the over-ordering cost of storage at an external warehouse is charged on a per-unit-period basis. The objective is to minimize the total cost, consisting of ordering, shortage, holding, and over-ordering costs. Demand and lead time are stochastic and discrete in nature. Demand during varying lead time is modeled using an empirical distribution so that the findings are not subject to assumptions of demand and lead time probability distributions. Due to the complexity of the developed mathematical expressions, the problems are solved using an iterative method. The method is tested with problem instances that use real data from industry. Optimal solutions of the problem instance are determined by performing exhaustive search. The proposed method can effectively find optimal solutions for continuous review policies and near optimal solutions for periodic review policies. Fundamental insights about the inventory policies are reported from a comparison between continuous review and periodic review solutions, as well as a comparison between backlog and lost sales cases.

1. Introduction

This paper focuses on the widely used (R, Q) inventory policy in distribution networks, where an order quantity Q is placed when the inventory position falls on or below the reorder point R at the time of a review. Inventory can be reviewed either continuously in real time or periodically at the end of some period. Continuous reviewing mostly requires a warehouse management system that supports it, whereas periodic reviewing can be performed without such a system. Mostly, the (R, Q) policy is designed for a continuous review, whereas an order-up-to level policy or basestock policy is for a periodic review. However, even without the technology to support a continuous review, some industrial users prefer the (R, Q) policy and use it with a periodic review. This is because some users would not like to change their order quantity every order. Also, their order quantity has to be a multiple of some pack sizes or container.

There are two types of behaviors of demands during stockout: lost sales, where the demands are lost; backlog, where the demands are fulfilled when inventories become available. This behavior depends on the characteristics of products, market, and relationships between suppliers and customers. Both types of demands during stockout are considered in this paper.

The inventory system under study is motivated by and therefore modeled after real distribution networks in various industries in Thailand. In such networks, internal storage space of an item in a warehouse or distribution center (DC) is preassigned and limited. When a replenishment order from a supplier arrives, there are instances when on-hand inventory exceeds the internal storage space. When this occurs, external space can be rented with an external storage cost charged on a per-unit per-period basis. In this system, the total cost of inventory management consists of four components: fixed ordering cost, inventory holding cost at internal storage, inventory holding cost at external storage (called over-ordering cost in this paper), and shortage cost. Item demands and replenishment lead time are stochastic and discrete in nature; therefore, the demands during varying lead time (i.e., the period of shortage risk) are of the same nature. Instead of assuming probability distribution(s) of demand, lead time, and demand during lead time, they are modeled using an empirical distribution. This is to overcome a problem found in real systems that many item demands do not follow widely used probability distributions.

The objective of the paper is to develop mathematical expressions for determining optimal or near optimal R and Q for the inventory system under study, so as to minimize the average total cost per period. The scope of the problem covers a wide range of real problems. The system considers four cost components (as mentioned above), discrete and varying demand and lead time that are not subject to assumptions of probability distribution, and four cases of review policies and demands during stockout. To the best of our knowledge, no mathematical expressions for a problem with this scope have been reported in the literature. The contributions of the paper are, therefore, the developed mathematical expressions, a method that implements them to solve a problem, and important fundamental insights gained from considering four cases of the problem.

There is a vast literature on inventory policy. Therefore, only relevant studies are included, that is, research studies that have involved storage space capacity. Those studies can be categorized into three groups: (1) storage space capacity is considered as a hard constraint, which means over-ordering storage is not allowed; (2) storage space capacity is a hard constraint, but with an additional cost of returning the over-ordered quantity to the supplier; and (3) storage space is a soft constraint, where an over-ordered amount is stored at an external, rented warehouse. These studies vary considerably in other aspects of the problem, such as inventory policies used, cost components considered, nature of demands during stockout, and the solution methodology.

In the first category, recent studies include Mandal et al. [1], Zhao et al. [2], Chou et al. [3], Zhao et al. [4], Zhong and Zhou [5], Pan et al. [6], Ghosh et al. [7], Beemsterboer et al. [8], and Zhang and Rajaram [9]. Among them, studies that focused on (R, Q) policy are Zhao et al. [2], Zhao et al. [4], and Pan et al. [6]. These studies examined a single product, continuous review (R, Q) policy, where demands are stochastic, shortages are backordered, and lead time is assumed to be constant. Among the three studies, only Zhao et al. [4] extended the problem to the case of multiple products. Heuristic algorithms were proposed to solve the problem: polynomial time algorithm in Zhao et al. [2], genetic algorithm in Pan et al. [6], and a solution approach consisting of an iterative part and local search in Zhao et al. [4]. For this set of problems, since storage space is a hard constraint, this implies that the solution methodology proposed in these studies would attempt to set the inventory parameters to be within the available space, which is not the same as this paper.

For the second category where over-ordered quantity is returned to the supplier, Hariga [10] considered the following problem: an (R, Q) policy for a single item problem using continuous review: demands during stockout are backordered, and cycle length does not include the stockout period. The cost function was derived and solved using a simple economic order quantity (EOQ) based heuristic solution.

