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Research Article

An Interval of No-Arbitrage Prices in Financial Markets with Volatility Uncertainty

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In financial markets with volatility uncertainty, we assume that their risks are caused by uncertain volatilities and their assets are effectively allocated in the risk-free asset and a risky stock, whose price process is supposed to follow a geometric G-Brownian motion rather than a classical Brownian motion. The concept of arbitrage is used to deal with this complex situation and we consider stock price dynamics with no-arbitrage opportunities. For general European contingent claims, we deduce the interval of no-arbitrage price and the clear results are derived in the Markovian case.

1. Introduction

Though many choice situations show uncertainty, owing to the Ellsberg Parasox, the impacts of ambiguity aversion on economic decisions are established and Beissner [1] considered general equilibrium economies with a primitive uncertainty model that features ambiguity about continuous-time volatility. Under uncertainty, multiple priors can be used to model decisions. Recently, these multiple priors models have attracted much attention. The decision theoretical setting of multiple priors was introduced by Gilboa and Schmeidler [2] and Artzner et al. [3] adapted it to monetary risk measures. Afterwards, Maccheroni et al. [4] generalized multiple priors to preferences. In diffusion models, Girsanov's theorem was employed to consider stochastic processes by Chen and Epstein [5], but these multiple priors can only lead to uncertainty. When these multiple priors are used in finance areas, they result in drift uncertainty for stock prices. In the risk-neutral world, when we assess financial claims, the uncertainty of this drift will disappear.

Under the assumption of no arbitrage and volatility uncertainty, Fernholz and Karatzas [6] considered to outperform the market. Compared with this, our paper is to model volatility uncertain financial markets which have no arbitrage. Epstein and Ji [7] or Vorbrink [8] used a specific example to illustrate an uncertain volatility model. On the basis of our predecessors, our paper solves a few basic problems of the volatility uncertainty in finance markets. Our aim is to analyze the volatility uncertain financial markets and we take advantage of the framework of sublinear expectation and G-Brownian motion which is introduced by Peng [9] to deal with the model in financial markets. The G-Brownian motion is no longer a classical Brownian motion. The construction of stochastic integration, $It\hat{o}$'s lemma, and martingale theory is utilized to the framework of G-Brownian motion. In order to control the model risk, the G-Brownian motion is employed to concern the model and evaluate claims by means of G-expectation which is a sublinear expectation.

In our financial markets with volatility uncertainty, the wealth is invested in risk-free asset and risky asset, in which the risky asset, that is, stock S_t and its price process S_t, is given by the following geometric G-Brownian motion:

$$dS_t = rS_t dt + \nu_t S_t dB_t, \quad S_0 = x_0 > 0, \tag{1}$$

where constant interest rate $r \geqslant 0$ is an expected instantaneous return of the stock and ν_t is the volatility of S which is associated with t. The canonical process $B = (B_t)$ is a G-Brownian motion relating to a sublinear expectation E_G, called G-expectation (see [9, 10] for a detailed construction). The stochastic calculus with respect to G-Brownian motion can also be established, especially $It\hat{o}$ integral [9]. The ordinary martingales are replaced by G-martingales. Denis et al. [11] developed the G-framework of Peng [10] (see [12]) in the framework of quasi-sure analysis. An upper expectation of classical expectations is used to represent the sublinear expectation E_G established by Denis et al. [11]; that is to say, there exists a set of probability measures $\mathbf{P}$ such that $E_G[X] = \sup_{P\in\mathbf{P}} \mathbb{E}^P[X]$.

In this paper, we prove that the considered financial market does not admit any arbitrage opportunity, but it allows for uncertain volatility. In our analysis, the notion of G-martingale which replaces the notion of martingale in classical probability theory plays a major role.

One of our aims is to solve

$$\sup_{P\in\mathbf{P}} \mathbb{E}^P \left(D_T V_T\right), \qquad \sup_{P\in\mathbf{P}} \mathbb{E}^P \left(-D_T V_T\right), \tag{2}$$

where V_T denotes the payoff of contingent claims at maturity T and D_T is a discounting. $\mathbf{P}$ presents a series of different probability measures.

The stochastic environment can bring about a set of probability measures that are not equivalent but even mutually singular. To illustrate this, let B be a Brownian motion under a measure P and think about the processes $S^{\underline{\sigma}} := (\underline{\sigma} B_t)$ and $S^{\overline{\sigma}} := (\overline{\sigma} B_t)$. Using $P^{\underline{\sigma}} = P \circ (S^{\underline{\sigma}})^{-1}$ and $P^{\overline{\sigma}} = P \circ (S^{\overline{\sigma}})^{-1}$, we describe the distributions over continuous trajectories which are induced by the two processes. These measures describe two possible hypotheses of real probability measure which drives the volatility uncertainty by (1). Therefore, we have

$$P^{\underline{\sigma}} \left(\left\{\langle B\rangle_T = \underline{\sigma}^2 T\right\}\right) = 1 = P^{\overline{\sigma}} \left(\left\{\langle B\rangle_T = \overline{\sigma}^2 T\right\}\right), \tag{3}$$

where both priors are mutually singular.

The definition of trading strategy and portfolio process is applied to obtain the wealth equation. Defining the concept of no-arbitrage in financial markets and the hedging classes, we gain the interval of no-arbitrage price for general European contingent claims. Finally, the connection of the lower and upper arbitrage prices is presented.

In such an ambiguous financial market, our subject is to analyze the European contingent claim concerning pricing and hedging. The asset pricing is extended to the financial markets with volatility uncertainty. The notion of no-arbitrage plays an important role in our analysis. Owing to the fact that the volatility uncertainty leads to additional source of risk, the classical definition of arbitrage will no longer be adequate. For this reason, a new arbitrage definition is presented to adjust our multiple priors model with mutually singular priors which are shown in (3). In this modified sense, we confirm that our volatility uncertain financial markets do not admit any arbitrage opportunity.

Utilizing the notion of no-arbitrage, we have obtained several results, which provide us with a better economic understanding of financial markets under volatility uncertainty. For general contingent claims, we determine an interval of no-arbitrage prices. The bounds of this interval are the upper and lower arbitrage prices v_{up} and v_{low}, which are obtained as the expected value of the claim's discounted payoff with respect to G-expectation (see (2)). They specify the lowest initial capital. We use the capital to hedge a short position in the claim or long position, respectively. Generally speaking, because E_G is a sublinear expectation, we have $v_{\text{low}} \neq v_{\text{up}}$. This verifies the market's incompleteness. In a few words, no arbitrage will be generated when price is in the interval $(v_{\text{low}}, v_{\text{up}})$ for a European contingent claim. In Section 4, when the contingent claim's payoff is only determined by the current stock price, we deduce a more clear structure about the upper and lower arbitrage prices by a partial differential equation (PDE for short). We derive an explicit representation for the corresponding supper-hedging strategies and consumption plans. Given the special situation when the payoff function shows convexity (concavity), the upper arbitrage price solves the classical Black-Scholes PDE with a volatility equal to $\underline{\sigma}(\overline{\sigma})$, and vice versa concerning the lower arbitrage price.

The novelties of this paper are that the volatility of S in our model is a variable which is related to t. This is different from works of Vorbrink [8] in which the volatility of S is a constant. We employ the G-framework including G-expectation, G-Brownian motion, and the concept of arbitrage to study the financial markets with volatility uncertainty; we gain the interval of no arbitrage, which is different from that in Denis and Martini [12].

This paper is organized as follows. Section 2 introduces the financial markets. We focus on and extend the terminology from mathematical finance. Section 3 applies a series of definitions and lemmas to derive the interval of no arbitrage. Section 4 restricts us to the Markovian case and derives results which are analogy to those in Avellaneda et al. [13] or Vorbrink [8]. Conclusions are given in Section 5.

