Hindawi
Mathematical Problems in Engineering
Volume 2017, Article ID 4687638, 9 pages
https://doi.org/10.1155/2017/4687638

Research Article

Forward Displacement Analysis of a General 6-3 Stewart Platform Using Conformal Geometric Algebra

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Received 30 March 2016; Revised 7 August 2016; Accepted 31 August 2016; Published 13 February 2017

Academic Editor: Eckhard Hitzer

In this paper, a new algorithm for the forward displacement analysis of a general 6-3 Stewart platform (6-3SPS) based on conformal geometric algebra (CGA) is presented. First, a 6-3SPS structure is changed into an equivalent 2RPS-2SPS structure. Then, two kinematic constraint equations are established based on the geometric characteristics, one of which is built according to the point characteristic four-ball intersection in CGA. A 16th-degree univariate polynomial equation is derived from the aforementioned two equations by the Sylvester resultant elimination. Finally, a numerical example is given to verify the algorithm.

1. Introduction

Stewart platforms [1] are six-degree-of-freedom parallel manipulators that generally consist of a base platform, a moving platform, and six legs connected to each other in parallel. They were first applied to animate flight simulator platforms. In these recent decades, Stewart mechanisms have attracted wide interest from researchers and engineers due to their advantages of simplicity, high stiffness, large load capacity, quick dynamic response, and excellent accuracy [2]. The 6-3 Stewart platform (6-3SPS) is a special form of Stewart platform. The basic geometric structure of the 6-3SPS consists of a moving platform connected to the adjacent links at three distinct points B_i, $i = 1, 2, 3$, by spherical kinematic pairs, which is shown in Figure 1. 6-3SPSs can be divided into either platform mechanisms or general mechanisms according to whether the six spherical points on the base platform are restricted to lie in plane or not.

It is well known that the forward displacement analysis (FDA) of parallel mechanisms, which is used to find the position and orientation of the moving platform given the actuator displacements, is much more challenging than the inverse displacement problem. Generally, the FDA problem involves highly nonlinear algebraic equations. There are two ways to solve these equations: numerical and closed-form solutions. Although numerical iterative approaches can obtain the forward kinematics solutions, they are not suitable for the general Stewart mechanism because they are computationally burdensome and require good initial values. A closed-form solution provides more information about the geometry and kinematic behavior over a numerical solution and the input-output closed-form univariate polynomial equation has significant theoretical values as it is fundamental to many other kinematic problems. Hence obtaining a closed-form solution to the FDA problem is more desirable in most cases.

Numerous studies have been conducted regarding closed-form solutions for the FDA of the 6-3SPS. Griffis and Duffy [3] obtained a closed-form solution to the platform mechanisms of the 6-3SPS in 1989, but their method is not suitable for the solution of the general mechanisms. In 1990, Innocenti and Parenti-Castelli [4] changed the 6-3SPS structure into the 3-3RPS structure based on the 6-3SPS model. The FDA problem can be reduced to the solution of a system of three second-order nonlinear equations. In 1991, Liang and Rong [5] derived the FDA solution to a Stewart triangle platform, 6-SPS parallel manipulator. In fact, the principle of Liang's method is the same as that in [4]. Both Nanua et al. [6] in 1990 and Akçali and Mutlu [7] in 2006 obtained the 16th-order input-output equation. However, the two methods are complex. Song and Kwon [8] obtained a

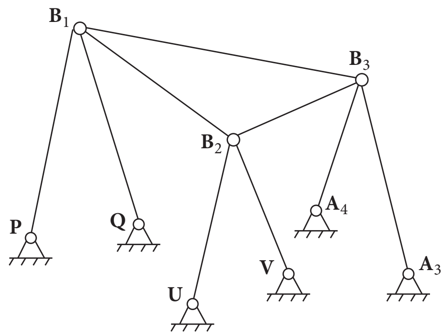

FIGURE 1: Structure of a 6-3SPS.

closed-form direct kinematic solution of the 6-3SPS using the tetrahedron approach in 2011. The closed-form solution in [8], however, is not polynomial form. Cheng et al. [9] obtained an eighth-order input-output equation based on the orthogonal complement method. Their method introduces errors due to rounding off the coefficient of the high-order term in the process of elimination.

In the process of carrying out a parallel mechanism kinematic analysis, the various components (links, joints, platforms, etc.) of the proposed parallel mechanism can be defined as three-dimensional Euclidean geometric objects such as points, lines, circles, planes, and spheres. However, it is difficult to represent and compute geometric constructions such as the intersection, extension, and contact of geometric objects. A solution for this problem is provided by conformal geometric algebra (CGA), which can represent the geometries of Euclidean space effectively by the outer product, inner product, and geometric product. At least three kinematic constraint equations, established in the previously mentioned literature, are solved based on two eliminations. This paper proposes a new process for obtaining the closed-form solution of the FDA of the 6-3SPS using CGA. Only two kinematic constraint equations are established based on the geometric characteristics and solved based on single elimination; as a result, the solution procedure is simpler and more efficient than that in the aforementioned works.

The paper is arranged as follows. In Section 2, we introduce the principle of CGA. In Section 3, the kinematic constraint equations are established using the CGA. In Section 4, we propose the elimination procedure. The numerical example to validate our new algorithm is given in Section 5. Conclusions are drawn in Section 6.

2. Conformal Geometric Algebra

CGA [10–12] is a mathematical language that integrates various mathematical theories, such as projective geometry and quaternion, and Lie Algebra. Therefore, CGA has been spotlighted as a new method in robotics [13, 14].

