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Research Article

Multiple Periodic Solutions for a Class of Second-Order Neutral Impulsive Functional Differential Equations

Jingli Xie,[1] Zhiguo Luo,[2] and Yuhua Zeng[3]

[1] *College of Mathematics and Statistics, Jishou University, Jishou, Hunan 416000, China*
[2] *Department of Mathematics, Hunan Normal University, Changsha, Hunan 410081, China*
[3] *Department of Mathematics, Hunan First Normal University, Changsha, Hunan 410205, China*

Correspondence should be addressed to Jingli Xie; xiejingli721124@163.com

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In this paper, we study a class of second-order neutral impulsive functional differential equations. Under certain conditions, we establish the existence of multiple periodic solutions by means of critical point theory and variational methods. We propose an example to illustrate the applicability of our result.

1. Introduction and Main Results

In this paper we consider a class of second-order neutral impulsive functional differential equations

$$\begin{aligned} & u''(t-\tau) - u(t-\tau) \\ & \quad + \lambda f(t, u(t), u(t-\tau), u(t-2\tau)) = 0, \\ & \qquad\qquad t \neq t_j,\ t \in J = [0, 2k\tau], \\ & \Delta u'(t_j) = I_j(u(t_j)), \quad j = 1, 2, \ldots, l, \\ & u(0) - u(2k\tau) = u'(0) - u'(2k\tau) = 0, \end{aligned} \tag{1}$$

where $f \in C(\mathbb{R}^4, \mathbb{R})$, $I_j \in C(\mathbb{R}, \mathbb{R})$, and $0 = t_0 < t_1 < t_2 < \cdots < t_l < t_{l+1} = 2k\tau$. The operator Δ is defined as $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-)$, where $u'(t_j^+)(u'(t_j^-))$ denotes the right-hand (left-hand) limit of u' at t_j. $\lambda \in \mathbb{R}$, τ is a constant with $\tau > 0$ and k is a given positive integer.

The necessity to study delay differential equations is due to the fact that these equations are useful mathematical tools in modeling many real processes and phenomena studied in biology, medicine, chemistry, physics, engineering, economics, and so forth [1, 2].

On the other hand, impulsive differential equation not only is richer than the corresponding theory of differential equations but also represents a more natural framework for mathematical modeling of real world phenomena. People generally consider impulses in positions u and u' for the second-order differential equation $u'' = f(t, u, u')$. However it is well known that in the motion of spacecraft instantaneous impulses depend on the position which result in jump discontinuities in velocity, with no change in position.

Thus, it is more realistic to consider the case of combined effects: impulses and time delays. This motivates us to consider neutral impulsive functional differential system (1).

The existence of periodic solutions of delay differential equations has been focused on by many researchers [3–6]. Several available approaches to tackle them include Lyapunov method, Fourier analysis method, fixed point theory, and coincidence degree theory [7–10]. Recently, some researchers have studied the existence of solutions for delay differential equations via variational methods [11–13]. In recent years, some researchers, by using critical point theory, have studied the existence of solutions for boundary value problems, periodic solutions, and homoclinic orbits of impulsive differential systems [14–19].

In this paper, we aim to establish existence of multiple periodic solutions for the second-order neutral impulsive functional differential equation (1) by using critical point theory and variational methods.

For (1) with $I_j = 0$, Shu and Xu [20] obtained the following periodic solutions result.

Theorem A. *Assume that the following conditions are satisfied.*

(H1) $\partial f(t, u_1, u_2, u_3)/\partial t \neq 0$.

(H2) *There exists a function* $F(t, u_1, u_2) \in C^1(\mathbb{R}^3, \mathbb{R})$ *such that*

$$\frac{\partial F(t, u_1, u_2)}{\partial u_2} + \frac{\partial F(t, u_2, u_3)}{\partial u_2} = f(t, u_1, u_2, u_3). \tag{2}$$

(H3) $F(t, u_1, u_2)$ *is* τ*-periodic in* t.

(H4) F *satisfies* $F(t, -u_1, -u_2) = F(t, u_1, u_2)$ *and* $f(t, -u_1, -u_2, -u_3) = -f(t, u_1, u_2, u_3)$.

(H5) $F(t, u_1, u_2) = 0$ *if and only if* $(u_1, u_2) = 0$, $\forall t \in [0, \tau]$.