2. The Market Model and the Mathematical Setting

2.1. G-Brownian Motion and the Multiple Priors Setting. In the whole paper, the one-dimensional case is considered and we fix an interval $[\underline{\sigma}, \overline{\sigma}]$ with $\underline{\sigma} > 0$. This interval describes the volatility uncertainty. $\underline{\sigma}$ and $\overline{\sigma}$ denote a lower and upper bound for volatility, respectively.

Definition 1 (see [9]). Let $\Omega \neq \emptyset$ be a given set. Let $\mathscr{H}$ be a linear space of real valued functions defined on Ω with $c \in \mathscr{H}$ for all constants c, and $|X| \in \mathscr{H}$ if $X \in \mathscr{H}$. ($\mathscr{H}$ is considered as the space of random variables.) A sublinear expectation $\widehat{E}$ on $\mathscr{H}$ is a functional $\widehat{E} : \mathscr{H} \to \mathbb{R}$ satisfying the following properties: for any $X, Y \in \mathscr{H}$, it has

(1) Monotonicity: if $X \geqslant Y$ then $\widehat{E}(X) \geqslant \widehat{E}(Y)$.

(2) Constant preserving: $\widehat{E}(c) = c$.

(3) Subadditivity: $\widehat{E}(X+Y) \leqslant \widehat{E}(X) + \widehat{E}(Y)$.

(4) Positive homogeneity: $\widehat{E}(\mu X) = \mu \widehat{E}(X)\ \forall \mu \geqslant 0$.

The triple $(\Omega, \mathscr{H}, \widehat{E})$ is called a sublinear expectation space.

Definition 2 (see [10] (G-normal distribution)). In a sublinear expectation space $(\Omega, \mathscr{H}, \widehat{E})$, a random variable X is called (centralized) G-normal distributed if for any $a, b \geqslant 0$

$$aX + b\overline{X} \sim \sqrt{a^2 + b^2}X, \tag{4}$$

where $\overline{X}$ is an independent copy of X. Here the letter G denotes the function

$$G(a) := \frac{1}{2}\widehat{E}\left(aX^2\right) : \mathbb{R} \longrightarrow \mathbb{R}. \tag{5}$$

Note that X has no mean-uncertainty; that is, it has $\widehat{E}(X) = \widehat{E}(-X) = 0$. Moreover, the following important identity holds:

$$G(a) = \frac{1}{2}\overline{\sigma}^2 a^+ - \frac{1}{2}\underline{\sigma}^2 a^- \tag{6}$$

with $\underline{\sigma}^2 := -\widehat{E}(-X^2)$ and $\overline{\sigma}^2 := \widehat{E}(X^2)$. We write that X is $N(\{0\}\times[\underline{\sigma}^2, \overline{\sigma}^2])$ distributed. Therefore, we say that G-normal distribution is characterized by the parameters $0 < \underline{\sigma} \leqslant \overline{\sigma}$.

Remark 3 (see [10]). The random variable X which is defined in (4) is generated by the following parabolic PDE defined in $[0, T] \times \mathbb{R}$.

For any $C_{l,Lip}(\mathbb{R})$, define $u(t,x) := \widehat{E}[\varphi(x + \sqrt{t}X)]$; then u is the unique (viscosity) solution of

$$\partial_t u - G(\partial_{xx} u) = 0, \quad u|_{t=0} = \varphi. \tag{7}$$

Equation (7) is called a G-equation.

Definition 4 (see [10] (G-Brownian motion)). A process $(B_t)_{t\geqslant 0}$ in a sublinear expectation space $(\Omega, \mathscr{H}, \widehat{E})$ is called a G-Brownian motion if the following properties are satisfied:

(i) $B_0 = 0$.

(ii) For each $t, s \geqslant 0$ the increment $B_{t+s} - B_t$ is $N(\{0\} \times [\underline{\sigma}^2 s, \overline{\sigma}^2 s])$ distributed and independent from $(B_{t_1}, B_{t_2}, \ldots, B_{t_n})$ for each $n \in \mathbb{N}$, $0 \leqslant t_1 \leqslant \cdots t_n \leqslant t$.

Condition (ii) can be replaced by the following three conditions giving a characterization of G-Brownian motion:

(i) For each $t, s \geqslant 0 : B_{t+s} - B_t \sim B_t$ and $\widehat{E}(|B_t|^3)/t \to 0$ as $t \to 0$.

(ii) The increment $B_{t+s} - B_t$ is independent from $(B_{t_1}, B_{t_2}, \ldots, B_{t_n})$ for each $n \in \mathbb{N}$, $0 \leqslant t_1 \leqslant \cdots t_n \leqslant t$.

(iii) $\widehat{E}(B_t) = -\widehat{E}(-B_t) = 0$, $\forall t \geqslant 0$.

For each $t_0 > 0$, it has that $(B_{t+t_0} - B_{t_0})_{t\geqslant 0}$ is a G-Brownian motion.

Let us briefly depict the construction of G-expectation and its corresponding G-Brownian motion. As in the previous sections, we fix a time horizon $T > 0$ and set $\Omega_T = C_0([0, T], \mathbb{R})$-the space of all real valued continuous paths starting at zero. Considering the canonical process $B_t(\omega) := \omega_t$, $t \leqslant T$, $\omega \in \Omega$, we define

$$\begin{aligned} L_{ip}(\Omega_T) := \Big\{\varphi\left(B_{t_1}, \ldots, B_{t_n}\right) \mid n \in \mathbb{N},\ t_1, \ldots, t_n \\ \in [0, T],\ \varphi \in C_{l,Lip}(\mathbb{R}^n)\Big\}. \end{aligned} \tag{8}$$

A G-Brownian motion is firstly constructed in $L_{ip}(\Omega_T)$. For this purpose, let $(\xi_i)_{i\in\mathbb{N}}$ be a sequence of random variables in a sublinear expectation space $(\widetilde{\Omega}, \widetilde{\mathscr{H}}, \widetilde{E})$ such that ξ_i is G-normal distributed and ξ_{i+1} is independent of $(\xi_1, \ldots, \xi_i)$ for each integer $i \geqslant 1$. Then a sublinear expectation in $L_{ip}(\Omega_T)$ is constructed by the following procedure: for each $X \in L_{ip}(\Omega_T)$ with $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})$ for some $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, $0 \leqslant t_0 < t_1 < \cdots < t_n \leqslant T$, set

$$\begin{aligned} &E_G\left[\varphi\left(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}\right)\right] \\ &\quad := \widetilde{E}\left[\varphi\left(\sqrt{t_1 - t_0}\xi_1, \sqrt{t_2 - t_1}\xi_2, \ldots, \sqrt{t_n - t_{n-1}}\xi_n\right)\right]. \end{aligned} \tag{9}$$

The related conditional expectation of $X \in L_{ip}(\Omega_T)$ as above under Ω_{t_i}, $i \in \mathbb{N}$, is defined by

$$\begin{aligned} &E_G\left[\varphi\left(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}\right) \mid \Omega_{t_i}\right] \\ &\quad := \psi\left(B_{t_1} - B_{t_0}, \ldots, B_{t_i} - B_{t_{i-1}}\right), \end{aligned} \tag{10}$$

where $\psi(x_1, \ldots, x_i) := \widetilde{E}[\varphi(x_1, \ldots, x_i, \sqrt{t_{i+1} - t_i}\xi_{i+1}, \ldots, \sqrt{t_n - t_{n-1}}\xi_n)]$. One checks that E_G consistently defines a sublinear expectation in $L_{ip}(\Omega_T)$ and the canonical process B represents a G-Brownian motion.