Studies that allowed external storage include Huang [11], Huang et al. [12], Hariga [13], Zhou et al. [14], Huang [15], Ouyang et al. [16], and Sana [17]. The inventory holding cost at an owned warehouse is normally smaller than at a rented warehouse. Huang [11] and Huang et al. [12] developed a retailer's inventory model when demand is known and constant and considered ordering cost and holding cost while shortages are not considered. The inventory policy was to find optimal time between order and order quantity, a (Q, T) policy. Huang [15] later extended their previous problem by adding the perishable characteristics of the product. Hariga [13] considered multiwarehouse systems to choose where to store excessive stock. The optimal solution consisted of the optimal order quantity and leased storage space. Some studies that focused on a problem where a supplier offers trade credit to a buyer to create an incentive to place larger orders are Zhou et al. [14] and Ouyang et al. [16]. For this problem, the optimal decision for the buyer is the order quantity. The optimal decisions for the supplier are the optimal trade credit period to offer and the corresponding order quantity. Sana [17] presented an EOQ model for stochastic demand without considering lead time and with storage space capacity. Demands during stockout are lost. It can be seen that among recent studies where external rented storage is allowed, no studies have focused on determining (R, Q) under the same settings as this paper.

2. Problem Characteristics and Notation

The mathematical model developed in this paper adheres to the following problem characteristics or assumptions.

(1) The inventory policy is an (R, Q) policy for a single continuously stocked item.

(2) Two review policies are considered: continuous review and periodic review.

(3) At the time of a review, if the inventory position (IP) falls on or below R, a replenishment order Q is placed to the supplier, and a fixed ordering cost is charged.

(4) In the periodic review case, it is assumed that the IP is reviewed frequently enough so that a review period is shorter than the replenishment lead time.

(5) The order arrives after a replenishment lead time that is stochastic and discrete.

(6) Demands are stochastic and discrete in nature.

(7) The item has a preassigned limited storage capacity. At the beginning of a cycle when a replenishment order

arrives, if the on-hand inventory (OH) exceeds the storage space, over-ordered inventories will be kept at an external warehouse.

(8) The over-ordered amount stored at an external warehouse is charged on a per-unit per-period basis.

(9) It is assumed that the rate of external storage cost is no less than the rate of internal storage cost.

(10) When demand arrives, if OH is less than the demand, shortage cost is charged on a per-unit basis.

(11) Two types of shortages are considered. If there exist shortages, demands during stockout may be backlogged and may be lost.

(12) An inventory cycle covers the elapsed time between replenishment orders, including the time the system has stock and the time the system is out of stock.

(13) The objective is to determine the optimal or near optimal R and Q which minimizes the total inventory management cost that consists of fixed ordering cost, inventory holding cost for internal storage space, shortage cost, and storage cost at an external warehouse (i.e., the so-called over-ordering cost in this paper).

(14) Given data include historical demand data, lead time data, all cost parameters, and internal storage space capacity. External storage space is always available.

Since both continuous review and periodic reviews as well as backlog and lost sales during stockout are considered, there are a total of four cases of the problem.

2.1. Notations

Q: replenishment order quantity, units

R: reorder point, units

C_P: fixed ordering cost, THB/order

C_H: inventory holding cost, THB/(unit-period)

C_S: shortage cost, THB/unit

C_O: over-ordering cost, THB/(unit-period)

X_D: random demand, units/period

μ_D: average demand, units/period

$f(x_d)$: probability mass function of X_D

X_L: random lead time, periods

μ_L: average lead time, periods

$f(x_l)$: probability mass function of X_L

X: random demand during varying lead time, units

μ: average demand during varying lead time, units, where $\mu = \mu_D \mu_L$

$f(x)$: probability mass function of X

W: storage space capacity, units

ES_C, ES_P: expected number of shortages, for a given R, under continuous and periodic reviews, respectively, units

PS_C, PS_P: probability of a shortage, for a given R, under continuous and periodic reviews, respectively, in cases that demand during stockout is backlogged and is lost, respectively, %

OH_{end}, $\text{OH}_{\text{ordering}}$, and OH_{begin}: random level of on-hand inventory at the end of a cycle, at the time of ordering, and at the beginning of a cycle, respectively, units

EOH_{end}, $\text{EOH}_{\text{ordering}}$, and $\text{EOH}_{\text{begin}}$, EOH: the expected OH at the end of a cycle, at the time of ordering, at the beginning, and over the length of time the item is in stock in a cycle, respectively

IP: random level of inventory position at the time of ordering, units

$\text{EO}_{C,L}$, $\text{EO}_{P,L}$, $\text{EO}_{C,B}$, and $\text{EO}_{P,B}$: expected over-ordered amount at the beginning of an order cycle for different cases: continuous (C) and periodic (P) reviews, in combination with lost sales (L) and backlog (B), units

$\text{PO}_{C,L}$, $\text{PO}_{P,L}$, $\text{PO}_{C,B}$, and $\text{PO}_{P,B}$: probability of over-ordering at the beginning of an order cycle for the four cases, %

$\text{TC}_{C,L}$, $\text{TC}_{P,L}$, $\text{TC}_{C,B}$, and $\text{TC}_{P,B}$: total inventory management cost per period for the four different cases, THB.