(H6) $\lim_{|u|\to 0}(F(t, u_1, u_2)/|u|^2) = 1$, *where* $|u| = (|u_1|^2 + |u_2|^2)^{1/2}$, $t \in [0, \tau]$.

(H7) *There exists a constant* $\alpha > 0$ *such that when* $|u_1|^2 + |u_2|^2 > \alpha^2$, $F(t, u_1, u_2) < 0$, $t \in [0, \tau]$.

Moreover, if there exists an integer $m > 0$ *such that* λ *satisfying*

$$\lambda > \frac{m^2\left(\pi^2 + k^2\tau^2\right)}{4k\tau^2}, \tag{3}$$

then the system

$$\begin{aligned} &u''(t-\tau) - u(t-\tau) \\ &\quad + \lambda f(t, u(t), u(t-\tau), u(t-2\tau)) = 0, \\ &u(0) - u(2k\tau) = u'(0) - u'(2k\tau) = 0 \end{aligned} \tag{4}$$

possesses at least $2m$ *nonzero solutions with the period* $2k\tau$.

Our main result is stated as follows.

Theorem 1. *Assume that (H1)–(H7) and the following condition are satisfied.*

(H8) I_j *is odd about* u, *and there exists a constant* $0 \le D < 1$ *such that* $|I_j(u)| \le D|u|$, *where* $j = 1, 2, \ldots, l$.

Moreover, if there exists an integer $m > 0$ *such that*

$$\lambda > \frac{m^2\left(\pi^2 + (1+D)k^2\tau^2\right)}{4k^2\tau^2}, \tag{5}$$

then system (1) admits at least $2m$ *nonzero solutions with the period* $2k\tau$.

Clearly, when $I_j = 0$, Theorem 1 generalizes Theorem A.

Note that the first equation of system (1) is equivalent to the following equation:

$$\begin{aligned} &u''(t-\tau) - u(t-\tau) + \lambda\Big(F_1'(t, u(t-\tau), u(t-2\tau)) \\ &\quad + F_2'(t, u(t), u(t-\tau))\Big) = 0, \end{aligned} \tag{6}$$

where $F_1'(t, u(t-\tau), u(t-2\tau)) = \partial F(t, u(t-\tau), u(t-2\tau))/\partial u(t-\tau)$ and $F_2'(t, u(t), u(t-\tau)) = \partial F(t, u(t), u(t-\tau))/\partial u(t-\tau)$.

The rest of this paper is organized as follows. In Section 2, we present some preliminaries, which will be used to prove our main result. In Section 3 we prove our main result and provide an example to illustrate the applicability of our results.

2. Some Preliminaries

Let

$$\begin{aligned} H^1_{2k\tau} = \big\{&u : \mathbb{R} \to \mathbb{R} \mid u, u' \in L^2(([0, 2k\tau]), \mathbb{R}),\ u(0) \\ &= u(2k\tau),\ u'(0) = u'(2k\tau)\big\}. \end{aligned} \tag{7}$$

Then $H^1_{2k\tau}$ is a separable and reflexive Banach space and the inner product

$$(u, v) = \int_0^{2k\tau} \left(u'(t)v'(t) + u(t)v(t)\right) dt \tag{8}$$

induces the norm

$$\|u\|_{H^1_{2k\tau}} = \left(\int_0^{2k\tau} \left|u'(t)\right|^2 + |u(t)|^2 dt\right)^{1/2}. \tag{9}$$

Definition 2. A function $u \in H^1_{2k\tau}$ is a solution of system (1) if the function u satisfies system (1).

Define a functional φ as

$$\begin{aligned} \varphi(u) = &\frac{1}{2}\int_0^{2k\tau} \left(|u(t)|^2 + \left|u'(t)\right|^2\right) dt \\ &- \lambda \int_0^{2k\tau} F(t, u(t), u(t-\tau))\, dt \\ &+ \sum_{j=1}^{l} \int_0^{u(t_j)} I_j(s)\, ds, \quad u \in H^1_{2k\tau}. \end{aligned} \tag{10}$$

Then φ is Fréchet differentiable at any $u \in H^1_{2k\tau}$. For any $v \in H^1_{2k\tau}$, by a simple calculation, we have

$$\begin{aligned} \varphi'(u)(v) = &\int_0^{2k\tau} \left(u'(t)v'(t) + u(t)v(t)\right) dt \\ &- \lambda \int_0^{2k\tau} \Big(F_1'(t, u(t), u(t-\tau))\, v(t) \\ &+ F_2'(t, u(t), u(t-\tau))\, v(t-\tau)\Big) dt \\ &+ \sum_{j=1}^{l} I_j\left(u\left(t_j\right)\right) v\left(t_j\right). \end{aligned} \tag{11}$$