2.1. Fundamental Theory. The fundamental algebraic operators [10] in geometric algebra are the inner product ($\mathbf{a}_c \cdot \mathbf{b}_c$), the outer product ($\mathbf{a}_c \wedge \mathbf{b}_c$), and the geometric product ($\mathbf{a}_c\mathbf{b}_c = \mathbf{a}_c \wedge \mathbf{b}_c + \mathbf{a}_c \cdot \mathbf{b}_c$). CGA, which is denoted by the base space $\mathbb{R}^{3+1,1}$, has a set of orthogonal bases given by e_1, e_2, e_3, e_+, and e_- with the properties

$$\begin{aligned} e_i^{\,2} &= 1, \quad i = 1, 2, 3, \\ e_+^2 &= 1, \\ e_-^2 &= -1, \\ e_i \cdot e_+ &= e_i \cdot e_- = e_+ \cdot e_- = 0. \end{aligned} \tag{1}$$

A set of null bases $\{e_\infty, e_0\}$ orthogonal to e_i can be introduced by $\{e_+, e_-\}$:

$$\begin{aligned} e_0 &= \frac{1}{2}\left(e_- - e_+\right), \\ e_\infty &= e_+ + e_-, \end{aligned} \tag{2}$$

with the properties

$$\begin{aligned} e_0^2 &= e_\infty^2 = 0, \\ e_\infty \cdot e_0 &= -1. \end{aligned} \tag{3}$$

2.2. Conformal Geometric Entities. Let $\mathbf{x} \in \mathbb{R}^3$ be a point expressed in the Euclidean space $\mathbb{R}^3$. The representation of the same point in $\mathbb{R}^{3+1,1}$ is given by

$$\mathbf{x}_c = \mathbf{x} + \frac{1}{2}\mathbf{x}^2 e_\infty + e_0, \tag{4}$$

where $\mathbf{x} = x_1e_1 + x_2e_2 + x_3e_3$, $\mathbf{x}^2 = x_1^2 + x_2^2 + x_3^2$.

In $\mathbb{R}^{3+1,1}$, the equation of a sphere of radius r centered at $\mathbf{p} \in \mathbb{R}^3$ can be written as

$$\mathbf{S}_c = \mathbf{p} + \frac{\mathbf{p}^2 - r^2}{2} e_\infty + e_0 = \mathbf{p}_c - \frac{1}{2} r^2 e_\infty. \tag{5}$$

2.3. Conformal Transformations. Rotation, which is called a rotor, can be written as a unit quaternion as follows:

$$\begin{aligned} \mathbf{R} &= \cos\frac{\phi}{2} - \mathbf{n}\sin\frac{\phi}{2}, \\ \mathbf{R}^* &= \cos\frac{\phi}{2} - \mathbf{n}\sin\frac{\phi}{2}, \end{aligned} \tag{6}$$

where $\mathbf{n}$ is a normalized bivector [10] representing the axis of rotation, ϕ is the rotation angle, and the superscript $*$ represents conjugation.

Translation in $\mathbb{R}^{3+1,1}$ can be defined as

$$\begin{aligned} \mathbf{T} &= 1 - \frac{1}{2}\mathbf{a}e_\infty, \\ \mathbf{T}^* &= 1 + \frac{1}{2}\mathbf{a}e_\infty, \end{aligned} \tag{7}$$

where $\mathbf{a} = 2d\mathbf{t}$, in which d denotes the displacement distance and $\mathbf{t}$ denotes the direction of movement. Therefore, the motion transformation of a rigid body in $\mathbb{R}^{3+1,1}$ can be denoted by

$$\mathbf{Q}_c = \mathbf{T}\mathbf{R}\mathbf{Q}_c'\mathbf{R}^*\mathbf{T}^*. \tag{8}$$

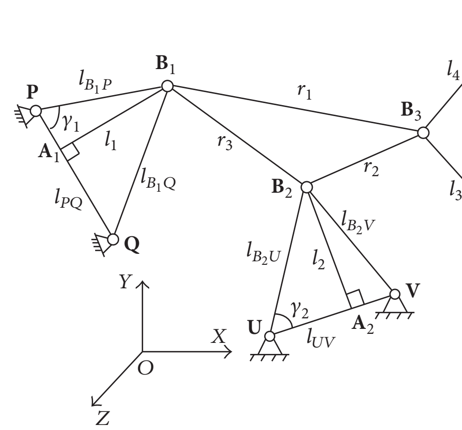

Figure 2: Geometric model of a general 6-3SPS.

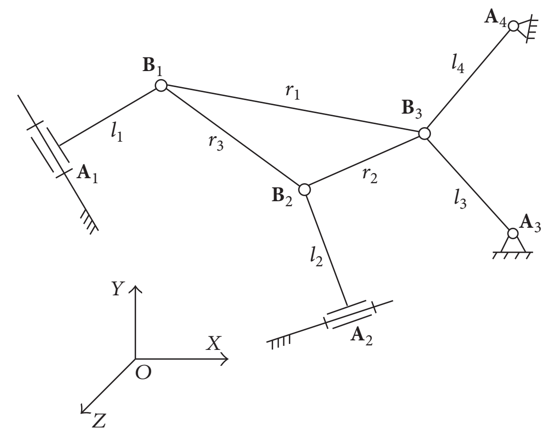

Figure 3: Equivalent 4-3 parallel platform.

2.4. Inner Product of CGA. In $\mathbb{R}^{3+1,1}$, the inner product of two conformal points is a direct representation of the Euclidean distance between the two points. For example, letting $\mathbf{p}_c$ and $\mathbf{s}_c$ be two points in $\mathbb{R}^{3+1,1}$, the inner product of two conformal points is represented by $\mathbf{p}$ and $\mathbf{s}$ in $\mathbb{R}^3$ as follows:

$$\mathbf{p}_c \cdot \mathbf{s}_c = -\frac{1}{2}\|\mathbf{s}-\mathbf{p}\|^2. \tag{9}$$

If the point $\mathbf{x}_c$ is on the surface of the sphere $\mathbf{S}_c$, we can obtain

$$\mathbf{x}_c \cdot \mathbf{S}_c = 0. \tag{10}$$

A more detailed description of CGA can be found in [15].

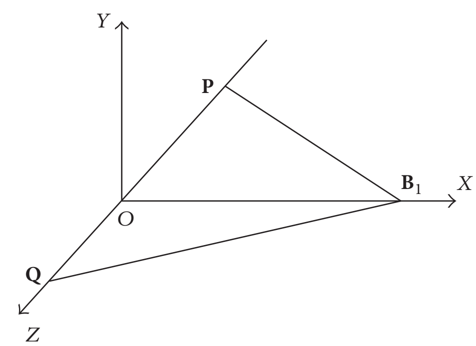

Figure 4: Position after transformation of the triangle $\mathbf{PB}_1\mathbf{Q}$.

3. Constraint Equations

A 6-3SPS has its six SPS (S, spherical joint; P, prismatic joint) legs connected at the five points in the moving platform and the five points in the base platform, shown in Figure 2. The six leg lengths provided by the prismatic joint in every leg are the six inputs to control the location and orientation of the moving platform. The six fixed spherical joints $\mathbf{P}, \mathbf{Q}, \mathbf{U}, \mathbf{V}, \mathbf{A}_3$, and $\mathbf{A}_4$ are not restricted to lie in a plane. The three moving joints $\mathbf{B}_1, \mathbf{B}_2$, and $\mathbf{B}_3$ are composite spherical joints.