3. Methodology

3.1. Probability Distribution of Demand during Varying Lead Time. The stochastic and discrete behaviors of the demand and lead time are modeled using the distribution of demand during varying lead time. Given historical demand data and lead time data, one can list all possible values of random demand, $X_D = x_d$, and random lead time, $X_L = x_l$, and empirically derive their probability mass functions, $f(x_d)$ and $f(x_l)$. Then, the demand during a given lead time $x_l(X = x \mid X_L = x_l)$ is the sum of the demands that occur during x_l, where $X = x = x_{d,1} + x_{d,2} + \cdots + x_{d,x_l} = \sum_{i=1}^{x_l} x_{d,i}$, and $x_{d,i}$ are independently and identically distributed random variables with $f(x_d)$. In addition, its probability is the product of the probability of all demands $P(X = x \mid X_L = x_l) = f(x_{d,1}) f(x_{d,2}) \cdots f(x_{d,x_l}) = \prod_{i=1}^{l} f(x_{d,i})$. Combining with $f(x_l)$ gives the probability of demand during varying lead time: $f(x) = \sum_l P(X = x \mid X_L = x_l) P(X_L = x_l)$.

3.2. Probability of Stockout and Probability of Over-Ordering. Given the values of Q, R, and W, in a replenishment cycle IP, OH, shortage amount, and over-ordered amount are shown in Figure 1, while the probability of stockout and the probability of over-ordering are shown in Figure 2.

In Figure 2, for illustration, suppose $f(x)$ is normal. The probability of stockout would occur at the right tail of $f(x)$ and towards the end of an order cycle, whereas the probability of over-ordering would occur at the left tail of $f(x)$, at the beginning of the next order cycle, and depends on the value of X in the previous cycle.

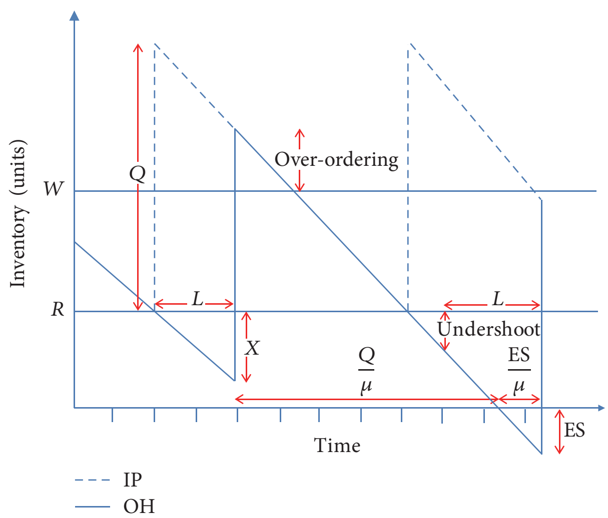

Figure 1: Inventory level in a replenishment cycle.

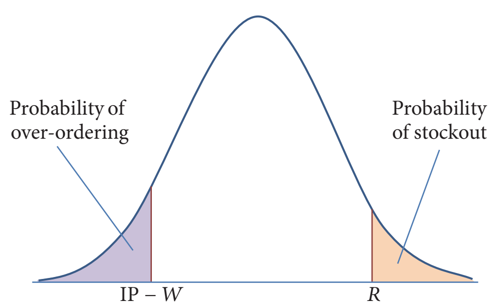

Figure 2: Probabilities of stockout and over-ordering under distribution of demand during lead time.

3.3. Inventory Review and Demands during Stockout. The key difference between continuous and periodic reviews is that in a continuous review a replenishment order is placed when OH falls on or below R, whereas in a periodic review those events usually occur between review periods, so there is an average delay of half a period before the order is placed. That delay leads to an undershoot amount of $\mu_D/2$ units, that is, the difference between R and OH at the point of ordering (i.e., the review time when an order is placed). The undershoot amount affects the probability of stockout and expected shortage as follows. With (1)–(4), it can be seen that both ES and PS are functions of R:

$$\mathrm{PS}_C = P\left(X > \mathrm{EOH}_{\mathrm{ordering}}\right) = P\left(X > R\right) = \sum_{x=R+1}^{x_{\max}} f\left(x\right), \tag{1}$$

$$\mathrm{ES}_C = \sum_{x=R+1}^{x_{\max}} \left(x - R\right) f\left(x\right), \tag{2}$$

$$\mathrm{PS}_C = P\left(X > \mathrm{EOH}_{\mathrm{ordering}}\right) = P\left(X > R - \frac{\mu_D}{2}\right) = \sum_{x=R-\mu_D/2+1}^{x_{\max}} f\left(x\right), \tag{3}$$

$$\mathrm{ES}_P = \sum_{x=R-\mu_D/2+1}^{x_{\max}} \left(x - R + \frac{\mu_D}{2}\right) f\left(x\right). \tag{4}$$

The two cases of demands during stockout, lost sales and backlog, are considered. The difference between the two cases is the amount of OH that is replenished at the beginning of a cycle, which would be Q for lost sales and $Q-\mathrm{ES}$ for backlog.