Let $\Theta := [\underline{\sigma}, \overline{\sigma}]$ and $\mathscr{A}^{\Theta}_{0,T}$ be the collection of all Θ-valued $(\mathscr{F}_t)$-adapted processes on $[0, T]$. We write

$$B^{0,\sigma}_t := \int_0^t \sigma_s dB_s, \tag{11}$$

and P^σ as the law of $B^{0,\sigma} = \int_0^{\cdot} \sigma_s dB_s$; that is, $P^\sigma = P_0 \circ (B^{0,\sigma})^{-1}$ is distribution over trajectories. Let the set of multiple priors $\mathbf{P}$ be the closure of $\{P^\sigma \mid \sigma \in \mathscr{A}^{\Theta}_{0,T}\}$ under the topology of weak convergence.

Theorem 5 (see [8]). *For any $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, $n \in \mathbb{N}$, $0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant T$, it holds that*

$$\begin{aligned} &E_G\left[\varphi\left(B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}\right)\right] \\ &\quad = \sup_{\theta \in \mathscr{A}^{\Theta}_{0,T}} \mathbb{E}^P\left[\varphi\left(B^{0,\theta}_{t_1}, \ldots, B^{t_{n-1},\theta}_{t_n}\right)\right] \\ &\quad = \sup_{\theta \in \mathscr{A}^{\Theta}_{0,T}} E^{P^\theta}\left[\varphi\left(B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}\right)\right] \\ &\quad = \sup_{P^\theta \in \mathbf{P}} E^{P^\theta}\left[\varphi\left(B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}\right)\right]. \end{aligned} \tag{12}$$

Furthermore,

$$E_G(X) = \sup_{P^\theta \in \mathbf{P}} \mathbb{E}^P(X), \quad \forall X \in L^1_G(\Omega_T). \tag{13}$$

We use the set of priors $\mathbf{P}$ to define the G-expectation E_G. It is given by

$$E_G(X) = \sup_{P \in \mathbf{P}} \mathbb{E}^P(X), \tag{14}$$

where X is any random variable. So the G-expectation can be defined. Relative to the G-expectation, the space of random variable is denoted by $L_G^1(\Omega_T)$.

In this paper, we consider the tuple as $(\Omega_T, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$ and the canonical process $B = (B_t)$ is a G-expectation motion with respect to $\mathbf{P}$ as given in the previous. The G-framework enables the analysis of stochastic processes for all priors of $\mathbf{P}$. The terminology of "*quasi-surely*" (q.s.) is proved to be very useful.

Unless there are special instructions, all equations should also be understood as "quasi-sure." This means a property almost surely for all conceivable scenarios.

As mentioned in the preceding, G-expectation can be defined in the space $L_G^p(\Omega_T)$, $p \geqslant 1$. It is the completion of $\mathcal{C}_b(\Omega_T)$, the set of bounded continuous functions on Ω_T under the norm $\|\xi\| := (E_G[|\xi|^2])^{1/2} < \infty$. Because the stochastic integrals are required to define trading strategies in the next sections, we briefly introduce the basic concepts about stochastic calculus and the construction of Itô integral with respect to G-Brownian motion.

For $p \geqslant 1$, let $M_G^{p,0}(0,T)$ be the collection of simple processes η of the following form: for a given partition $[0,T]$, $\{t_0, t_1, \ldots, t_N\}$, $N \in \mathbb{N}$, for any $t \in [0,T]$ the process η is defined by

$$\eta_t(\omega) := \sum_{j=0}^{N-1} \xi_i(\omega) \mathbf{1}_{[t_j, t_{j+1})}(t), \tag{15}$$

where $\xi_i(\omega) \in L_G^p(\Omega_{t_i})$, $i = 0, 1, \ldots, N-1$. For each $\eta \in M_G^{p,0}(0,T)$, let

$$\|\eta\|_{M_G^p} := \left(E_G \int_0^T |\eta_s|^p \, ds\right)^{1/p}. \tag{16}$$

We denote by $M_G^{p,0}(0,T)$ the completion of $M_G^{p,0}(0,T)$ under the norm $\|\cdot\|_{M_G^p}$.

Definition 6 (see [14]). For $\eta \in M_G^{2,0}(0,T)$ with the presentation in (15), the integral mapping is defined by $I : M_G^{2,0}(0,T) \to L_G^2(\Omega_T)$ and

$$I(\eta) = \int_0^T \eta(s) \, dB_s := \sum_{j=0}^{N-1} \xi_i \left(B_{t_{j+1}} - B_{t_j}\right). \tag{17}$$

We consider the quadratic variation process $(\langle B \rangle_t)$ of G-Brownian motion. It has

$$\langle B \rangle_t = B_t^2 - 2\int_0^t B_s dB_s, \quad \forall t \leqslant T. \tag{18}$$

It is a continuous, increasing process, absolutely continuous with respect to dt. It contains all the statistical uncertainty of the G-Brownian motion. For $s, t \geqslant 0$ we have $\langle B \rangle_{s+t} - \langle B \rangle_s \sim \langle B \rangle_t$ and it is independent of Ω_s.

Definition 7 (see [9]). Let $x \in \mathbb{R}$, $z \in M_G^2(0,T)$ and $\eta \in M_G^1(0,T)$. Then the process

$$M_t := x + \int_0^t z_s dB_s + \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s)\, ds, \qquad t \leqslant T, \tag{19}$$

is a G-martingale.

Specially, the nonsymmetric part $-K_t := \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s) ds$, $t \in [0,T]$, is a G-martingale, which is quite a surprising result because $(-K_t)$ is continuous, nonincreasing with quadratic variation equal to zero.

Remark 8 (see [15]). M is a symmetric G-martingale if and only if $K \equiv 0$.

Theorem 9 (see [15] (martingale representation)). *Let $\alpha \geqslant 0$ and $\xi \in L_G^\alpha(\Omega_T)$. Then the G-martingale X with $X_t := E_G(\xi \mid \mathcal{F}_t)$, $t \in [0,T]$, has the following unique representation:*

$$X_t = X_0 + \int_0^t z_s dB_s - K_t, \tag{20}$$

where K is a continuous, increasing process with $K_0 = 0$, $K_T \in L_G^\beta(\Omega_T)$, $z \in H_G^\beta(0,T)$, $\forall \beta \in [1, \alpha)$, and $-K$ a G-martingale.

If $\alpha = 2$ and ξ bounded from above, $z \in M_G^2(0,T)$ and $K_T \in L_G^2(\Omega_T)$ (see [14]).

A construction of the stochastic integral for the domain $H_G^p(0,T)$, $p \geqslant 1$ is established by Song [15]. Although the structure of these spaces is similar as before, the norm for completion is different and the random variables $\xi_i(\omega)$ in (15) are elements of a subset of $L_G^p(\Omega_{t_i})$. We will also use the domain $H_G^1(0,T)$ which is necessary for the martingale representation in the G-framework (see Theorem 9). For $p = 2$, both domains coincide (see Song [15]). As a consequence, we can define the stochastic integral since $M_G^2(0,T)$ is contained in $H_G^1(0,T)$. In financial fields, more trading strategies will be feasible.

2.2. The Financial Market Model. We consider the following financial market $\mathcal{M}$ which includes a risk-free asset and a single risky asset and two assets are traded continuously over $[0,T]$. Assume that the risk-free asset is a bond and its interest rate is r. So the discount process D_t can be defined to satisfy the following formula:

$$dD_t = -rD_t dt, \quad D_0 = 1, \tag{21}$$

where constant $r \geqslant 0$ is the interest rate of the riskless bond as in the classical theory.