3.1. The Structure Transformation. According to [4, 5], in Figure 2 the point $\mathbf{B}_1$ can be only located on a circle with axis $\mathbf{PQ}$. Therefore, we can replace the two SPS links $\mathbf{B}_1\mathbf{P}$ and $\mathbf{B}_1\mathbf{Q}$ by one RPS (R, revolute joint) link $\mathbf{B}_1\mathbf{A}_1$, where point A_1 denotes the foot point on $\mathbf{PQ}$ with respect to point $\mathbf{B}_1$. The triangle $\mathbf{PB}_1\mathbf{Q}$ is equivalent to a rotation of $\mathbf{B}_1$ with respect to $\mathbf{PQ}$, with θ_1 denoting the rotation angle with axis $\mathbf{PQ}$. Analogously, the two SPS links $\mathbf{B}_2\mathbf{U}$ and $\mathbf{B}_2\mathbf{V}$ can be replaced by one RPS link $\mathbf{B}_2\mathbf{A}_2$, where point $\mathbf{B}_2$ denotes the foot point on $\mathbf{UV}$ with respect to point $\mathbf{B}_2$. The triangle $\mathbf{UB}_2\mathbf{V}$ is equivalent to a rotation of $\mathbf{B}_2$ with respect to $\mathbf{UV}$, with θ_2 denoting the rotation angle with axis $\mathbf{UV}$. Now clearly, the FDA of the 6-3SPS is equivalent to that of the special 4-3 parallel platform 2RPS-2SPS seen in Figure 3, where two revolute pairs are introduced in place of the pairs of legs converging at points $\mathbf{B}_1$ and $\mathbf{B}_2$.

3.2. Basic Closure Equations. We locate a reference frame O-XYZ with its origin at random on the base platform $\mathbf{PQUVA}_3\mathbf{A}_4$, as shown in Figure 2. The coordinates of points $\mathbf{P}, \mathbf{Q}, \mathbf{U}, \mathbf{V}, \mathbf{A}_3$, and $\mathbf{A}_4$ are known, respectively, in fixed frame; that is to say, the coordinates of vectors $\mathbf{P}, \mathbf{Q}, \mathbf{U}, \mathbf{V}, \mathbf{A}_3$, and $\mathbf{A}_4$ are known. The distances of three points $\mathbf{B}_1, \mathbf{B}_2$, and $\mathbf{B}_3$, namely, $l_{B_1B_2}$, $l_{B_1B_3}$, and $l_{B_2B_3}$, are known. The lengths of all six legs, l_{B_1P}, l_{B_1Q}, l_{B_2U}, l_{B_2V}, $l_{B_3A_3}$, and $l_{B_3A_4}$, are known.

For the triangle $\mathbf{PB}_1\mathbf{Q}$ in Figure 2, the movement locus of the point $\mathbf{B}_1$ is a circle with its center at $\mathbf{A}_1$ and $l_{B_1A_1}$ as its radius after removing the coupling of the joint $\mathbf{B}_1$ and the moving platform, as shown in Figure 3. Thus, the constraint of the spherical joints $\mathbf{P}$ and $\mathbf{Q}$ with respect to $\mathbf{B}_1$ is equivalent to that of the revolute joint $\mathbf{A}_1$ with respect to $\mathbf{B}_1$. Translating the point $\mathbf{A}_1$ to the origin of the frame O-XYZ, and rotating the triangle $\mathbf{PB}_1\mathbf{Q}$ with $-\beta_1$, $-\alpha_1$, and $-\theta_1$ about the axes Y, X, and Z, the result is that the point $\mathbf{B}_1$ locates the axis X and $\mathbf{QP}$ overlaps the axis Z, as shown in Figure 4.

Now, according to (8), the point $\mathbf{B}_1$ and the vector $\mathbf{QP}$ in $\mathbb{R}^{3+1,1}$ can be written as follows:

$$\mathbf{B}_{1c} = \mathbf{M}_1\mathbf{B}'_{1c}\mathbf{M}_1^*, \tag{11}$$

$$\begin{aligned}\mathbf{QP}_c &= \mathbf{QP} + \frac{1}{2}\mathbf{QP}^2 e_0 + e_\infty \\ &= \mathbf{R}_y(\beta_1)\,\mathbf{R}_x(\alpha_1)\,\mathbf{QP}'_c\mathbf{R}_x^*(\alpha_1)\,\mathbf{R}_y^*(\beta_1),\end{aligned} \tag{12}$$

where

$$
\begin{aligned}
\mathbf{B}_{1c}' &= l_1 e_1 + \frac{1}{2} l_1^2 e_\infty + e_0, \\
\mathbf{M}_1 &= \mathbf{T}(\mathbf{A}_1)\mathbf{R}_y(\beta_1)\mathbf{R}_x(\alpha_1)\mathbf{R}_z(\theta_1), \\
\mathbf{R}_z(\theta_1) &= \cos\frac{\theta_1}{2} - e_3 \sin\frac{\theta_1}{2}, \\
\mathbf{R}_x(\alpha_1) &= \cos\frac{\alpha_1}{2} - e_1 \sin\frac{\alpha_1}{2}, \\
\mathbf{R}_y(\beta_1) &= \cos\frac{\beta_1}{2} - e_2 \sin\frac{\beta_1}{2}, \\
\mathbf{T}(\mathbf{A}_1) &= 1 - \left(A_{1x}e_1 + A_{1y}e_2 + A_{1y}e_3\right)e_\infty, \\
\mathbf{QP} &= QP_x e_1 + QP_y e_2 + QP_z e_3, \\
\mathbf{QP}_c' &= l_{QP} e_3 + \frac{1}{2} l_{QP}^2 e_\infty + e_0,
\end{aligned}
\tag{13}
$$

and l_1 denotes $l_{B_1A_1}$.