3.4. Expected Over-Ordered Amount. To derive EO, first consider the expected OH at the end of a cycle, $\mathrm{EOH}_{\mathrm{end}}$, for continuous review: $\mathrm{EOH}_{\mathrm{end}} = \sum_{x=0}^{R}(R-x)f(x) = \sum_{x=0}^{x_{\max}}(R-x)f(x) - \sum_{x=R}^{x_{\max}}(R-x)f(x) = R-\mu+\mathrm{ES}_C$. For a periodic review, $\mathrm{EOH}_{\mathrm{end}}$ is adjusted with the undershoot amount, $\mathrm{EOH}_{\mathrm{end}} = R-\mu_D/2-\mu+\mathrm{ES}_P$. Hence, a random OH at different points in an order cycle can be derived by simply replacing μ with X. Expressions of the random OH for the four cases under different review policies and demands during stockout are shown in Table 1.

Now, EO can be derived from the difference between the random OH at the beginning of a cycle ($\mathrm{OH}_{\mathrm{begin}}$) and W, for the four cases of policies in (5). The expected inventory position at the time of ordering (IP) is used to generalize the formula for the four cases. Similarly, PO is generalized according to (6):

$$\mathrm{EO} = \begin{cases} 0 & \mathrm{IP} - W \le 0 \\ \displaystyle\sum_{x=0}^{\mathrm{IP}-W} \left(\mathrm{IP} - W - x\right) f\left(x\right) & \mathrm{IP} - W > 0, \end{cases} \tag{5}$$

$$\mathrm{PO} = \begin{cases} 0 & \mathrm{IP} - W \le 0 \\ \displaystyle\sum_{x=0}^{\mathrm{IP}-W} f\left(x\right) & \mathrm{IP} - W > 0 \end{cases} \tag{6}$$

EO and PO for the four cases use the corresponding IP, where $\mathrm{IP}_{C,L} = Q+R+\mathrm{ES}_C$, $\mathrm{IP}_{P,L} = Q+R-\mu_D/2+\mathrm{ES}_P$, $\mathrm{IP}_{C,B} = Q+R$, and $\mathrm{IP}_{P,B} = Q+R-\mu_D/2$. With (5) and (6), it can be seen that both EO and PO are functions of Q and R.

3.5. Average Inventory Management Cost per Cycle. In a cycle, the inventory management cost has four components: ordering, shortage, holding, and over-ordering costs. The ordering cost and shortage cost are simply C_P and $C_S\mathrm{ES}$, respectively. The over-ordering cost is the storage cost at an external warehouse, which occurs at the beginning of an order cycle, and is subject to C_O. Based on EO, that is, the expected over-ordered amount at the beginning of a cycle, the over-ordering cost is charged on average EO/2 for the length of EO/μ_D periods. Therefore, the expected over-ordering cost is $C_O(\mathrm{EO}^2/2\mu_D)$. Finally, the holding cost at internal storage is charged, based on the average OH at the beginning and at the end of a cycle for the length of Q/μ_D, taking into account the expected over-ordered amount. The sum of the four cost components becomes the total inventory management cost. Dividing the total cost by the average cycle length $(Q+\mathrm{ES})/\mu_D$, which takes into account the length of shortage period, gives the average total cost per period for the four cases in (7).

Table 1: On-hand inventory in a cycle for four cases of (R, Q) inventory policies.

Point in a cycle	Continuous review	Periodic review
OH falls on or below R	$\mathrm{OH} = R$	$\mathrm{OH} = R$
At the time of ordering	$\mathrm{OH}_{\mathrm{ordering}} = R$	$\mathrm{OH}_{\mathrm{ordering}} = R - \dfrac{\mu_D}{2}$
At the end of a cycle	$\mathrm{OH}_{\mathrm{end}} = R - X + \mathrm{ES}_C$	$\mathrm{OH}_{\mathrm{end}} = R - \dfrac{\mu_D}{2} - X + \mathrm{ES}_P$
At the beginning of the next cycle		
Lost sales	$\mathrm{OH}_{\mathrm{begin}} = Q + R - X + \mathrm{ES}_C$	$\mathrm{OH}_{\mathrm{begin}} = Q + R - \dfrac{\mu_D}{2} - X + \mathrm{ES}_P$
Backlog	$\mathrm{OH}_{\mathrm{begin}} = Q + R - X$	$\mathrm{OH}_{\mathrm{begin}} = Q + R - \dfrac{\mu_D}{2} - X$

