Research Article

Boundary Value Methods for Second-Order PDEs via the Lanczos-Chebyshev Reduction Technique

T. A. Biala,[1] S. N. Jator,[2] and R. B. Adeniyi[3]

[1]*Department of Mathematics and Computer Science, Sule Lamido University, PMB 048, Kafin Hausa, Nigeria*
[2]*Department of Mathematics and Statistics, Austin Peay State University, Clarksville, TN 37044, USA*
[3]*Department of Mathematics, University of Ilorin, Ilorin, Nigeria*

Correspondence should be addressed to T. A. Biala; bialatoheeb@yahoo.com

In this paper, we study the performance of Boundary Value Methods (BVMs) on second-order PDEs. The PDEs are transformed into a system of second-order ordinary differential equations (ODEs) using the Lanczos-Chebyshev reduction technique. The conditions under which the BVMs converge and the computational complexities of the algorithms are discussed. Numerical illustrations are given to show the simplicity and high accuracy of the approach.

1. Introduction

The study of methods which approximates the solution of partial differential equations is an important issue in numerical analysis. This is because most modeled problems in science and engineering results in PDEs such as the problem of the elastic torsion of prismatic bars, axisymmetric ideal flow around sphere, the heat equation, the wave equations and so on. In this paper, we are interested in the approximate solution of the initial or boundary value problems of second-order PDEs

$$\mathbf{L}u = f, \tag{1}$$

where $\mathbf{L}$ is a fixed elliptic or hyperbolic operator of order $2m$, $f \in F$, and F is a space function defined on a domain D.

The reduction of (1) to a lower dimensional problem results, in most cases, to either a system of second-order ordinary differential equation (ODE) or an algebraic system. The reduction technique described in this paper is based on the Lanczos τ-method [1, 2]. In this approach, polynomials are used as the trial functions with an important advantage that the "economized" solutions which can be readily integrated or differentiated are obtained [1, 2].

In the last few years, Boundary Value Methods (BVMs) have been used for the solution of the first-order initial and boundary value problems and their convergence and linear stability properties have been fully discussed in [3–8]. These BVMs are also used to solve higher order initial and boundary value problems by first reducing the higher order differential equations into an equivalent first-order system which increases the computational cost and time. Biala [9] and Biala et al. [10–12] developed BVMs for the direct solution of systems of the general second-order ODEs and PDEs with initial or boundary conditions. The boundary value technique simultaneously generates approximate solution $(y_1, y_2, \ldots, y_N)^T$ to the exact solution $(y(x_1), y(x_2), \ldots, y(x_N))^T$ on the entire interval of integration. This approach has the advantage of producing smaller global errors (at the end of the range of integration) than those produced by the step-by-step methods due to the presence of accumulated errors at the each step in the step-by-step method. The novel property of this paper is in the ease with which the second-order PDEs are transformed into a system of ODEs.

The paper is organized as follows: In Section 2, we derive a continuous approximation $U(x)$ of the exact solution $y(x)$ from which the class of BVMs are developed. The convergence and computational complexities of the methods are discussed in Section 3. In Section 4, the Lanczos-Chebyshev reduction technique is introduced. Several numerical illustrations, given in Section 5, are given to show the efficacy of

this approach. Finally, we give a concluding remark in Section 6.

2. Boundary Value Methods

The BVMs given in [9] are a class of methods for the general second-order ODEs based on the Linear Multistep Formulas. Most formulas [4] used for second-order ODEs are implemented by reducing them into a system of first-order ODEs. The BVMs in [9] were introduced in order to lessen the computational cost and time and also to utilize additional information associated with specific differential equations such as the oscillatory nature of solutions. One main feature of the BVMs in [9] is that they can be used in the same way for solving both initial and boundary value problems with some slight change in the code used. Therefore such methods are the best candidates for solving second-order PDEs after reduction into an equivalent second-order system.