Assume that the risky asset is a stock with price S_t at time t, whose price process S_t is given by the following equation:

$$dS_t = rS_t dt + \nu_t S_t dB_t, \quad S_0 = x_0 > 0, \tag{22}$$

where $B = (B_t)$ denotes the canonical process which is a G-Brownian motion under E_G or $\mathbf{P}$, respectively, with parameters $\overline{\sigma} > \underline{\sigma} > 0$.

Since $B = (B_t)$ is a G-Brownian motion, the volatility of S is related to t which is different from that of Vorbrink [8], where the volatility of stock price is a constant 1. Consequently, the stock price evolution involves not only risk modeled by the noise part but also ambiguity about the risk due to the unknown deviation of the process B from its mean. According to financial fields, this ambiguity is called volatility uncertainty.

Compared with the classical stock price process, (22) does not contain any volatility parameter σ. This is due to the characteristics of the G-Brownian motion B. Apparently, if we choose $\sigma = \underline{\sigma} = \overline{\sigma}$, then we will be in the classical Black-Scholes model.

Remark 10. Take notice of the discounted stock price process $(D_t S_t)$ which is a symmetric G-martingale relative to the corresponding G-expectation E_G. As everyone knows, both the pricing and hedging of contingent claims are treated under a risk-neutral measure. This leads to a favorable situation in which the discounted stock price process is a (local) martingale [16]. In our ambiguous setting, this is also allowed. In order to model $(D_t S_t)$ as a symmetric G-martingale (see Definition 7), we do not need to change the sublinear expectation. A symmetric G-martingale is required to make sure that the stock is the same for all participants, whether they sell or buy.

Definition 11 (see [17]). In the market $\mathscr{M}$, a trading strategy is an $(\mathscr{F}_t)$-adapted vector process $(\alpha, \beta) = (\alpha_t, \beta_t)$, β a member of $H_G^1(0,T)$ such that $(\beta_t S_t) \in H_G^1(0,T)$, and $\alpha_t \in \mathbb{R}$ for all $t \leqslant T$.

A cumulative consumption process $\{C = (C_t),\ 0 \leqslant t \leqslant T\}$ is a nonnegative $(\mathscr{F}_t)$-adapted process with values in $L_G^1(\Omega_T)$, and with increasing, right-continuous, and $C_0 = 0$, $C_T < \infty$ q.s.

A basic assumption in the market $\mathscr{M}$ is that the stock price process S_t defined by (22) is an element of $M_G^2(0,T) = H_G^2(0,T)$. We impose the so-called self-financing condition. In other words, consumption and trading in $\mathscr{M}$ satisfy

$$\begin{aligned} H_t &:= D_t^{-1}\alpha_t + \beta_t S_t \\ &= \alpha_0 D_0^{-1} + \beta_0 S_0 + \int_0^t \alpha_u dD_u^{-1} + \int_0^t \beta_u dS_u - C_t, \\ &\qquad \forall t \leqslant T \text{ q.s.}, \end{aligned} \tag{23}$$

where H_t denotes the value of the trading strategy at time t. The meaning of (23) is that, starting with an initial amount $D_0^{-1}\alpha_0 + \beta_0 S_0$ of wealth, all changes in wealth are due to capital gains (appreciation of stocks and interest from the bond), minus the amount consumed. The q.s. means quasi-surely, which is the same as before.

For economic and mathematical considerations, it is more appropriate to introduce wealth and a portfolio process which presents the proportions of wealth invested in the risky stock.

Remark 12 (see [17]). A portfolio process π represents proportions of a wealth X which is invested in the stock. If we define

$$\begin{aligned} \beta_t &:= \frac{X_t \pi_t}{S_t}, \\ \alpha_t &:= X_t D_t (1 - \pi_t), \\ &\qquad \forall t \leqslant T, \end{aligned} \tag{24}$$

then we have $X_t = H_t$. As long as π constitutes a portfolio process with corresponding wealth process X, the (α, β) is a trading strategy in the sense of (23).

Definition 13 (see [8]). A portfolio process is an $(\mathscr{F}_t)$-adapted real valued process if $\pi = \pi_t$ with values in $L_G^1(\Omega_T)$.

Definition 14. For a given initial capital m, a portfolio process π, and a cumulative consumption process C, consider the wealth equation

$$\begin{aligned} dX_t &= X_t (1 - \pi_t) D_t dD_t^{-1} + X_t \pi_t \frac{dS_t}{S_t} - dC_t \\ &= X_t r dt + X_t \pi_t \nu_t dB_t - dC_t \end{aligned} \tag{25}$$

with initial wealth $X_0 = m$. Or equivalently,

$$\begin{aligned} D_t X_t &= m - \int_0^t D_u dC_u + \int_0^t D_u X_u \pi_u \nu_u dB_u, \\ &\qquad \forall t \leqslant T. \end{aligned} \tag{26}$$

If this equation has a unique solution $X = (X_t) := X^{m,\pi,C}$, then it is called the wealth process corresponding to the triple (m, π, C).

In the setup of Definition 14, notice that the $\int_0^T {X_t}^2 {\pi_t}^2 {\nu_t}^2 dt < \infty$ must hold quasi-surely. Thus, we need to impose requirements $(\pi_t X_t \nu_t) \in H_G^p(0,T)$ $p \geqslant 1$, or $(\pi_t X_t \nu_t) \in M_G^p(0,T)$, $p \geqslant 2$.

Definition 15. A portfolio/consumption process pair (π, C) is called admissible for an initial capital $m \in \mathbb{R}$ if

(i) the pair obeys the conditions of Definitions 11, 13, and 14,

(ii) $(\pi_t X_t^{m,\pi,C} \nu_t) \in H_G^1(0,T)$,

(iii) the solution $X_t^{m,\pi,C}$ satisfies

$$X_t^{m,\pi,C} \geqslant -J, \quad \forall t \leqslant T, \text{ q.s.}, \tag{27}$$

where J is a nonnegative random variable in $L_G^2(\Omega_T)$.

We then have $(\pi, C) \in \mathscr{A}(m)$.

In the above Definitions 11 and 13–15, it is necessary to guarantee that the financial fields and related stochastic analysis can be well defined. In particular, condition (ii) of Definition 15 makes sure that the mathematical framework does not collapse by allowing for many portfolio processes.

3. Arbitrage and Contingent Claims

Definition 16 (see [8] (arbitrage in $\mathscr{M}$)). We say that there is an arbitrage opportunity in $\mathscr{M}$ if there exist an initial wealth $m \leqslant 0$ and an admissible pair $(\pi, C) \in \mathscr{A}(m)$ with $C \equiv 0$ such that, at some time $T > 0$,

$$\begin{aligned} X_T^{m,\pi,0} &\geqslant 0 \quad \text{q.s.,} \\ P\left(X_T^{m,\pi,0} > 0\right) &> 0 \quad \text{for at least one } P \in \mathbf{P}. \end{aligned} \tag{28}$$

Lemma 17 (no arbitrage). *In the financial market $\mathscr{M}$, there does not exist any arbitrage opportunity.*

Proof. Assume that there exists an arbitrage opportunity; that is to say, there exist $m \leqslant 0$ and a pair $(\pi, C) \in \mathscr{A}(m)$ with $C \equiv 0$ such that $X_T^{m,\pi,0} \geqslant 0$ quasi-surely for some $T > 0$. Then we have $E_G(X_T^{m,\pi,0}) \geqslant 0$. By definition of the wealth process, it has

$$\begin{aligned} 0 &\leqslant E_G\left(D_T X_T^{m,\pi,0}\right) \\ &\leqslant m + E_G\left(\int_0^T D_t X_t^{m,\pi,0} \pi_t \nu_t dB_t\right) = m. \end{aligned} \tag{29}$$

Since the G-expectation of an integral with respect to G-Brownian motion is zero, we have $E_G(D_T X_T^{m,\pi,0}) = 0$. This implies $D_T X_T^{m,\pi,0} = 0$ q.s. Therefore, $(m, \pi, 0)$ cannot constitute an arbitrage. □

In the financial market $\mathscr{M}$, we consider a European contingent claim V and assume that its payoff at maturity time T is V_T. Here, V_T represents a nonnegative, $\mathscr{F}_t$-adapted random variable. Regardless of any time, we impose the assumption $V_T \in L_G^2(\Omega_T)$. The price of the claim at time 0 is denoted by V_0. For the sake of finding reasonable prices for V, we need to utilize the concept of arbitrage. Considering that the financial market $(\mathscr{M}, V)$ contains the original market $\mathscr{M}$ and the contingent claim V. Similar to the above, an arbitrage opportunity needs to be defined in the financial market $(\mathscr{M}, V)$.