To save space, EOH in the formula is used to denote the expected on-hand inventory during the period of Q/μ_D for the inventory holding cost component:

$$\mathrm{TC} = \frac{\mu_D\left(C_P + C_S\mathrm{ES} + (C_H Q/\mu_D)\,\mathrm{EOH} + (C_O - C_H)\left(\mathrm{EO}^2/2\mu_D\right)\right)}{Q + \mathrm{ES}}, \tag{7}$$

where TC for each policy needs ES to be computed using (2) or (4), EO using (5), $\mathrm{EOH}_{C,L} = Q/2 + R - \mu + \mathrm{ES}_C$, $\mathrm{EOH}_{P,L} = Q/2 + R - \mu_D/2 - \mu + \mathrm{ES}_P$, $\mathrm{EOH}_{C,B} = Q/2 + R - \mu$, and $\mathrm{EOH}_{P,B} = Q/2 + R - \mu_D/2 - \mu$.

3.6. Mathematical Model to Find R and Q. The mathematical model to determine R and Q consists of two mathematical expressions that are derived by taking the first derivative of TC with respect to both R and Q, setting both equations to zero, and then solving them. The general form of $d\mathrm{TC}/dQ$ for the four cases is shown in (8). Solving the equations results in the general expression for Q in (9). Note that, from (5) and (6), since EOs are functions of IPs, which are functions of Q, we have $d\mathrm{EO}/dQ = \mathrm{PO}$:

$$\frac{d\mathrm{TC}}{dQ} = \frac{\mu_D\left((C_H/\mu_D)(Q/2 + \mathrm{EOH}) + (C_O - C_H)(\mathrm{EO}/\mu_D)\,\mathrm{PO}\right) - \mathrm{TC}}{Q + \mathrm{ES}} = 0, \tag{8}$$

$$Q = \sqrt{\frac{2\mu_D}{C_H}\left[C_P + C_S\mathrm{ES} + \frac{C_H\mathrm{ES}}{\mu_D}G + (C_O - C_H)\left(1 + \frac{(C_O - C_H)\,\mathrm{PO}^2}{C_H}\right)\frac{\mathrm{EO}^2}{2\mu_D}\right]} - \mathrm{ES} - \frac{(C_O - C_H)\,\mathrm{PO}}{C_H}\mathrm{EO}. \tag{9}$$

To save space, an expression G is used in (9) for the four cases: $G_{C,L} = \mu - R - \mathrm{ES}_C/2$, $G_{P,L} = \mu + \mu_D/2 - R - \mathrm{ES}_P/2$, $G_{C,B} = \mu - R + \mathrm{ES}_C/2$, and $G_{P,B} = \mu + \mu_D/2 - R + \mathrm{ES}_P/2$.

Similarly, the $d\mathrm{TC}/dR$ for the four cases is shown in (10). Solving them results in the expression for R in (11). Note that $d\mathrm{TC}/dR$ is based on the results of three derivatives: (1) $d\mathrm{ES}/dR = -\mathrm{PS}$, which is based on (1)–(4) where PS and ES are both functions of R, (2) $d\mathrm{EO}/dR = \mathrm{PO}(1 - \mathrm{PS})$ for lost sales, which is based on (5) and (6) where EO (or IP) and ES are functions of R, and (3) $d\mathrm{EO}/dR = \mathrm{PO}$ for backlog, where their IPs in (5) and (6) do not contain ES:

$$\begin{aligned}\frac{d\mathrm{TC}}{dR} = {} & \frac{\mu_D}{Q + \mathrm{ES}}\left(-C_S\mathrm{PS} + C_H H + (C_O - C_H)(I)\right) \\ & + \frac{\mathrm{PS}}{Q + \mathrm{ES}}\mathrm{TC} = 0.\end{aligned} \tag{10}$$

To save space, expressions H and I are used in (10), where $H_L = (Q/\mu_D)(1 - \mathrm{PS})$ for lost sales; $H_B = Q/\mu_D$ for backlog; $I_L = \mathrm{PO}(1 - \mathrm{PS})(\mathrm{EO}/\mu_D)$ for lost sales; and $I_B = \mathrm{PO}(\mathrm{EO}/\mu_D)$ for backlog:

$$R = a - b, \tag{11}$$

where $a = (\mu_D(Q + \mathrm{ES})/(\mathrm{PS})(C_H Q))(C_S P_S - C_H J - (C_O - C_H)K)$. Substitute J and K with $J_L = (Q/\mu_D)(1 - \mathrm{PS})$ and $K_L = \mathrm{PO}(1 - \mathrm{PS})(\mathrm{EO}/\mu_D)$ for lost sales, $J_B = Q/\mu_D$ and $K_B = \mathrm{PO}(\mathrm{EO}/\mu_D)$ for backlog, and $b = (\mu_D/C_H Q)(C_P + C_S\mathrm{ES} + (C_H Q/\mu_D)N + (C_O - C_H)(\mathrm{EO}^2/2\mu_D))$. Substitute N with $N_{C,L} = Q/2 - \mu + \mathrm{ES}_C$, $N_{P,L} = Q/2 - \mu_D/2 - \mu + \mathrm{ES}_P$, $N_{C,B} = Q/2 - \mu$, and $N_{P,B} = Q/2 - \mu_D/2 - \mu$, accordingly.