In what follows, we consider the general system of second-order boundary value problems

$$y'' = f\left(x, y \cdot y'\right), \quad x \in [a,b], \tag{2}$$

subject to the mixed boundary conditions

$$
\begin{aligned}
a_0 y(a) + b_0 y'(a) &= d_o, \\
a_1 y(b) + b_1 y'(b) &= d_1,
\end{aligned}
\tag{3}
$$

where $f : \mathbb{R} \times \mathbb{R}^{2m} \to \mathbb{R}^m$ are continuous functions, y, y', $y'' \in \mathbb{R}^m$, and m is the dimension of the system. The BVMs for solving (2) take the general form

$$y_{n+j} + \alpha_v^{(j)} y_{n+v} + \alpha_0^{(j)} y_n = h^2 \sum_{i=0}^{2v} \beta_i^{(j)} f_{n+i}, \tag{4}$$

$$j = 1, \ldots, v-1, v+1, \ldots, 2v$$

with the derivative formulas

$$h y'_{n+j} + \alpha_v^{\prime(j)} y_{n+v} + \alpha_0^{\prime(j)} y_n = h^2 \sum_{i=0}^{2v} \beta_i^{\prime(j)} f_{n+i}, \tag{5}$$

$$j = 0, 1, \ldots, 2v,$$

where h is the constant stepsize and $\alpha_v^{(j)}$, $\alpha_0^{(j)}$, $\beta_i^{(j)}$, $\alpha_v^{\prime(j)}$, $\alpha_0^{\prime(j)}$, and $\beta_i^{\prime(j)}$ are chosen so that (4) and (5) have order $2v + 2$ and $2v + 1$, respectively.

2.1. Derivation of the BVMs. In this section, we shall use the interpolation and collocation approach [9–15] to construct a $2v$-step continuous LMM (CLMM) which will be used to produce formulas for solving (2). The CLMM has the general form

$$U(x) = \alpha_v(x) y_{n+v} + \alpha_0(x) y_n + h^2 \sum_{i=0}^{2v} \beta_i(x) f_{n+i}. \tag{6}$$

Evaluating (6) at x_j, $j = 1, \ldots, v-1, v+1, \ldots, 2v$ we obtain the formulas of the form

$$y_{n+j} + 2y_{n+v} - y_n = h^2 \sum_{i=0}^{2v} \beta_i f_{n+i}, \tag{7}$$

$$j = 1, \ldots, v-1, v+1, \ldots, 2v$$

whose derivative formula is

$$h y'_{n+k} + \alpha'_v y_{n+v} + \alpha'_0 y_n = h^2 \sum_{i=0}^{2v} \beta'_i f_{n+i}, \quad k = 0\,(1)\,(2v). \tag{8}$$

To obtain (6), we seek an approximation $U(x)$ to the exact solution $y(x)$ of the form

$$U(x) = \sum_{r=0}^{p+q-1} b_r x^r, \tag{9}$$

where $x \in [a,b]$ and b_r are coefficients to be uniquely determined so that the method has order $2v + 2$ and p and q are the number of interpolation and collocation points, respectively. We impose that the interpolating function (9) satisfies the following conditions:

$$
\begin{aligned}
U(x_v) &= y_v, \\
U(x_0) &= y_0, \\
U''(x_i) &= f_i, \quad i = 0\,(1)\,(2v),
\end{aligned}
\tag{10}
$$

which lead to a system of $2v + 3$ equations which is solved using a Computer Algebra System (CAS) such as Mathematica to obtain b_r, $r = 0(1)(p + q - 1)$.

The CLMM is developed by substituting the values of b_r into (9). After some algebraic manipulations, the CLMM is expressed in form (6). The $2v$-step CLMM is used to generate the main methods (7) and derivative formulas of form (8) which are combined to solve (2). For example, the BVMs of order 6 used are as follows.