Definition 18 (see [17] (arbitrage in $(\mathscr{M}, V)$)). We say that there is an arbitrage opportunity in $(\mathscr{M}, V)$ if there exist an initial wealth $m \geqslant 0$ ($m \leqslant 0$, resp.), an admissible pair $(\pi, C) \in \mathscr{A}(m)$, and a constant $a = -1$ ($a = 1$, resp.), such that

$$m + a \cdot V_0 \leqslant 0 \tag{30}$$

at time 0, and

$$\begin{aligned} X_T^{m,\pi,C} + a \cdot V_T &\geqslant 0 \quad \text{q.s.,} \\ P\left(X_T^{m,\pi,C} + a \cdot V_T > 0\right) &> 0 \quad \text{for at least one } P \in \mathbf{P} \end{aligned} \tag{31}$$

at time T.

The values $a = \pm 1$ in Definition 18 indicate short or long positions in the claims V, respectively. This definition of arbitrage is standard in the literature [17]. For the same reasons as before, we again require quasi-sure dominance for the wealth at time T and again with positive probability for only one possible scenario.

In the following, we show that there exist no-arbitrage prices for a claim V. Under these prices, there is no-arbitrage opportunity. Because the uncertainty caused by the quadratic variation cannot be dispelled, generally speaking, there is no self-financing portfolio strategy which replicates the European claim or a risk-free hedge for the claim in our ambiguous market $\mathscr{M}$.

Roughly stated, since there is only one kind of situation where stocks will be traded, the measures induced by the G-framework result in market's incompleteness.

Definition 19 (see [17]). Given a European contingent claim V, the upper hedging class is defined by

$$\mathscr{U} := \left\{m \geqslant 0 \mid \exists (\pi, C) \in \mathscr{A}(m) : X_T^{m,\pi,C} \geqslant V_T \text{ q.s.}\right\} \tag{32}$$

and the lower hedging class is defined by

$$\begin{aligned} \mathscr{L} := \{m \geqslant 0 \mid \exists (\pi, C) \in \mathscr{A}(-m) : X_T^{-m,\pi,C} \\ \geqslant -V_T \text{ q.s.}\}. \end{aligned} \tag{33}$$

In addition, the upper arbitrage price is defined by

$$\nu_{\text{up}} := \inf \{m \mid m \in \mathscr{U}\} \tag{34}$$

and the lower arbitrage price is defined by

$$\nu_{\text{low}} := \sup \{m \mid m \in \mathscr{L}\}. \tag{35}$$

Lemma 20 (see [17]). *$m_1 \in \mathscr{L}$ and $0 \leqslant n_1 \leqslant m_1$ implies $n_1 \in \mathscr{L}$. Analogously, $m_2 \in \mathscr{U}$ and $n_2 \geqslant m_2$ implies $n_2 \in \mathscr{U}$.*

The proof uses the idea that one "just consumes immediately the difference between the two initial wealth" (see [17] for the complete proof process).

For any $\sigma \in [\underline{\sigma}, \overline{\sigma}]$, we define the Black-Scholes price of a European contingent claim V as follows:

$$u_0^\sigma = E^{P^\sigma}\left(D_T V_T\right). \tag{36}$$

Similar to the constrained circumstances [17], we prove the next three lemmas which are related to the European contingent claim V.

Lemma 21. *For any $\sigma \in [\underline{\sigma}, \overline{\sigma}]$, it holds that u_0^σ belongs to the interval $[\nu_{low}, \nu_{up}]$.*

Proof. Let $m \in \mathscr{U}$. From the definition of $\mathscr{U}$, we know that there exists a pair $(\pi, C) \in \mathscr{A}(m)$ such that $X_T^{m,\pi,C} \geqslant V_T$

q.s. Employing the properties of G-expectation as stated in Definition 1, we obtain for any $\sigma \in [\underline{\sigma}, \overline{\sigma}]$ that

$$\begin{aligned} m &= E_G\left(m + \int_0^T D_t X_t^{m,\pi,C} \pi_t \nu_t dB_t\right) \\ &\geqslant E_G\left(m + \int_0^T D_t X_t^{m,\pi,C} \pi_t \nu_t dB_t - \int_0^T D_t dC_t\right) \\ &= E_G\left(D_T X_T^{m,\pi,C}\right) \geqslant E_G\left(D_T V_T\right) = \sup_{P\in\mathbf{P}} \mathbb{E}^P\left(D_T V_T\right) \\ &\geqslant u_0^\sigma. \end{aligned} \tag{37}$$

Therefore, $u_0^\sigma \leqslant m$. We know $\nu_{\text{up}} := \inf\{m \mid m \in \mathscr{U}\}$; hence, $u_0^\sigma \leqslant \nu_{\text{up}}$.

Analogously, let $m \in \mathscr{L}$. By definition of $\mathscr{L}$, there exists a pair $(\pi, C) \in \mathscr{A}(-m)$ such that $X_T^{-m,\pi,C} \geqslant -V_T$ q.s. For the same reason, we obtain for any $\sigma \in [\underline{\sigma}, \overline{\sigma}]$ that

$$\begin{aligned} -m &= E_G\left(-m + \int_0^T D_t X_t^{-m,\pi,C} \pi_t \nu_t dB_t\right) \\ &\geqslant E_G\left(-m + \int_0^T D_t X_t^{-m,\pi,C} \pi_t \nu_t dB_t - \int_0^T D_t dC_t\right) \\ &= E_G\left(D_T X_T^{-m,\pi,C}\right) \geqslant E_G\left(-D_T V_T\right) \\ &\geqslant -E^{P^\sigma}\left(-D_T V_T\right) = -u_0^\sigma. \end{aligned} \tag{38}$$

Therefore, $m \leqslant u_0^\sigma$. We know $\nu_{\text{low}} := \sup\{m \mid m \in \mathscr{L}\}$; hence, $\nu_{\text{low}} \leqslant u_0^\sigma$. □

Lemma 22. *For any price $V_0 > \nu_{up}$, there exists an arbitrage opportunity. Also for any price $V_0 < \nu_{low}$, there exists an arbitrage opportunity.*

Proof. The idea of proving this lemma comes from [8]. We only consider the case $V_0 < \nu_{\text{low}}$ since the argument is similar. Assume $V_0 < \nu_{\text{low}}$, $m \leqslant 0$ and let $-m \in (V_0, \nu_{\text{low}})$. By definition of ν_{low} and Lemma 20, we deduce that $-m \in \mathscr{L}$. Hence, there exists a pair $(\pi, C) \in \mathscr{A}(-m)$ with

$$\begin{aligned} X_T^{-m,\pi,C} &\geqslant -V_T \quad \text{q.s.}, \\ -m - V_0 &> 0. \end{aligned} \tag{39}$$