3.7. Solving the Model for R and Q. Given all the input data, C_P, C_S, C_H, C_O, $f(x)$, W, μ_D, μ, the two mathematical expressions for R and Q are recursive functions of each other. Therefore, they must be solved iteratively until their values converge. In each iteration, given an initial value Q, solving for a new R requires a search in the range of possible values of R because some input parameters in (11), including ES, PS, EO, and PO, are themselves functions of R. In other words, the search for R_{new} begins with computing ES, PS, EO, and PO using R (a variable) and Q (fixed at an initial value for this search step). Then, compute R_{new} using (11) and compute $|R - R_{\mathrm{new}}|$. It is important to note that the calculations of ES, PS, EO, and PO must be done in a sequence because EO uses ES as one of the inputs. While holding Q fixed and moving R in the direction that reduces $|R - R_{\mathrm{new}}|$, the best R_{new} is the value that minimizes $|R - R_{new}|$. Solving for Q_{new} using (10) is also performed in a similar manner. We now state the algorithm to solve for R and Q.

Step 1 (initialization).

(i) Input data are $C_P, C_S, C_H, C_O, W, d, l$; initialize TC = M (a large number).

(ii) From x_d and x_l, compute $\mu_D = \overline{x}_d$, $\mu_L = \overline{x}_l$, and fit $f(x_d)$ and $f(x_l)$.

(iii) Compute all x for each x_l and the probability: $x = \sum_{i=1}^{x_l} x_{d,i}$, $P(X = x \mid X_L = x_l) = \prod_{i=1}^{l} f(x_{d,i})$.

(iv) Compute $\mu = \mu_D \mu_L$, $x_{\max}$, and $f(x) = \sum_l P(X = x \mid X_L = x_l) P(X_L = x_l)$.

(v) Compute initial $Q = \sqrt{2\mu_D C_P / C_H}$.

Step 2. Compute new R

(i) For $R = 0$ to $x_{\max}$, the following steps are considered.

(a) Compute $\mathrm{PS}_C = \sum_{x=R+1}^{x_{\max}} f(x)$ and $\mathrm{ES}_C = \sum_{x=R+1}^{x_{\max}} (x - R) f(x)$, if it is a continuous review; or $\mathrm{PS}_P = \sum_{x=R-\mu_D/2+1}^{x_{\max}} f(x)$ and $\mathrm{ES}_P = \sum_{x=R-\mu_D/2+1}^{x_{\max}} (x - R + \mu_D/2) f(x)$, if it is a periodic review.

(b) Compute $\mathrm{EO} = \sum_{x=0}^{\mathrm{IP}-W} (\mathrm{IP} - W - x) f(x)$ if $\mathrm{IP} - W > 0$ and $\mathrm{PO} = \sum_{x=0}^{\mathrm{IP}-W} f(x)$ if $\mathrm{IP} - W > 0$, where IP takes on one of the following: $\mathrm{IP}_{C,L} = Q + R + \mathrm{ES}_C$, $\mathrm{IP}_{P,L} = Q + R - \mu_D/2 + \mathrm{ES}_P$, $\mathrm{IP}_{C,B} = Q + R$, and $\mathrm{IP}_{P,B} = Q + R - \mu_D/2$, depending on the case.

(c) Compute $R_{new}(R) = a - b$, where $a = (\mu_D(Q + \mathrm{ES})/(\mathrm{PS})(C_H Q))(C_S P_S - C_H J - (C_O - C_H)K)$, $J_L = (Q/\mu_D)(1 - \mathrm{PS})$, and $K_L = \mathrm{PO}(1 - \mathrm{PS})(\mathrm{EO}/\mu_D)$ for lost sales, or $J_B = Q/\mu_D$ and $K_B = \mathrm{PO}(\mathrm{EO}/\mu_D)$ for backlog; $b = (\mu_D / C_H Q)(C_P + C_S \mathrm{ES} + (C_H Q/\mu_D)N + (C_O - C_H)(\mathrm{EO}^2/2\mu_D))$, and N takes on one of the following: $N_{C,L} = Q/2 - \mu + \mathrm{ES}_C$, $N_{P,L} = Q/2 - \mu_D/2 - \mu + \mathrm{ES}_P$, $N_{C,B} = Q/2 - \mu$, and $N_{P,B} = Q/2 - \mu_D/2 - \mu$, depending on the case.

(d) Compute $|R - R_{new}(R)|$.

(ii) Update $R = R_{new}(R)$ for $\min |R - R_{new}(R)|$.

Step 3. Compute new Q.

(i) Compute ES with R from Step 2.