BVM order 6:

$$
\begin{aligned}
y_{n+1} - \frac{1}{2} y_{n+2} - \frac{1}{2} y_n &= \frac{h^2}{480}\left(-19 f_n - 204 f_{n+1}\right. \\
&\quad \left. - 14 f_{n+2} - 4 f_{n+3} + f_{n+4}\right),
\end{aligned}
$$

$$
\begin{aligned}
y_{n+3} - \frac{3}{2} y_{n+2} + \frac{1}{2} y_n &= \frac{h^2}{480}\left(17 f_n + 252 f_{n+1}\right. \\
&\quad \left. + 402 f_{n+2} + 52 f_{n+3} - 3 f_{n+4}\right),
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
y_{n+4} - 2 y_{n+2} - y_n &= \frac{h^2}{15}\left(f_n + 16 f_{n+1} + 26 f_{n+2}\right. \\
&\quad \left. + 16 f_{n+3} + f_{n+4}\right)
\end{aligned}
$$

$$n = 0\,(4)\,(N - 4),$$

with the derivative formulas

$$hy_n' - \frac{1}{2}y_{n+2} + \frac{1}{2}y_n = \frac{h^2}{180}\left(-53f_n - 144f_{n+1}\right.$$
$$\left. + 30f_{n+2} - 16f_{n+3} + 3f_{n+4}\right),$$

$$hy_{n+1}' - \frac{1}{2}y_{n+2} + \frac{1}{2}y_n = \frac{h^2}{720}\left(39f_n + 70f_{n+1}\right.$$
$$\left. - 144f_{n+2} + 42f_{n+3} - 7f_{n+4}\right),$$

$$hy_{n+2}' - \frac{1}{2}y_{n+2} + \frac{1}{2}y_n = \frac{h^2}{180}\left(5f_n + 104f_{n+1} + 78f_{n+2}\right. \quad (12)$$
$$\left. - 8f_{n+3} + f_{n+4}\right),$$

$$hy_{n+3}' - \frac{1}{2}y_{n+2} + \frac{1}{2}y_n = \frac{h^2}{720}\left(31f_n + 342f_{n+1}\right.$$
$$\left. + 768f_{n+2} + 314f_{n+3} - 15f_{n+4}\right),$$

$$hy_{n+4}' - \frac{1}{2}y_{n+2} + \frac{1}{2}y_n = \frac{h^2}{180}\left(3f_n + 112f_{n+1} + 56f_{n+2}\right.$$
$$\left. + 240f_{n+3} + 59f_{n+4}\right).$$

Also, the BVMs of order 8 are as follows.

BVM order 8:

$$y_{n+1} - \frac{1}{3}y_{n+3} - \frac{2}{3}y_n = \frac{h^2}{60480}\left(-2803f_n - 37950f_{n+1}\right.$$
$$\left. - 14913f_{n+2} - 7108f_{n+3} + 3147f_{n+4} - 990f_{n+5}\right.$$
$$\left. + 137f_{n+6}\right),$$

$$y_{n+2} - \frac{2}{3}y_{n+3} - \frac{2}{3}y_n = \frac{h^2}{60480}\left(-1291f_n - 21906f_{n+1}\right.$$
$$\left. - 32133f_{n+2} - 6288f_{n+3} + 1467f_{n+4} - 402f_{n+5}\right.$$
$$\left. + 53f_{n+6}\right),$$

$$y_{n+4} - \frac{4}{3}y_{n+3} + \frac{1}{3}y_n = \frac{h^2}{30240}\left(661f_n + 10734f_{n+1}\right.$$
$$\left. + 19323f_{n+2} + 27268f_{n+3} + 2523f_{n+4} - 18f_{n+5}\right. \quad (13)$$
$$\left. - 11f_{n+6}\right),$$

$$y_{n+5} - \frac{5}{3}y_{n+3} + \frac{2}{3}y_n = \frac{h^2}{12096}\left(535f_n + 8550f_{n+1}\right.$$
$$\left. + 15501f_{n+2} + 22900f_{n+3} + 11889f_{n+4} + 1158f_{n+5}\right.$$
$$\left. - 53f_{n+6}\right),$$

$$y_{n+6} - 2y_{n+3} + y_n = \frac{h^2}{2240}\left(141f_n + 2430f_{n+1}\right.$$
$$\left. + 4131f_{n+2} + 6756f_{n+3} + 4131f_{n+4} + 2430f_{n+5}\right.$$
$$\left. + 141f_{n+6}\right)$$