This implies the existence of arbitrage in the sense of Definition 18. If $\exists 0 < a < 1$ with $-am = V_0$, then $(\pi, aC) \in \mathscr{A}(-am)$ and $X_T^{-am,\pi,aC} = aX_T^{-m,\pi,C}$. Let $P \in \mathbf{P}$; without loss of generality, we may assume $P(V_T > 0) > 0$. Due to

$$\begin{aligned} 1 &= P\left(X_T^{-m,\pi,C} \geqslant -V_T\right) \\ &\leqslant P\left(X_T^{-m,\pi,C} = -V_T\right) + P\left(aX_T^{-m,\pi,C} > -V_T\right) \leqslant 1, \end{aligned} \tag{40}$$

we deduce

$$P\left(X_T^{-m,\pi,C} = -V_T\right) + P\left(aX_T^{-m,\pi,C} > -V_T\right) = 1. \tag{41}$$

Assume $P(X_T^{-m,\pi,C} = -V_T) = 1$ and we deduce $X_T^{-m,\pi,C} = -V_T$, q.s. This contradicts $X_T^{-m,\pi,C} \geqslant -V_T$ q.s. Hence, $P(X_T^{-am,\pi,aC} > -V_T) > 0$. Hence, $(-am, \pi, aC)$ constitutes an arbitrage. □

Lemma 23. *For any $V_0 \notin \mathscr{L} \cup \mathscr{U}$, there is no arbitrage in the financial market $(\mathscr{M}, V)$.*

Proof. The idea of proving this lemma also comes from [8]. We prove it by contradiction. Assume $V_0 \notin \mathscr{U}$, $V_0 \notin \mathscr{L}$ and that there exists an arbitrage opportunity in $(\mathscr{M}, V)$. We suppose that it satisfies Definition 18 for $a = 1$. The case $a = -1$ works similarly.

By definition of arbitrage, there exists $m \leqslant 0$, $(\pi, C)\mathscr{A} \in (-m)$ with

$$\begin{aligned} m = X_0^{-m,\pi,C} &\leqslant -V_0, \\ X_T^{-m,\pi,C} &\geqslant -V_T \quad \text{q.s.} \end{aligned} \tag{42}$$

Therefore, $m \in \mathscr{L}$. By Lemma 21, it has $V_0 \in \mathscr{L}$. This contradicts our assumption. □

Theorem 24. *For the financial market $(\mathscr{M}, V)$, the following identities hold:*

$$\begin{aligned} \nu_{\text{up}} &= E_G\left(D_T V_T\right), \\ \nu_{\text{low}} &= -E_G\left(-D_T V_T\right). \end{aligned} \tag{43}$$

Proof. Firstly, let us begin with the identity $\nu_{\text{up}} = E_G(D_T V_T)$. As seen in the proof of Lemma 22, for any $m \in \mathscr{U}$ we have $m \geqslant E_G(D_T V_T)$. Therefore, $\nu_{\text{up}} = \inf\{m \mid m \in \mathscr{U}\} \geqslant E_G(D_T V_T)$.

To show the opposite inequality we need to define the G-martingale M by

$$M_t := E_G\left(D_T V_T \mid \mathscr{F}_t\right), \quad \forall t \leqslant T. \tag{44}$$

By the martingale representation theorem [15] (see Theorem 9), we know there exists $z \in H_G^1(0,T)$ and continuous, increasing processes $K = (K_t)$ with $K_T \in L_G^1(\Omega_T)$ such that for any $t \leqslant T$

$$M_t = E_G\left(D_T V_T\right) + \int_0^t z_s dB_s - K_t \quad \text{q.s.} \tag{45}$$

For any $t \leqslant T$, we set $m = E_G(D_T V_T) \geqslant 0$, $X_t \pi_t \nu_t = z_t D_t^{-1} \in H_G^1(0,T)$, and $C_t = \int_0^t D_s^{-1} dK_s \in L_G^1(\Omega_T)$. Then the wealth process $X^{m,\pi,C}$ satisfies

$$\begin{aligned} D_t X_t^{m,\pi,C} &= m + \int_0^t D_s X_s^{m,\pi,C} \pi_s \nu_s dB_s - \int_0^t D_s dC_s \\ &= M_t. \end{aligned} \tag{46}$$

The properties of K and C obey the conditions of a cumulative consumption process in the sense of Definition 11. Due to $D_t X_t^{m,\pi,C} = M_t \geqslant 0$, for $\forall t \leqslant T$, the wealth process is bounded from below, where (π, C) is admissible for m.

As $X_T^{m,\pi,C} = D_T^{-1} M_T = V_T$ quasi-surely, we have $m = E_G(D_T V_T) \in \mathscr{U}$. Due to the definition of $\mathscr{U}$, we conclude that $\nu_{\text{up}} \leqslant E_G(D_T V_T)$.

The proof for the second identity is analogous. Again, using the proof of Lemma 22, we obtain $m \leqslant -E_G(-D_T V_T)$ for any $m \in \mathscr{L}$. Hence, $v_{\mathrm{low}} \leqslant -E_G(-D_T V_T)$.

In order to obtain $v_{\mathrm{low}} \geqslant -E_G(-D_T V_T)$, we define a G-martingale M by

$$M_t = E_G\left(-D_T V_T \mid \mathscr{F}_t\right), \quad \forall t \leqslant T. \tag{47}$$

The remaining part is almost a copy of the above. By the martingale representation theorem [15], there exist $z \in H_G^1(0,T)$, and a continuous, increasing process $K = (K_t)$ with $K_T \in L_G^1(\Omega_T)$ such that, for any $t \leqslant T$,

$$M_t = E_G\left(-D_T V_T\right) + \int_0^t z_s dB_s - K_t \quad \text{q.s.} \tag{48}$$

As the above, for any $t \leqslant T$, let

$$\begin{aligned} -m &= E_G\left(-D_T V_T\right) \geqslant 0, \\ X_t \pi_t v_t &= z_t D_t^{-1} \in H_G^1(0,T), \end{aligned} \tag{49}$$

and $C_t = \int_0^t D_s^{-1} dK_s \in L_G^1(\Omega_T)$. Then the wealth process $X^{-m,\pi,C}$ satisfies

$$\begin{aligned} D_t X_t^{-m,\pi,C} &= -m + \int_0^t D_s X_s^{-m,\pi,C} \pi_s v_s dB_s - \int_0^t D_s dC_s \\ &= M_t, \end{aligned} \tag{50}$$

where C obeys the condition of a cumulative consumption process due to the properties of K. Moreover, for any $t \leqslant T$, it has

$$D_t X_t^{-m,\pi,C} = E_G\left(-D_T V_T \mid \mathscr{F}_t\right) \geqslant E_G\left(-V_T \mid \mathscr{F}_t\right), \tag{51}$$

which is bounded from below in the sense of item (iii) in Definition 15 because $-V_T \in L_G^2(\Omega_T)$. Therefore, the wealth process is bounded from below. Consequently, (π, C) is admissible for $-m$.

Since $X_T^{-m,\pi,C} = D_T^{-1} M_T = -V_T$ q.s., it has $m = -E_G(-D_T V_T) \in \mathscr{L}$.

Due to the definition of $\mathscr{L}$, we conclude $v_{\mathrm{low}} \geqslant -E_G(-D_T V_T)$. So far, we have completed the proof of Theorem 24. □

The proof of Theorem 24 is different from that of Vorbrink [8], because the volatility of stock price is related to t rather than a constant.