According to (12), α_1 and β_1 can be expressed as

$$a_1 = \sin^{-1}\left(\frac{-QP_y}{l_{QP}}\right), \tag{14}$$

$$\beta_1 = 2\tan^{-1}\left(\frac{QP_x}{l_{QP}\cos\alpha_1 + QP_z}\right). \tag{15}$$

In Figure 2, let $\gamma_1 = \angle B_1PQ$. According to the law of cosines, γ_1 can be expressed as

$$\cos\gamma_1 = \frac{l_{B_1P}^2 + l_{QP}^2 - l_{B_1Q}^2}{2 l_{B_1P} l_{QP}}. \tag{16}$$

Similar to the solution process of $\mathbf{B}_{1c}$, $\mathbf{B}_{2c}$ can also be obtained by

$$\mathbf{B}_{2c} = \mathbf{M}_2 \mathbf{B}_{2c}' \mathbf{M}_2^*, \tag{17}$$

where

$$\mathbf{B}_{2c}' = l_2 e_1 + \frac{1}{2} l_2^2 e_\infty + e_0, \tag{18}$$

$$\mathbf{M}_2 = \mathbf{T}(\mathbf{A}_2)\mathbf{R}_y(\beta_2)\mathbf{R}_x(\alpha_2)\mathbf{R}_z(\theta_2), \tag{19}$$

and l_2 denotes $l_{B_2A_2}$.

According to (4), in $\mathbb{R}^{3+1,1}$ $\mathbf{A}_3$ and $\mathbf{A}_4$ can be written:

$$
\begin{aligned}
\mathbf{A}_{3c} &= \mathbf{A}_3 + \frac{1}{2}\mathbf{A}_3^2 e_\infty + e_0, \\
\mathbf{A}_{4c} &= \mathbf{A}_4 + \frac{1}{2}\mathbf{A}_4^2 e_\infty + e_0.
\end{aligned}
\tag{20}
$$

According to (9), the distance between $\mathbf{B}_1$ and $\mathbf{B}_2$ is written:

$$\mathbf{B}_{1c}\cdot\mathbf{B}_{2c} = -\frac{1}{2}\left\|\mathbf{B}_1 - \mathbf{B}_2\right\|^2 = -\frac{1}{2} l_{B_1B_2}^2 = -\frac{1}{2} r_3^2, \tag{21}$$

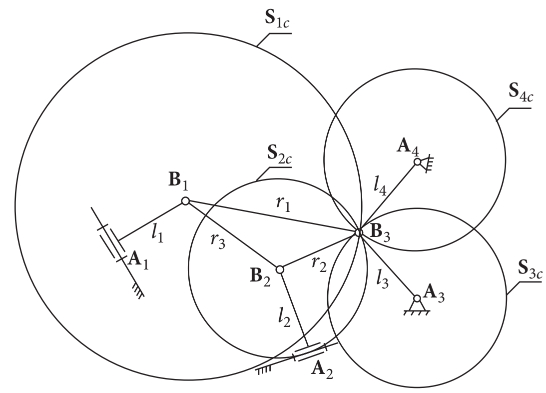

Figure 5: Intersection of conformal geometric spheres.

where r_3 denotes $l_{B_1B_2}$. Substituting (11) and (17) into (21) and rewriting it, we have

$$
\begin{aligned}
& l_1 l_2 \left[c(\beta_1-\beta_2)c\theta_1 c\theta_2 \right. \\
&\quad + s(\beta_1-\beta_2)(s\alpha_1 s\theta_1 c\theta_2 - s\alpha_2 c\theta_1 s\theta_2) \\
&\quad \left. + (c(\beta_1-\beta_2)s\alpha_1 s\alpha_2 + c\alpha_1 c\alpha_2)s\theta_1 s\theta_2\right] + \frac{1}{2}\left(r_3^2\right. \\
&\quad \left. - l_1^2 - l_2^2\right) = 0,
\end{aligned}
\tag{22}
$$

where $c(\beta_1-\beta_2)$, $s(\beta_1-\beta_2)$, $c\alpha_1$, $s\alpha_1$, $s\theta_1$, and $c\theta_1$ denote $\cos(\beta_1-\beta_2)$, $\sin(\beta_1-\beta_2)$, $\cos\alpha_1$, $\sin\alpha_1$, $\sin\theta_1$, and $\cos\theta_1$, respectively.

Letting $r_1 = l_{B_1B_3}$, $r_2 = l_{B_2B_3}$, $l_3 = l_{B_3A_3}$, and $l_4 = l_{B_3A_4}$, from Figure 5 we can see that the joint $\mathbf{B}_3$ is on the intersection of four spheres, that is, the sphere $\mathbf{S}_{1c}$ with its center at $\mathbf{B}_1$ and r_1 as its radius, the sphere $\mathbf{S}_{2c}$ with its center at $\mathbf{B}_2$ and r_2 as its radius, the sphere $\mathbf{S}_{3c}$ with its center at $\mathbf{A}_3$ and l_3 as its radius, and the sphere $\mathbf{S}_{4c}$ with its center at $\mathbf{A}_4$ and l_4 as its radius. Therefore, $\mathbf{B}_3$ can be expressed as

$$\mathbf{B}_{3c'} = \mathbf{S}_{1c}\wedge\mathbf{S}_{2c}\wedge\mathbf{S}_{3c}\wedge\mathbf{S}_{4c}, \tag{23}$$

where

$$
\begin{aligned}
\mathbf{S}_{1c} &= \mathbf{B}_{1c} - \frac{1}{2} r_1^2 e_\infty, \\
\mathbf{S}_{2c} &= \mathbf{B}_{2c} - \frac{1}{2} r_2^2 e_\infty, \\
\mathbf{S}_{3c} &= \mathbf{A}_{3c} - \frac{1}{2} l_3^2 e_\infty, \\
\mathbf{S}_{4c} &= \mathbf{A}_{4c} - \frac{1}{2} l_4^2 e_\infty.
\end{aligned}
\tag{24}
$$

Expanding (23) yields

$$\mathbf{B}_{3c'} = W_1 e_1 + W_2 e_2 + W_3 e_3 + W_4 e_\infty + W_5 e_0, \tag{25}$$

where

$$
\begin{aligned}
W_i &= {}^i a_1 s\theta_1 s\theta_2 + {}^i a_2 s\theta_1 c\theta_2 + {}^i a_3 c\theta_1 s\theta_2 + {}^i a_4 c\theta_1 c\theta_2 \\
&\quad + {}^i a_5 s\theta_1 + {}^i a_6 s\theta_1 + {}^i a_7 s\theta_2 + {}^i a_8 c\theta_2 + {}^i a_9, \\
&\qquad i = 1,\ldots,5.
\end{aligned}
\tag{26}
$$

In the above polynomials, the coefficients ${}^{i}a_4 = 0$ $(i = 1, 3, 5)$ and ${}^{i}a_j$ $(i = 1, \ldots, 5;\ j = 1, \ldots, 9)$ are reported in Appendix.