From (H3), we get

$$\begin{aligned} \varphi'(u)(v) = &\int_0^{2k\tau} \left(-u''(t) + u(t)\right) v(t)\, dt \\ &- \lambda \int_0^{2k\tau} \Big(F_1'(t, u(t), u(t-\tau))\, v(t) \\ &+ F_2'(t, u(t+\tau), u(t))\, v(t)\Big) dt. \end{aligned} \tag{12}$$

Therefore, the corresponding Euler equation of functional φ is

$$\begin{aligned} &u''(t) - u(t) \\ &\quad + \lambda\left(F_1'(t, u(t), u(t-\tau)) + F_2'(t, u(t+\tau), u(t))\right) \\ &= 0. \end{aligned} \tag{13}$$

Note that (6) is equivalent to system (13) and critical points of the functional φ are classical $2k\tau$-periodic solutions of system (1).

Definition 3 (see [21]). Let E be a real reflexive Banach space, and

$$\Sigma = \{A \mid A \subset E \setminus \{0\} \text{ is closed, symmetric set}\}. \tag{14}$$

Define $\gamma : \Sigma \to \mathbb{Z}^+ \cup \{+\infty\}$ as follows:

$$\gamma(A) = \begin{cases} \min\{n \in \mathbb{Z}^+ : \text{ there exists an odd continuous map } \varphi : A \to \mathbb{R}^n \setminus \{0\}\}; \\ 0, \quad \text{if } A = \emptyset; \\ +\infty, \quad \text{if there is no odd continuous map } \varphi : A \to \mathbb{R}^n \setminus \{0\} \text{ for any } n \in \mathbb{Z}^+. \end{cases} \tag{15}$$

Then we say γ is the genus of Σ.

Denote $i_1(\varphi) = \lim_{a\to -0}\gamma(\varphi_a)$ and $i_2(\varphi) = \lim_{a\to -\infty}\gamma(\varphi_a)$, where $\varphi_a = \{u \in E \mid \varphi(u) \le a\}$.

Lemma 4 (see [22]). *Let E be a real Banach space and $\varphi \in C^1(E, \mathbb{R})$ with φ even functional and satisfying the Palais-Smale (PS) condition. Suppose $\varphi(0) = 0$ and*

(i) *if there exist an m-dimensional subspace X of E and a constant $r > 0$ such that*

$$\sup_{u \in X \cap B_r} \varphi(u) < 0, \tag{16}$$

where B_r is an open ball of radius r in E centered at 0, then we have $i_1(\varphi) \ge m$;

(ii) *if there exists j-dimensional subspace V of E such that*

$$\inf_{u \in V^\perp} \varphi(u) > -\infty, \tag{17}$$

then we have $i_2(\varphi) \le j$.

Moreover, if $m \ge j$, then φ possesses at least $2(m - j)$ distinct critical points.

3. Proof of Theorem 1 and an Example

We apply Lemma 4 to finish the proof. Under assumption (H4), it is easy to see that if function u is a solution of system (1), then function $-u$ is also a solution of system (1). Therefore, the solutions of system (1) are a set which is symmetric with respect to the origin in $H^1_{2k\tau}$. It follows directly from (10), (H5), and (H8) that φ is even in u and $\varphi(0) = 0$. The rest of the proof is divided into three steps.

Step 1. We show that the functional φ satisfies assumption (ii) of Lemma 4.

It follows from (H7) that there exists a constant $M > 0$ such that

$$\max_{t\in\mathbb{R}} F(t, u(t), u(t-\tau)) \le \max_{(t,u_1,u_2)\in\Omega} F(t, u_1, u_2) \le M, \tag{18}$$

where $\Omega = [0, \tau] \times [-\alpha, \alpha] \times [-\alpha, \alpha]$. Combining (10) and (18), we get