(ii) For $Q = 0$ to $x_{\max}$, the following are considered.

(a) Compute PO and EO, and compute Q according to (9).

(b) Compute $|Q - Q_{new}(Q)|$.

(iii) Update $Q = Q_{new}(Q)$ with $\min |Q - Q_{new}(Q)|$, and update PO, EO.

Table 2: Iteration steps to obtain the optimal R and Q.

Iteration	C, B		P, B		C, L		P, L	
	R	Q	R	Q	R	Q	R	Q
Initial		868		868		868		868
(1)	2,963	948	3,042	1,044	2,964	943	3,041	1,041
(2)	2,912	995			2,915	989		
(3)	2,882	1,025			2,886	1,017		
(4)	2,864	1,042			2,870	1,032		
(5)	2,854	1,052			2,861	1,041		
(6)	2,848	1,058			2,856	1,046		
(7)	2,844	1,062			2,853	1,049		
(8)	2,841	1,065			2,851	1,051		
(9)	2,840	1,066			2,850	1,052		
(10)					2,849	1,053		

(iv) Update $TC = \mu_D(C_P + C_S \mathrm{ES} + (C_H Q/\mu_D)\mathrm{EOH} + (C_O - C_H)(\mathrm{EO}^2/2\mu_D))/(Q + \mathrm{ES})$ where EOH is one of the following: $\mathrm{EOH}_{C,L} = Q/2 + R - \mu + \mathrm{ES}_C$, $\mathrm{EOH}_{P,L} = Q/2 + R - \mu_D/2 - \mu + \mathrm{ES}_P$, $\mathrm{EOH}_{C,B} = Q/2 + R - \mu$, and $\mathrm{EOH}_{P,B} = Q/2 + R - \mu_D/2 - \mu$, depending on the case.

Step 4. Repeat Steps 2 and 3 if TC improves from the previous value; Otherwise, stop.

4. Numerical Example

A numerical example that presents the developed method uses the following data: C_P = 12.55 THB, C_S = 4 THB/unit, C_H = 0.012 THB/(unit-day), C_O = 0.104 THB/(unit-day), W = 3,300 units, $f(l) = \{(1, 0.365), (2, 0.234), (3, 0.257), (4, 0.144)\}$, μ_L= 2.18 days, and μ_D = 360.27 units/day. Historical daily demand and lead time data are used to empirically construct $f(d)$ and $f(x)$. Note that these are real data of a fast moving item at a distribution center of a healthcare company in Thailand.

To evaluate the performance of the model, first the optimal solutions for all four cases are determined. Based on the data, the maximum possible demand during lead time is $x_{\max}$ = 8,116 units (demand data contain a few extreme outliers). For each policy, a complete enumeration of R and Q is performed, that is, evaluating $(8{,}117)^2$ = 66,885,689 solutions. This takes approximately two hours of computing time, which is not a practical approach for an industrial user, and the distribution centers may have hundreds or thousands of items to manage.

Initial testing shows that the proposed method can generate optimal solutions for a continuous review policy and near optimal solutions for a periodic review policy. Table 2 contains iteration results of the proposed method. For periodic review cases, a detailed investigation of the iterations indicates that the search stops in one iteration, and a move from the first iteration solution yields worse total costs. Table 3 shows the results from using the developed method. For near optimal cases, the maximum off-optimal percentage of the total cost per period is only 0.0143% for a periodic

Table 3: Performance of the developed method, optimal and off-optimal.

Continuous review with backlog (CPU time = 43.69 s)				Continuous review with lost sales (CPU time = 48.27 s)			
Iter.	R	Q	TC	Iter.	R	Q	TC
9	2,840	1,066	46.0347	10	2,849	1,053	46.1756
Opt.	2,840	1,066	46.0347	Opt.	2,849	1,053	46.1756
Off-opt	0%	0%	0%	Off-opt	0%	0%	0%
Periodic review with backlog (CPU time = 9.14 s)				Periodic review with lost sales (CPU time = 9.13 s)			
Iter.	R	Q	TC	Iter.	R	Q	TC
1	3,042	1,044	46.0413	1	3,041	1,041	46.1774
Opt.	3,020	1,066	46.0347	Opt.	3,029	1,053	46,1756
Off-opt	0.73%	2.06%	0.0143%	Off-opt	0.40%	1.14%	0.0039%

Table 4: Optimal solutions and their cost components.