From (4), the expression of $\mathbf{B}_{3c}$ is written as

$$\mathbf{B}_{3c} = B_{3x}e_1 + B_{3y}e_2 + B_{3z}e_3 + \frac{1}{2}\left(B_{3x}^2 + B_{3y}^2 + B_{3z}^2\right)e_\infty + e_0. \tag{27}$$

According to (25), $\mathbf{B}_{3c}$ can be obtained by

$$\mathbf{B}_{3c} = \frac{W_1}{W_5}e_1 + \frac{W_2}{W_5}e_2 + \frac{W_3}{W_5}e_3 + \frac{W_4}{W_5}e_\infty + e_0. \tag{28}$$

According to (9), we have

$$\mathbf{B}_{3c} \cdot \mathbf{B}_{3c} = W_1^2 + W_2^2 + W_3^2 - 2W_4W_5 = 0. \tag{29}$$

Substituting for the sine and cosine the well-known expressions

$$\cos\theta_i = \frac{1 - t_i^2}{1 + t_i^2}, \qquad \sin\theta_i = \frac{2t_i}{1 + t_i^2}, \tag{30}$$

where $t_i = \tan(\theta_i/2)$, $i = 1, 2$, (22) and (29) can be rewritten as follows:

$$\begin{gathered} n_1t_1^2t_2^2 + n_2t_1^2t_2 + n_3t_1t_2^2 + n_4t_1t_2 + n_5t_1^2 + n_6t_2^2 + n_7t_1 + n_8t_2 + n_0 = 0, \\ \sum_{\substack{i=0,\ldots,4 \\ j=0,\ldots,4}} a_{ij}t_1^it_2^j = 0, \end{gathered} \tag{31}$$

where

$$\begin{aligned} n_0 &= n_1 = r_3^2 - l_1^2 - l_2^2 + 2l_1l_2c\left(\beta_1 - \beta_2\right), \\ n_2 &= -n_8 = 4l_1l_2s\alpha_2s\left(\beta_1 - \beta_2\right), \\ n_3 &= -n_7 = -4l_1l_2s\alpha_1s\left(\beta_1 - \beta_2\right), \\ n_4 &= 8l_1l_2\left(c\alpha_1c\alpha_2 + s\left(\beta_1 - \beta_2\right)s\alpha_1s\alpha_2\right), \\ n_5 &= n_6 = r_3^2 - l_1^2 - l_2^2 - 2l_1l_2c\left(\beta_1 - \beta_2\right). \end{aligned} \tag{32}$$

and a_{ij} are real constants that depend only on ${}^{i}a_j$ $(i = 1, \ldots, 5;\ j = 1, \ldots, 9)$.

Equations (31) are the kinematics constraint equations in the FDA of the 6-3SPS.

4. Solution Procedure

Equations (31) represent a system of two equations in two unknown variables, t_1 and t_2. t_2 can be eliminated between them obtaining one equation in the only unknown, t_1. Equations (31) can be rearranged as follows:

$$\begin{gathered} M_1t_2^2 + M_2t_2 + M_3 = 0, \\ G_1t_2^4 + G_2t_2^3 + G_3t_2^2 + G_4t_2 + G_5 = 0, \end{gathered} \tag{33}$$

where

$$\begin{aligned} M_1 &= n_1t_1^2 + n_3t_1 + n_6, \\ M_2 &= n_2t_1^2 + n_4t_1 + n_8, \\ M_3 &= n_5t_1^2 + n_7t_1 + n_0, \\ G_1 &= \sum_{i=0,\ldots,4} a_{i4}t_1^i, \\ G_2 &= \sum_{i=0,\ldots,4} a_{i3}t_1^i, \\ G_3 &= \sum_{i=0,\ldots,4} a_{i2}t_1^i, \\ G_4 &= \sum_{i=0,\ldots,4} a_{i1}t_1^i, \\ G_5 &= \sum_{i=0,\ldots,4} a_{i0}t_1^i. \end{aligned} \tag{34}$$

Constructing the Sylvester resultant, the eliminant of (33) is the following:

$$\begin{vmatrix} 0 & G_1 & G_2 & G_3 & G_4 & G_5 \\ G_1 & G_2 & G_3 & G_4 & G_5 & 0 \\ 0 & 0 & 0 & M_1 & M_2 & M_3 \\ 0 & 0 & M_1 & M_2 & M_3 & 0 \\ 0 & M_1 & M_2 & M_3 & 0 & 0 \\ M_1 & M_2 & M_3 & 0 & 0 & 0 \end{vmatrix} = 0. \tag{35}$$

Expanding (35), a 16th-order polynomial equation with the variable t_1 can be obtained as follows:

$$\sum_{i=0}^{16} b_it_1^i = 0, \tag{36}$$

where b_i are real constants that depend only on input data. Therefore 16 real and complex solutions for t_1 are possible.

For $t = t_1$, (33) are indeed algebraic equations in the only unknown t_2, and these have a common root, t_2, whose value can be found by equating to zero the first-degree greatest common divisor (GCD) of the polynomials on the left-hand sides of (33).

Hence, a unique solution (t_1, t_2) of (31) is derived for every solution t_1 of (36) and, consequently, through (11), (17), and (28) a unique location of $\mathbf{B}_1$, $\mathbf{B}_2$, and $\mathbf{B}_3$ is obtained. Therefore, the FDA of the 6-3SPS provides 16 solutions in the field of complex numbers.

5. Numerical Verification

In order to validate the new algorithm, the input data of a numerical example are given in Table 1. The data are equivalent to the data of the example discussed in [4]. All 16 sets of output data obtained by the proposed method agree with those given in [4]; the outputs of four real solutions are listed in Table 2.

Table 1: Input data.