$$\begin{aligned} \varphi(u) &= \frac{1}{2}\int_0^{2k\tau}\left(|u(t)|^2 + \left|u'(t)\right|^2\right)dt \\ &\quad - \lambda\int_0^{2k\tau} F(t, u(t), u(t-\tau))\,dt \\ &\quad + \sum_{j=1}^{l}\int_0^{u(t_j)} I_j(s)\,ds \\ &\ge \frac{1}{2}\int_0^{2k\tau}\left|u'(t)\right|^2 dt + \frac{1-D}{2}\int_0^{2k\tau}|u(t)|^2\,dt \\ &\quad - 2\lambda Mk\tau \ge \frac{1-D}{2}\|u\|^2_{H^1_{2k\tau}} - 2\lambda Mk\tau > -\infty, \end{aligned} \tag{19}$$

which implies that φ is bounded from below. By condition (ii) of Lemma 4, we have $i_2(\varphi) = 0$.

Step 2. We show that the functional φ satisfies the PS condition.

For any given sequence $\{u_n\} \in H^1_{2k\tau}$ such that $\{\varphi(u_n)\}$ is bounded and $\lim_{n\to\infty}\varphi'(u_n) = 0$, there exists a constant C_1 such that

$$\begin{gathered} |\varphi(u_n)| \le C_1, \\ \left\|\varphi'(u_n)\right\|_{(H^1_{2k\tau})^*} \le C_1, \\ \forall n \in \mathbb{N}, \end{gathered} \tag{20}$$

where $(H^1_{2k\tau})^*$ is the dual space of $H^1_{2k\tau}$.

Combining (19) and (20), we have

$$\frac{1}{2}\|u\|^2_{H^1_{2k\tau}} \le C_1 + 2\lambda Mk\tau. \tag{21}$$

It follows that $\|u_n\|_{H^1_{2k\tau}}$ is bounded.

Since $H^1_{2k\tau}$ is a reflexive Banach space, so we may extract a weakly convergent subsequence, for simplicity, we also note again by $\{u_n\}$, $u_n \rightharpoonup u$ in $H^1_{2k\tau}$. So we have

$$\begin{aligned}
&\int_0^{2k\tau} \left(F_1'(t,u_n(t),u_n(t-\tau)) - F_1'(t,u(t),u(t-\tau))\right) \\
&\quad\cdot (u_n(t)-u(t))\,dt \longrightarrow 0, \\
&\int_0^{2k\tau} \left(F_2'(t,u_n(t),u_n(t-\tau)) - F_2'(t,u(t),u(t-\tau))\right) \\
&\quad\cdot (u_n(t-\tau)-u(t-\tau))\,dt \longrightarrow 0, \\
&\sum_{j=1}^{l} \left(I_j(u_n(t_j)) - I_j(u(t_j))\right)(u_n(t_j)-u(t_j)) \\
&\quad\longrightarrow 0, \\
&\qquad u_n(t)-u(t) \longrightarrow 0 \text{ as } n\longrightarrow\infty,\ t\in[0,2k\tau].
\end{aligned} \tag{22}$$

Therefore, by (22), we have $\|u_n - u\|_{H^1_{2k\tau}} \to 0$. Hence the functional φ satisfies the PS condition.

Step 3. We show that the functional φ satisfies assumption (i) of Lemma 4.

Let $\beta_j(t) = (k\tau/j\pi)\sin(j\pi/\kappa\tau)t$, $j = 1,2,\ldots,m$. By calculations, we obtain

$$\begin{aligned}
\int_0^{2k\tau} \left|\beta_j(t)\right|^2 dt &= \left(\frac{k\tau}{j\pi}\right)^2 k\tau, \\
\int_0^{2k\tau} \left|\beta_j'(t)\right|^2 dt &= k\tau.
\end{aligned} \tag{23}$$

Define the m-dimensional linear subspace as follows:

$$E_m = \operatorname{span}\{\beta_1(t),\beta_2(t),\ldots,\beta_m(t)\}. \tag{24}$$

It is clear to see that E_m is a symmetric set. Take $r > 0$, when $u(t) \in E_m \cap S_r$, where S_r denotes boundary of B_r, $u(t)$ has expansion $u(t) = \sum_{j=1}^m b_j\beta_j(t)$, $b_j \in \mathbb{R}$, and

$$\begin{aligned}
r^2 = \|u(t)\|^2_{H^1_{2k\tau}} &= \int_0^{2k\tau} \left(\left|u'(t)\right|^2 + |u(t)|^2\right) dt \\
&\le k\tau\sum_{j=1}^m b_j^2\left(1+\frac{k^2\tau^2}{j^2\pi^2}\right).
\end{aligned} \tag{25}$$