Solution performance	C, B	P, B	C, L	P, L
R	2,840	3,020	2,849	3,029
Q	1,066	1,066	1,053	1,053
ES (units)	7.0931	7.0952	6.9565	6.9586
EO (units)	183.8970	183.8274	185.4028	185.3339
PS	0.0153	0.0153	0.0150	0.0151
PO	0.5077	0.6080	0.5102	0.6064
Fixed ordering cost (THB/cycle)	12.55	12.55	12.55	12.55
Shortage cost (THB/cycle)	28.3724	28.3808	27.8261	27.8343
Inventory holding cost (THB/cycle)	91.3130	91.3086	90.5150	90.5107
Over-ordering cost (THB/cycle)	4.8811	4.8774	4.9614	4.9577
Total cost (THB/cycle)	137.1165	137.1168	135.8525	135.8527
Cycle length (days)	2.9785	2.9786	2.9421	2.9421
Total cost per day (THB)	46.0347	46.0347	46.1756	46.1756
Annual cost (THB)	5,641.23	5,641.22	5,728.62	5,728.61

review policy with backlog. The computational times for all four cases are also reported in Table 3.

From examining the optimal solutions, the differences between continuous review solutions and periodic review solutions are only in R, but not in Q, and it is not coincidental that the difference is equal to the expected undershoot amount of $\mu_D/2$. Therefore, for a periodic review, one can simply solve the continuous review problem and add $\mu_D/2$ to R, while Q remains unchanged, to obtain the optimal solution. Table 4 shows details of the optimal solutions.

From the table, a comparison between continuous review and periodic review leads to the same findings for both backlog and lost sales cases. At the optimal solutions, (1) both policies have the same Q, (2) a periodic review has a higher R in order to compensate for the undershoot, but ES and shortage cost remain slightly higher, (3) holding cost and over-ordering cost are higher for a continuous review, and (4) a periodic review has a slightly higher total cost and longer cycle length, and together they lead to the same average total cost per day. In summary, there is no difference between continuous and periodic reviews, in terms of inventory management cost and performance. This is because of the inclusion of the undershoot amount to inventory OH in the calculation for the periodic policy; that is, a periodic review without considering undershoot would yield an inferior solution.

Another comparison between backlog and lost sales cases leads to the same findings for both a continuous review and a periodic review. Thus, it seems that there is no interaction between review policy and the nature of demands during stockout. At the optimal solutions, (1) R is higher for lost sales (as expected, because in our example, backlog cost and lost sales cost are the same, so it is only natural that the cycle service level would be set higher (less PS) to try to prevent shortages for lost sales), (2) Q is higher for backlog because it must satisfy demands during stockout, (3) higher R leads to higher EO and PO for lost sales, whereas higher Q leads to higher holding cost for backlog, and (4) although backlog has a higher total cost, it also has a longer cycle length, which leads to a lower average total cost per period. In summary, for the same cost components, backlog is more cost effective than lost sales.

5. Discussion

As previously mentioned, the problem under study is derived from a real inventory management problem at a distribution center of a healthcare company in Thailand. The operating conditions are as follows: The DC uses an (R, Q) policy under periodic (daily) review for a few hundred continuously stocked healthcare products, whose stocks are independently managed. For each item, storage space location and capacity are assigned. Replenishment lead time from the manufacturer varies between one and four days. This variation is caused by the difference between the day at which an order is placed and the delivery day. For example, an order placed on Monday and received by Friday has a four-day lead time, while an order placed on Tuesday and received by Friday has a three-day lead time, and so on. Items are classified by their movement (fast, medium, and slow). Most items' demand data do not fit with known probability distributions. In a cycle where the order is received in a relatively short time and the demand during lead time is low, internal storage space capacity is exceeded. When this occurs, external storage space is rented to keep excessive inventory. In this system, shortages are mostly backlogged because the DC customers are retail stores of various types and sizes. The proposed mathematical model is applicable not only to the described system, but also to other systems with similar characteristics. Based on the authors' experience, other industries in Thailand that face the same problems are distribution centers of retail businesses, authorized distributors of heavy equipment (agricultural tools and equipment, cooling systems, and compressors), spare part retailers and wholesalers, distribution networks for medicine, and medical supplies serving hospital systems.

6. Conclusions

This paper proposes a method to find optimal R and Q for both continuous and periodic reviews as well as for backlog and lost sales cases. The method is based on two newly developed mathematical expressions to find R and Q that minimize the average total cost per period and explicitly consider the over-ordering storage cost. Due to the complexity of the mathematical expressions, they must be solved iteratively. The method has a very promising performance based on the numerical test; that is, the solutions found are optimal for a continuous review and near optimal for a periodic review.

Three important fundamental insights are gained. First, one can solve the periodic review problem by simply solving the continuous review problem and adding the undershoot amount to the optimal R, while the optimal Q remains the same for both review policies. Continuous review and periodic review policies perform nearly the same, given that the undershoot amount is explicitly considered. For the same cost structure and cost components, cases of backlogged demand during stockout are more cost effective than those of lost demand. These findings are not subject to assumptions of the probability distribution of demand during varying lead time, because an empirical distribution fitted from real demand and lead time data is used.

Future research directions are as follows: (1) For cases of periodic review, our mathematical model can give only a near optimal solution. This calls for further investigation, which could lead to potential improvement of the method to handle periodic review cases. (2) The problem can be extended to multiple products sharing the same storage space.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors would like to thank the industrial user, who provided the demand, cost, and operational data of their products to this study.

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