W_a	$\mathbf{P}$	$\mathbf{Q}$	$\mathbf{U}$	$\mathbf{V}$	$\mathbf{A}_3$	$\mathbf{A}_4$
X	50	−25	80	−50	48	36
Y	0	−2	20	−20	15	31
Z	100	40	50	70	68	−93
Distance between three points of $\mathbf{B}_i$ $(i = 1, 2, 3)$	$l_{B_1B_3} = 190, l_{B_2B_3} = 135, l_{B_1B_2} = 141$					
Length of six legs	$l_{B_1P} = 76, l_{B_1Q} = 160, l_{B_2U} = 139, l_{B_2V} = 55, l_{B_3A_3} = 128, l_{B_3A_4} = 217$					

Table 2: Four real solutions from 16 sets of output solutions.

i	t_1	t_2		B_1	B_2	B_3
			X	79.5353	−26.0942	−70.9222
1	−0.6258	−1.0743	Y	−45.8809	−68.9457	54.3104
			Z	152.9018	62.3955	94.3853
			X	68.8676	−47.0215	21.2493
2	−0.3717	0.442	Y	−33.0062	21.0886	137.0612
			Z	165.8073	106.4396	95.7395
			X	82.5389	−48.8261	14.0067
3	0.9609	0.240	Y	51.0783	24.7276	−108.359
			Z	145.9154	101.9852	71.8847
			X	90.9016	−40.7763	−6.7822
4	0.7363	0.5102	Y	53.3944	6.2851	−100.5172
			Z	135.3847	117.4237	74.218

6. Conclusions

This paper presents a closed-form solution for the FDA of the 6-3SPS using the CGA. The 16th-order input-output equation is derived from two equations with two variables by using the Sylvester resultant method. The two equations are derived by using a computer algebra symbolic calculation. The number of solutions agrees with that of [4]. The procedure is unique in that the final resultant matrix is in the symbolic form of geometric parameters. This symbolic formulation not only can provide insights into the nature of the problem, but also greatly simplifies the implementation of computer programming. To compute solutions for a new set of structural parameter and limb actuation data, one only needs to substitute numerical values of these parameters into the symbolic resultant matrix without requiring any algebraic manipulation. Compared with the previously reported method, the novelty of the proposed method lies in that the problem is modeled based on the geometric characteristics and is solved based on single elimination; as a result, the solution procedure is simpler, more efficient, and more readily programmed. The computation time using the new solution presented in this paper is about 0.15 s in Maple 16 running on a PC with Intel Pentium Dual-Core E5800 3.2 GHz and 2 GB Ram. However, employing the same computer system and software, the computation time using the method developed in [4] is 0.3 s.

Appendix

$$
\begin{aligned}
{}^1a_1 &= -\frac{1}{2} l_1 l_2 \left(K_3 - K_4 - l_3^2 + l_4^2\right) (c\alpha_2 c\beta_1 s\alpha_1 \\
&\quad - c\alpha_1 c\beta_2 s\alpha_2), \\
{}^1a_2 &= -\frac{1}{2} l_1 l_2 \left(K_3 - K_4 - l_3^2 + l_4^2\right) c\alpha_1 s\beta_2, \\
{}^1a_3 &= \frac{1}{2} l_1 l_2 \left(K_3 - K_4 - l_3^2 + l_4^2\right) c\alpha_2 s\beta_1, \\
{}^1a_5 &= \frac{1}{2} l_1 \left[\left(K_4 - l_4^2 - l_2^2 + r_2^2\right) A_{3z}\right. \\
&\quad \left. + \left(l_3^2 + l_2^2 - K_3 - r_2^2\right) A_{4z}\right] c\alpha_1 + \frac{1}{2} \\
&\quad \cdot l_1 \left[\left(l_4^2 - K_4 + l_2^2 - r_2^2\right) A_{3y}\right. \\
&\quad \left. + \left(K_3 - l_3^2 - l_2^2 + r_2^2\right) A_{4y}\right] c\beta_1 s\alpha_1, \\
{}^1a_6 &= \frac{1}{2} l_1 \left[\left(K_4 - l_4^2 - l_2^2 + r_2^2\right) A_{3y}\right. \\
&\quad \left. + \left(l_3^2 + l_2^2 - K_3 - r_2^2\right) A_{4y}\right] s\beta_1, \\
{}^1a_7 &= \frac{1}{2} l_2 \left[\left(l_4^2 - K_4 + l_1^2 - r_1^2\right) A_{3z}\right.
\end{aligned}
$$

$$+\left(K_3-l_3^2-l_1^2+r_1^2\right)A_{4z}\Big]c\alpha_2+\frac{1}{2}$$
$$\cdot l_2\Big[\left(K_4-l_4^2-l_1^2+r_1^2\right)A_{3y}$$
$$+\left(l_3^2-K_3+l_1^2-r_1^2\right)A_{4y}\Big]c\beta_2 s\alpha_2,$$

$${}^1a_8=-\frac{1}{2}l_2\Big[\left(K_4-l_4^2-l_1^2+r_1^2\right)A_{3y}$$
$$+\left(l_3^2+l_1^2-K_3-r_1^2\right)A_{4y}\Big]s\beta_2,$$

$${}^1a_9=-\frac{1}{2}\left(A_{3z}A_{4y}-A_{3y}A_{4z}\right)\left(l_1^2-l_2^2-r_1^2+r_2^2\right),$$

$${}^2a_1=-\frac{1}{2}l_1l_2\left(K_3-K_4-l_3^2+l_4^2\right)s\alpha_1 s\alpha_2 s\left(\beta_1-\beta_2\right),$$

$${}^2a_2=\frac{1}{2}l_1l_2\left(K_3-K_4-l_3^2+l_4^2\right)s\alpha_1 c\left(\beta_1-\beta_2\right),$$

$${}^2a_3=-\frac{1}{2}l_1l_2\left(K_3-K_4-l_3^2+l_4^2\right)s\alpha_2 c\left(\beta_1-\beta_2\right),$$

$${}^2a_4=-\frac{1}{2}l_1l_2\left(K_3-K_4-l_3^2+l_4^2\right)s\left(\beta_1-\beta_2\right),$$

$${}^2a_5=\frac{1}{2}l_1\Big[\left(K_4-l_4^2-l_2^2+r_2^2\right)A_{3x}$$
$$+\left(l_3^2+l_2^2-K_3-r_2^2\right)A_{4x}\Big]s\alpha_1 c\beta_1$$
$$+\frac{1}{2}l_1\Big[\left(l_4^2-K_4+l_2^2-r_2^2\right)A_{3z}$$
$$+\left(K_3-l_3^2-l_2^2+r_2^2\right)A_{4y}\Big]s\alpha_1 s\beta_1,$$

$${}^2a_6=-\frac{1}{2}l_1\Big[\left(K_4-l_4^2-l_2^2+r_2^2\right)A_{3z}$$
$$+\left(l_3^2+l_2^2-K_3-r_2^2\right)A_{4z}\Big]c\beta_1-\frac{1}{2}$$
$$\cdot l_1\Big[\left(K_4-l_4^2-l_2^2+r_2^2\right)A_{3x}$$
$$+\left(l_3^2+l_2^2-K_3-r_2^2\right)A_{4x}\Big]s\beta_1,$$