Remark 25. Because of sublinear expectation E_G, by Theorem 24 we have $v_{\mathrm{low}} \neq v_{\mathrm{up}}$. This means that the market is not complete implying that not all claims can be hedged perfectly. Therefore, there are many no-arbitrage prices for V. As long as $(E_G[D_T V_T \mid \mathscr{F}_t])$ is not a symmetric G-martingale, it has $v_{\mathrm{low}} \neq v_{\mathrm{up}}$. Under other circumstances, the process K is identically equal to zero (see Remark 8), meaning that $(E_G[D_T V_T \mid \mathscr{F}_t])$ is symmetric and V_T can be hedged perfectly owing to Remark 8 and Theorem 9.

Theorem 26. *For any price $V_0 \in (v_{low}, v_{up}) \neq \emptyset$ of a European contingent claim at time zero, there does not exist any arbitrage opportunity in $(\mathscr{M}, V)$. For any price $V_0 \notin (v_{low}, v_{up}) \neq \emptyset$ there exists an arbitrage in the market.*

Proof. The first part directly follows from Lemma 23. From Lemma 22, we know that $V_0 \notin [v_{\mathrm{low}}, v_{\mathrm{up}}]$ implies the existence of an arbitrage opportunity. Thus, we only need to show that $V_0 = v_{\mathrm{up}}$ and $V_0 = v_{\mathrm{low}}$ admit an arbitrage opportunity.

We only treat the case $V_0 = v_{\mathrm{up}} = \inf\{m \mid m \in \mathscr{U}\}$. Then $m \geqslant V_0$; that is, $-m + V_0 \leqslant 0$. The second case is similar. Comparing the proof of Theorem 24 and letting $m > 0$, for $-m = E_G(-D_T V_T)$, there exists a pair $(\pi, C) \in \mathscr{A}(-m)$ such that

$$\begin{aligned} D_t X_t^{-m,\pi,C} &= -m + \int_0^t D_s X_s^{-m,\pi,C} \pi_s v_s dB_s - \int_0^t D_s dC_s \\ &= M_t = -D_T V_T \quad \text{q.s.} \end{aligned} \tag{52}$$

Then $K_T = \int_0^t D_s dC_s$, where K is an increasing, continuous process with $E_G(-K_T) = 0$. So we can select $P \in \mathbf{P}$ such that $E_P(-K_T) < 0$ (see Remark 25). Then the pair $(\pi, C) \in \mathscr{A}(-m)$ satisfies

$$E^P\left(D_T X_T^{-m,\pi,0}\right) > E^P\left(D_T X_T^{-m,\pi,C}\right) = E^P\left(-D_T V_T\right). \tag{53}$$

Thus, $P(X_T^{m,\pi,0} > -V_T) > 0$ and we conclude that $(\pi, C) \in \mathscr{A}(-m)$ constitutes an arbitrage. □

On account of Theorem 26, we call $(v_{\mathrm{low}}, v_{\mathrm{up}}) \neq \emptyset$ the arbitrage free interval. Particularly, in the Markovian case where $V_T = \Phi(S_T)$ for some Lipschitz function $\Phi : \mathbb{R} \to \mathbb{R}$, we can give more structural details about the bounds v_{up} and v_{low}. We investigate this issue in Section 4.

4. The Markovian Case

For the European contingent claims V, we have the form $V_T = \Phi(S_T)$ for some Lipschitz function $\Phi : \mathbb{R} \to \mathbb{R}$. We use a nonlinear Feynman-Kac formula which is established in Peng [9]. Let us rewrite the dynamics of S in (22) as

$$\begin{aligned} dS_u^{t,x} &= rS_u^{t,x} du + v_u S_u^{t,x} dB_u, \\ & u \in [t,T], \ S_t^{t,x} = x > 0. \end{aligned} \tag{54}$$

Analogy to the lower and upper arbitrage prices at time 0, at time $t \in [0,T]$, the lower and upper arbitrage prices are noted by $v_{\mathrm{low}}^t(x)$ and $v_{\mathrm{up}}^t(x)$, respectively. At a considered time t, the stock price S_t is replaced by the variable x. That is, $S_t = x$.

Theorem 27. *Given a European contingent claim $V = \Phi(S_T)$, its upper arbitrage price $v_{up}^t(x)$ is given by $u(t,x)$, where $u : [0,T] \times \mathbb{R}_+ \to \mathbb{R}$ is the unique solution of the following PDE:*

$$\begin{aligned} \partial_t u + rx\partial_x u + G\left(v^2 x^2 \partial_{xx} u\right) &= ru, \\ u(T,x) &= \Phi(x). \end{aligned} \tag{55}$$

A precise representation for the corresponding trading strategy in the stock and the cumulative consumption process is given by

$$\beta_t = \partial_x u(t, S_t), \quad \forall t \in [0, T],$$
$$C_t = -\frac{1}{2}\int_0^t {\nu_s}^2 {S_s}^2 \partial_{xx} u(s, S_s)\, d\langle B\rangle_s + \int_0^t {\nu_s}^2 {S_s}^2 G(\partial_{xx} u(s, S_s))\, ds, \quad \forall t \in [0, T]. \tag{56}$$

Analogously, $-\underline{u}(t, x)$ is the lower arbitrage price $v^t_{\text{low}}(x)$, where $\underline{u} : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$. $v^t_{\text{low}}(x)$ solves (55) but with terminal condition $\underline{u}(t, x) = -\Phi(x)$, $\forall x \in \mathbb{R}_+$.

Proof. Firstly, we consider the Backward Stochastic Differential Equation (in short BSDE):

$$Y^{t,x}_s = E_G\left(\Phi\left(S^{t,x}_T\right) + \int_s^T f\left(S^{t,x}_r, Y^{t,x}_r\right) dr \mid \mathcal{F}_s\right), \quad s \in [t, T], \tag{57}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given Lipschitz function. Peng [9] showed that the BSDE has a unique solution. So we can define a function $u : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$ by $u(t, x) := Y^{t,x}_t$, $(t, x) \in [0, T] \times \mathbb{R}_+$. In the light of the knowledge of the nonlinear Feynman-Kac formula [9], the function u is a viscosity solution of the following PDE:

$$\partial_t u + rx\partial_x u + G\left(\nu^2 x^2 \partial_{xx} u\right) + f(x, u) = 0, \qquad u(T, x) = \Phi(x). \tag{58}$$

We define the function

$$\hat{u}(t, x) := E_G\left(\Phi\left(S^{t,x}_T\right) D_T\right). \tag{59}$$

According to the above definition, for $f \equiv 0$, $\hat{u}$ solves (58). Since the function G is nondegenerate, $\hat{u}$ turns into a classical $C^{1,2}$-solution (see page 19 in [9]). Consequently, together with Itô's formula (Theorem 5.4 in [18]), it has

$$\begin{aligned}
\hat{u}\left(t, S^{0,x}_t\right) - \hat{u}(0, x) &= \int_0^t \left[\partial_t \hat{u}\left(s, S^{0,x}_s\right) + rS^{0,x}_s \partial_x \hat{u}\left(s, S^{0,x}_s\right)\right] ds + \int_0^t \nu_s S^{0,x}_s \partial_x \hat{u}\left(s, S^{0,x}_s\right) dB_s \\
&\quad + \int_0^t \frac{1}{2}\left(\nu_s S^{0,x}_s\right)^2 \partial_{xx} \hat{u}\left(s, S^{0,x}_s\right) d\langle B\rangle_s \\
&= \int_0^t \nu_s S^{0,x}_s \partial_x \hat{u}\left(s, S^{0,x}_s\right) dB_s + \underbrace{\frac{1}{2}\int_0^t \left(\nu_s S^{0,x}_s\right)^2 \partial_{xx} \hat{u}\left(s, S^{0,x}_s\right) d\langle B\rangle_s - \int_0^t \left(\nu_s S^{0,x}_s\right)^2 G\left(\partial_{xx} \hat{u}\left(s, S^{0,x}_s\right)\right) ds.}_{-K_t = -\int_0^t D_s dC_s}
\end{aligned} \tag{60}$$