By (H6), for given ε with $0 < \varepsilon < (\lambda m^2/4k^2\tau^2)(4k^2\tau^2/m^2 - (\pi^2 + (1+D)k^2\tau^2)/\lambda)$, there exists $0 < \delta < 1$ such that when $(|u(t)|^2 + |u(t-\tau)|^2)^{1/2} < \delta$, we have

$$\begin{aligned}
&\lambda F(t,u(t),u(t-\tau)) \\
&\quad > (\lambda-\varepsilon)\left(|u(t)|^2 + |u(t-\tau)|^2\right).
\end{aligned} \tag{26}$$

Combining (10), (25), and (26), when $u(t) \in E_m \cap S_r$, we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_0^{2k\tau} \left(|u(t)|^2 + \left|u'(t)\right|^2\right) dt \\
&\quad - \lambda\int_0^{2k\tau} F(t,u(t),u(t-\tau))\,dt \\
&\quad + \sum_{j=1}^{l}\int_0^{u(t_j)} I_j(s)\,ds \le \frac{1}{2}\|u\|^2_{H^1_{2k\tau}} - (\lambda-\varepsilon) \\
&\quad \cdot \int_0^{2k\tau} \left(|u(t)|^2 + |u(t-\tau)|^2\right) dt + \frac{D}{2} \\
&\quad \cdot \int_0^{2k\tau} |u(t)|^2\,dt \le \frac{k\tau}{2}\sum_{j=1}^m b_j^2\left(1+\frac{(1+D)k^2\tau^2}{j^2\pi^2}\right) \\
&\quad - \frac{2(\lambda-\varepsilon)k^2\tau^2}{m^2\pi^2}k\tau\sum_{j=1}^m b_j^2 \le \frac{k\tau}{2\pi^2} \\
&\quad \cdot \sum_{j=1}^m b_j^2\left(\pi^2 + (1+D)k^2\tau^2 - \frac{4(\lambda-\varepsilon)k^2\tau^2}{m^2}\right) \\
&= \frac{\lambda k\tau}{2\pi^2} \\
&\quad \cdot \sum_{j=1}^m b_j^2\left(\frac{\pi^2+(1+D)k^2\tau^2}{\lambda} - \frac{4k^2\tau^2}{m^2} + \varepsilon\frac{4k^2\tau^2}{\lambda m^2}\right) \\
&< 0.
\end{aligned} \tag{27}$$

Therefore $i_1(\varphi) \ge m$. Consequently, system (1) admits at least $2m$ nonzero $2k\tau$-periodic solutions.

We conclude this section with the following example.

Example 5. Consider (1) with

$$\begin{aligned}
&f(t,u(t),u(t-\tau),u(t-2\tau)) \\
&\quad = 4u(t-\tau) - 4\left(2+\cos\frac{2\pi t}{\tau}\right)u(t-2\tau) \\
&\qquad \cdot \left(u^2(t) + 2u^2(t-\tau) + u^2(t-2\tau)\right), \\
&F(t,u_1,u_2) = u_1^2 + u_2^2 - \left(2+\cos\frac{2\pi t}{\tau}\right)\left(u_1^2+u_2^2\right)^2, \\
&\qquad\qquad I_j(u) = 0.5u.
\end{aligned} \tag{28}$$

It is easy to see that $\partial f(t,u_1,u_2,u_3)/\partial t \neq 0$ and when $(u_1,u_2) = 0$, $F(t,u_1,u_2) = 0$; then (H1) and (H5) hold. Set $u_1 = u(t)$, $u_2 = u(t-\tau)$, $u_3 = u(t-2\tau)$, and then $\partial F(t,u(t),u(t-\tau))/\partial u(t-\tau) + \partial F(t,u(t-\tau),u(t-2\tau))/\partial u(t-\tau) = f(t,u(t),u(t-\tau),u(t-2\tau))$. By a simple computation, we have $F(t+\tau,u_1,u_2) = F(t,u_1,u_2)$, $F(t,-u_1,-u_2) = F(t,u_1,u_2)$, and $f(t,-u_1,-u_2,-u_3) = -f(t,u_1,u_2,u_3)$. So conditions (H2)–(H4) hold. Clearly, the conditions (H6)–(H8) hold. Therefore system (1) admits at least $2m$ nonzero solutions with the period $2k\tau$.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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