$${}^2a_7=\frac{1}{2}l_2\Big[\left(l_4^2-K_4+l_1^2-r_1^2\right)A_{3x}$$
$$+\left(K_3-l_3^2-l_1^2+r_1^2\right)A_{4x}\Big]s\alpha_2 c\beta_2+\frac{1}{2}$$
$$\cdot l_2\Big[\left(K_4-l_4^2-l_1^2+r_1^2\right)A_{3z}$$
$$+\left(l_3^2-K_3+l_1^2-r_1^2\right)A_{4z}\Big]s\alpha_2 s\beta_2,$$

$${}^2a_8=\frac{1}{2}l_2\Big[\left(K_4-l_4^2-l_1^2+r_1^2\right)A_{3z}$$
$$+\left(l_3^2+l_1^2-K_3-r_1^2\right)A_{4z}\Big]c\beta_2+\frac{1}{2}$$
$$\cdot l_2\Big[\left(K_4-l_4^2-l_1^2+r_1^2\right)A_{3x}$$
$$+\left(l_3^2+l_1^2-K_3-r_1^2\right)A_{4x}\Big]c\beta_2,$$

$${}^2a_9=\frac{1}{2}\left(A_{3z}A_{4x}-A_{3x}A_{4z}\right)\left(l_1^2-l_2^2-r_1^2+r_2^2\right),$$

$${}^3a_1=-\frac{1}{2}l_1l_2\left(K_3-K_4-l_3^2+l_4^2\right)\left(c\alpha_2 s\alpha_1 s\beta_1\right.$$
$$\left.-c\alpha_1 s\alpha_2 s\beta_2\right),$$

$${}^3a_2=-\frac{1}{2}l_1l_2\left(K_3-K_4-l_3^2+l_4^2\right)c\alpha_1 c\beta_2,$$

$${}^3a_3=\frac{1}{2}l_1l_2\left(K_3-K_4-l_3^2+l_4^2\right)c\alpha_2 c\beta_1,$$

$${}^3a_5=-\frac{1}{2}l_1\Big[\left(K_4-l_4^2-l_2^2+r_2^2\right)A_{3x}$$
$$+\left(l_3^2+l_2^2-K_3-r_2^2\right)A_{4x}\Big]c\alpha_1-\frac{1}{2}$$
$$\cdot l_1\Big[\left(l_4^2-K_4+l_2^2-r_2^2\right)A_{3y}$$
$$+\left(K_3-l_3^2-l_2^2+r_2^2\right)A_{4y}\Big]s\alpha_1 s\beta_1,$$

$${}^3a_6=\frac{1}{2}l_1\Big[\left(K_4-l_4^2-l_2^2+r_2^2\right)A_{3y}$$
$$+\left(l_3^2+l_2^2-K_3-r_2^2\right)A_{4y}\Big]c\beta_1,$$

$${}^3a_7=-\frac{1}{2}l_2\Big[\left(l_4^2-K_4+l_1^2-r_1^2\right)A_{3x}$$
$$+\left(K_3-l_3^2-l_1^2+r_1^2\right)A_{4x}\Big]c\alpha_2-\frac{1}{2}$$
$$\cdot l_2\Big[\left(K_4-l_4^2-l_1^2+r_1^2\right)A_{3y}$$
$$+\left(l_3^2-K_3+l_1^2-r_1^2\right)A_{4y}\Big]s\alpha_2 s\beta_2,$$

$${}^3a_8=-\frac{1}{2}l_2\Big[\left(K_4-l_4^2-l_1^2+r_1^2\right)A_{3y}$$
$$+\left(l_3^2+l_1^2-K_3-r_1^2\right)A_{4y}\Big]c\beta_2,$$

$${}^3a_9=-\frac{1}{2}\left(A_{3y}A_{4x}-A_{3x}A_{4y}\right)\left(l_1^2-l_2^2-r_1^2+r_2^2\right),$$

$${}^4a_1=l_1l_2c\alpha_2 s\alpha_1\left(K_4A_{3x}-l_4^2A_{3x}-K_3A_{4x}+l_3^2A_{4x}\right)$$
$$\cdot c\beta_1-l_1l_2c\alpha_2 s\alpha_1\left(K_4A_{3z}-l_4^2A_{3z}-K_3A_{4z}\right.$$
$$\left.+l_3^2A_{4z}\right)s\beta_1+l_1l_2s\alpha_2\left(K_4A_{3y}-l_4^2A_{3y}-K_3A_{4y}\right.$$
$$\left.+l_3^2A_{4y}\right)s\alpha_1 s\left(\beta_1-\beta_2\right)+l_1l_2s\alpha_2 c\alpha_1\left(K_4A_{3z}\right.$$
$$\left.-l_4^2A_{3z}-K_3A_{4z}+l_3^2A_{4z}\right)s\beta_2$$
$$-l_1l_2s\alpha_2 c\alpha_1\left(K_4A_{3x}-l_4^2A_{3x}-K_3A_{4x}+l_3^2A_{4x}\right)$$
$$\cdot c\beta_2,$$

$${}^4a_2=l_1l_2\left(-K_4A_{3y}+l_4^2A_{3y}+K_3A_{4y}-l_3^2A_{4y}\right)$$
$$\cdot s\alpha_1 c\left(\beta_1-\beta_2\right)+l_1l_2c\alpha_1\left(K_4A_{3z}-l_4^2A_{3z}\right.$$

$$- K_3 A_{4z} + l_3^2 A_{4z}\right) c\beta_2 + l_1 l_2 c\alpha_1 \left(K_4 A_{3x} - l_4^2 A_{3x}$$

$$- K_3 A_{4x} + l_3^2 A_{4x}\right) s\beta_2,$$

$$^4a_3 = l_1 l_2 \left(K_4 A_{3y} - l_4^2 A_{3y} - K_3 A_{4y} + l_3^2 A_{4y}\right)$$

$$\cdot s\alpha_2 c\left(\beta_1 - \beta_2\right) - l_1 l_2 c\alpha_2 \left(K_4 A_{3z} - l_4^2 A_{3z}\right.$$

$$- K_3 A_{4z} + l_3^2 A_{4z}\right) c\beta_1 - l_1 l_2 c\alpha_2 \left(K_4 A_{3x} - l_4^2 A_{3x}$$

$$- K_3 A_{4x} + l_3^2 A_{4x}\right) s\beta_1,$$

$$^4a_4 = l_1 l_2 \left(K_4 A_{3y} - l_4^2 A_{3y} - K_3 A_{4y} + l_3^2 A_{4y}\right) s\left(\beta_1\right.$$