Now, consider the function

$$\tilde{u}(t, x) := D_t^{-1} \hat{u}(t, x), \quad \forall (t, x) \in [0, T] \times \mathbb{R}_+. \tag{61}$$

For $t = 0$, based on Theorem 24, it has

$$\tilde{u}(t, x) = \hat{u}(t, x) = E_G\left(\Phi\left(S^{t,x}_T\right) D_T\right) = E_G\left(D_T V_T\right) = v^t_{\text{up}}(x), \quad \forall (t, x) \in [0, T] \times \mathbb{R}_+. \tag{62}$$

Moreover, $\tilde{u}$ can be used as a solution of (55). In addition, the function u defined by

$$u(t, x) := Y^{t,x}_t = E_G\left(\Phi\left(S^{t,x}_T\right) - \int_t^T rY^{t,x}_s ds \mid \mathcal{F}_t\right), \quad \forall (t, x) \in [0, T] \times \mathbb{R}_+, \tag{63}$$

solves (55) owing to the nonlinear Feynman-Kac formula since $f(x, y) = -ry$. By uniqueness of the solution in (55) (see [19]; f is obviously bounded in x), we have $\tilde{u} = u$. Thus,

$$u(t, x) = E_G\left(\Phi\left(S^{t,x}_T\right) D_{T-t}\right) = v^t_{\text{up}}(x), \quad \forall (t, x) \in [0, T] \times \mathbb{R}_+, \tag{64}$$

and it uniquely solves (55).

In combination with the proof of Theorem 24, by using its notions and Remark 12, we obtain the precise expressions for the trading strategy β and the cumulative consumption process C. That is, it has $z_t = \nu_t S^{0,x}_t \partial_x \hat{u}(t, S^{0,x}_t) = S^{0,x}_t \beta_t \nu_t D_t$. Therefore, $\beta_t = \partial_x u(t, S_t)$, $\forall t \in [0, T]$.

Analogously, we derive

$$\begin{aligned}
C_t &= \int_0^t D_t^{-1} dK_s \\
&= -\frac{1}{2}\int_0^t {\nu_s}^2 {S_s}^2 \partial_{xx} u(s, S_s)\, d\langle B\rangle_s \\
&\quad + \int_0^t {\nu_s}^2 {S_s}^2 G(\partial_{xx} u(s, S_s))\, ds, \quad \forall t \in [0, T].
\end{aligned} \tag{65}$$

□

Due to Theorem 27, the functions $u(t, x) = v^t_{\text{up}}(x)$ and $\underline{u}(t, x) = -v^t_{\text{low}}(x)$ can be characterized as the unique solutions of (55). Under the circumstances of Φ being a convex or concave function, respectively, (55) simplifies greatly.

Lemma 28. *(1) If Φ is concave, then $u(t,\cdot)$ is convex for any $t \leqslant T$.*

(2) If Φ is concave, then $u(t,\cdot)$ is concave for any $t \leqslant T$. Similarly, if Φ is convex, then $\underline{u}(t,\cdot)$ is concave for any $t \leqslant T$. If Φ is concave, then $\underline{u}(t,\cdot)$ is convex for any $t \leqslant T$.

Proof. We only need to take into account the upper arbitrage price which is determined by the function

$$u(t,x) = E_G\left(\Phi\left(S_T^{t,x}\right) D_{T-t}\right) = v_{\mathrm{up}}^t(x), \quad \forall (t,x) \in [0,T] \times \mathbb{R}_+. \tag{66}$$

First of all, let Φ be convex, $t \in [0,T]$, and $x, y \in \mathbb{R}_+$. For any $\theta \in [0,1]$, it has

$$\begin{aligned}
u(t,\theta x + (1-\theta) y) &= E_G\left[\Phi\left(S_T^{t,\theta x+(1-\theta)y}\right) e^{-r(T-t)}\right] \\
&= E_G\left[\Phi\left((\theta x + (1-\theta) y) \cdot e^{-r(T-t)-(1/2)\int_t^T {\nu_u}^2 d\langle B\rangle_u + \int_t^T \nu_u dB_u}\right) e^{-r(T-t)}\right] \\
&\leqslant E_G\left[\theta\Phi\left(x e^{-r(T-t)-(1/2)\int_t^T {\nu_u}^2 d\langle B\rangle_u + \int_t^T \nu_u dB_u}\right) + (1-\theta)\Phi\left(y e^{-r(T-t)-(1/2)\int_t^T {\nu_u}^2 d\langle B\rangle_u + \int_t^T \nu_u dB_u}\right)\right] e^{-r(T-t)} \\
&\leqslant E_G\left[\theta\Phi\left(x e^{-r(T-t)-(1/2)\int_t^T {\nu_u}^2 d\langle B\rangle_u + \int_t^T \nu_u dB_u}\right) \cdot e^{-r(T-t)}\right] + E_G\left[(1-\theta) \cdot \Phi\left(y e^{-r(T-t)-(1/2)\int_t^T {\nu_u}^2 d\langle B\rangle_u + \int_t^T \nu_u dB_u}\right)\right] \\
&= \theta E_G\left[\Phi\left(S_T^{t,x}\right) e^{-r(T-t)}\right] + (1-\theta) E_G\left[\Phi\left(S_T^{t,y}\right) \cdot e^{-r(T-t)}\right] = \theta u(t,x) + (1-\theta) u(t,y),
\end{aligned} \tag{67}$$

where we used the convexity of Φ, the monotonicity of E_G, and, in the second inequality, the sublinearity of E_G. Therefore, $u(t,\cdot)$ is convex for all $t \in [0,T]$.

Secondly, let Φ be concave. For any $(t,x) \in [0,T] \times \mathbb{R}_+$, we define

$$g(t,x) := E^P\left[\Phi\left(\widetilde{S}_T^{t,x}\right) e^{-r(T-t)}\right], \tag{68}$$

where

$$d\widetilde{S}_s^{t,x} = r\widetilde{S}_s^{t,x} ds + \underline{\sigma}\widetilde{S}_s^{t,x} dB_s, \quad s \in [t,T], \ \widetilde{S}_t^{t,x} = x. \tag{69}$$

Since $B = (B_t)$ is a classical Brownian motion under P_0, g solves the Black-Scholes PDE (7) with $\overline{\sigma}$ replaced by $\underline{\sigma}$.

Because E^P is linear, this is straightforward to mean that $g(t,\cdot)$ is concave for any $t \in [0,T]$. Consequently, g also solves (55). By uniqueness, we conclude that $g = u$. Therefore, $u(t,\cdot)$ is concave for any $t \in [0,T]$. □

5. Conclusion

In order to analyse the financial markets with volatility uncertainty, we consider a stock price modeled by a geometric G-Brownian motion which features volatility uncertainty. This is all based on the structure of a G-Brownian motion. The "G-framework" is summarized by Peng [9] which gives us a useful mathematical setting. A little new arbitrage free concept is utilized to obtain the detailed results which give us an economically better understanding of financial markets under volatility uncertainty. We establish the connection of the lower and supper arbitrage prices by means of partial differential equations. The outcomes in this paper are only applied to European contingent claims. For other cases, we would extend these results to American contingent claims in our forthcoming paper.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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