$$\left.- \beta_2\right),$$

$$^4a_5 = -l_1 \left(l_2^2 - r_2^2\right)\left(A_{3z}A_{4x} - A_{3x}A_{4z}\right) c\alpha_1 - l_1 \left(l_2^2\right.$$

$$\left.- r_2^2\right) s\alpha_1 \left[\left(-A_{3y}A_{4x} + A_{3x}A_{4y}\right) c\beta_1\right.$$

$$\left.+ \left(-A_{3z}A_{4y} + A_{3y}A_{4z}\right) s\beta_1\right],$$

$$^4a_6 = l_1 \left(l_2^2 - r_2^2\right)\left[\left(A_{3z}A_{4y} - A_{3y}A_{4z}\right) c\beta_1\right.$$

$$\left.+ \left(-A_{3y}A_{4x} + A_{3x}A_{4y}\right) s\beta_1\right],$$

$$^4a_7 = l_2 \left(l_1^2 - r_1^2\right)\left(A_{3z}A_{4x} - A_{3x}A_{4z}\right) c\alpha_2 + l_2 \left(l_1^2\right.$$

$$\left.- r_1^2\right) s\alpha_2 \left[\left(-A_{3y}A_{4x} + A_{3x}A_{4y}\right) c\beta_2\right.$$

$$\left.+ \left(-A_{3z}A_{4y} + A_{3y}A_{4z}\right) s\beta_2\right],$$

$$^4a_8 = -l_2 \left(l_1^2 - r_1^2\right)\left[\left(A_{3z}A_{4y} - A_{3y}A_{4z}\right) c\beta_2\right.$$

$$\left.+ \left(-A_{3y}A_{4x} + A_{3x}A_{4y}\right) s\beta_2\right],$$

$$^4a_9 = 0,$$

$$^5a_1 = -\frac{1}{2} l_1 l_2 \left(c\alpha_2 s\alpha_1 \left(A_{3x} - A_{4x}\right) c\beta_1\right.$$

$$\left.+ c\alpha_2 s\alpha_1 \left(-A_{3z} + A_{4z}\right) s\beta_1\right) - \frac{1}{2} l_1 l_2 s\alpha_2 \left(A_{3y}\right.$$

$$\left.- A_{4y}\right) s\alpha_1 s\left(\beta_1 - \beta_2\right) - \frac{1}{2}$$

$$\cdot l_1 l_2 s\alpha_2 c\alpha_1 \left[\left(-A_{3x} + A_{4x}\right) c\beta_2\right.$$

$$\left.+ \left(A_{3z} - A_{4z}\right) s\beta_2\right],$$

$$^5a_2 = \frac{1}{2} l_1 l_2 \left(A_{3y} - A_{4y}\right) c\left(\beta_1 - \beta_2\right) s\alpha_1 + \frac{1}{2}$$

$$\cdot l_1 l_2 c\alpha_1 \left(-A_{3z} + A_{4z}\right) c\beta_2 + \frac{1}{2} l_1 l_2 c\alpha_1 \left(-A_{3x}\right.$$

$$\left.+ A_{4x}\right) s\beta_2,$$

$$^5a_3 = \frac{1}{2} l_1 l_2 \left(-A_{3y} + A_{4y}\right) c\left(\beta_1 - \beta_2\right) s\alpha_2 + \frac{1}{2}$$

$$\cdot l_1 l_2 c\alpha_2 \left(A_{3z} - A_{4z}\right) c\beta_1 + \frac{1}{2} l_1 l_2 c\alpha_2 \left(A_{3x} - A_{4x}\right)$$

$$\cdot s\beta_1,$$

$$^5a_5 = -\frac{1}{2} l_1 l_2 \left(A_{3y} - A_{4y}\right) s\left(\beta_1 - \beta_2\right),$$

$$^5a_6 = \frac{1}{2} l_1 \left(A_{3z}A_{4x} - A_{3x}A_{4z}\right) c\alpha_1 + \frac{1}{2}$$

$$\cdot l_1 s\alpha_1 \left(-A_{3y}A_{4x} + A_{3x}A_{4y}\right) c\beta_1 + \frac{1}{2}$$

$$\cdot l_1 s\alpha_1 \left(-A_{3z}A_{4y} + A_{3y}A_{4z}\right) s\beta_1,$$

$$^5a_7 = -\frac{1}{2} l_1 \left(A_{3z}A_{4y} - A_{3y}A_{4z}\right) c\beta_1 - \frac{1}{2}$$

$$\cdot l_1 \left(-A_{3y}A_{4x} + A_{3x}A_{4y}\right) s\beta_1,$$

$$^5a_8 = \frac{1}{2} l_2 \left(-A_{3z}A_{4x} + A_{3x}A_{4z}\right) c\alpha_2 + \frac{1}{2}$$

$$\cdot l_2 s\alpha_2 \left(A_{3y}A_{4x} - A_{3x}A_{4y}\right) c\beta_2 + \frac{1}{2}$$

$$\cdot l_2 s\alpha_2 \left(A_{3z}A_{4y} - A_{3y}A_{4z}\right) s\beta_2,$$

$$^5a_9 = \frac{1}{2} l_2 \left(A_{3z}A_{4y} - A_{3y}A_{4z}\right) c\beta_2 + \frac{1}{2} l_2 \left(-A_{3y}A_{4x}\right.$$

$$\left.+ A_{3x}A_{4y}\right) s\beta_2, \tag{A.1}$$

where, $K_3 = A_{3x}^2 + A_{3y}^2 + A_{3z}^2$, $K_4 = A_{4x}^2 + A_{4y}^2 + A_{4z}^2$.

Notations

$\mathbb{R}^3$: Euclidean space
$\mathbb{R}^{3+1,1}$: Conformal geometric space
c: The subscript c represents the vector in $\mathbb{R}^{3+1,1}$
$*$: The superscript $*$ represents conjugation
$\mathbf{x}$: A point expressed in $\mathbb{R}^3$; that is, $\mathbf{x} = x_1 e_1 + x_2 e_2 + x_3 e_3$.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This research is sponsored by the National Natural Science Foundation of China (Grant no. 51375059), the Special Fund for Agro-Scientific Research in the Public Interest of China (Grant no. 201313009-06), and the National Key Technology R&D Program of the Ministry of Science and Technology of China (Grant no. 2013BAD17B06).

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