$$n = 0\,(6)\,(N - 6),$$

with the derivatives

$$hy_n' - \frac{1}{3}y_{n+3} + \frac{1}{3}y_n = \frac{h^2}{13440}\left(-3795f_n - 14850f_{n+1}\right.$$
$$\left. + 2403f_{n+2} - 6300f_{n+3} + 3267f_{n+4} - 1026f_{n+5}\right.$$
$$\left. + 141f_{n+6}\right),$$

$$hy_{n+1}' - \frac{1}{3}y_{n+3} + \frac{1}{3}y_n = \frac{h^2}{120960}\left(4019f_n - 3426f_{n+1}\right.$$
$$\left. - 7125f_{n+2} + 18308f_{n+3} - 11019f_{n+4} + 3390f_{n+5}\right.$$

$$\left. - 457f_{n+6}\right),$$

$$hy_{n+2}' - \frac{1}{3}y_{n+3} + \frac{1}{3}y_n = \frac{h^2}{120960}\left(2293f_n\right.$$
$$\left. + 46830f_{n+1} + 22683f_{n+2} - 14204f_{n+3} + 3579f_{n+4}\right.$$
$$\left. - 786f_{n+5} + 85f_{n+6}\right),$$

$$hy_{n+3}' - \frac{1}{3}y_{n+3} + \frac{1}{3}y_n = \frac{h^2}{13440}\left(315f_n + 4590f_{n+1}\right.$$
$$\left. + 9369f_{n+2} + 6576f_{n+3} - 1107f_{n+4} + 270f_{n+5}\right.$$
$$\left. - 33f_{n+6}\right),$$

$$hy_{n+4}' - \frac{1}{3}y_{n+3} + \frac{1}{3}y_n = \frac{h^2}{120960}\left(2453f_n\right.$$
$$\left. + 44526f_{n+1} + 70779f_{n+2} + 135812f_{n+3}\right.$$
$$\left. + 51675f_{n+4} - 3090f_{n+5} + 245f_{n+6}\right),$$

$$hy_{n+5}' - \frac{1}{3}y_{n+3} + \frac{1}{3}y_n = \frac{h^2}{120960}\left(2995f_n\right.$$
$$\left. + 40350f_{n+1} + 85377f_{n+2} + 103300f_{n+3}\right.$$
$$\left. + 145653f_{n+4} + 47166f_{n+5} - 1481f_{n+6}\right),$$

$$hy_{n+6}' - \frac{1}{3}y_{n+3} + \frac{1}{3}y_n = \frac{h^2}{13440}\left(141f_n + 5886f_{n+1}\right.$$
$$\left. + 4995f_{n+2} + 19812f_{n+3} + 5859f_{n+4} + 19710f_{n+5}\right.$$
$$\left. + 4077f_{n+6}\right)$$

$$n = 0\,(6)\,(N - 6).$$
$$(14)$$

Remark 1. We note here that, for higher orders (higher degree polynomials of (9)), the accuracy of the scheme becomes lower and tends to integrate the scheme less accurately. This is due to the truncation errors in the numerical solution.

3. Convergence of the BVMs

In this section, we shall establish the convergence of the BVMs derived in the previous section. We emphasize that we evaluate (6) at $x_1, x_2, \ldots, x_{v-1}, x_{v+1}, \ldots, x_{2v}$ to obtain

$$y_{n+1} + \alpha_v^{(1)}y_{n+v} + \alpha_0^{(1)}y_n = h^2\sum_{i=0}^{2v}\beta_i^{(1)}f_{n+i},$$

$$y_{n+2} + \alpha_v^{(2)}y_{n+v} + \alpha_0^{(2)}y_n = h^2\sum_{i=0}^{2v}\beta_i^{(2)}f_{n+i},$$

$$\vdots$$

$$y_{n+v-1} + \alpha_v^{(v-1)}y_{n+v} + \alpha_0^{(v-1)}y_n = h^2\sum_{i=0}^{2v}\beta_i^{(v-1)}f_{n+i},$$

$$y_{n+v+1} + \alpha_v^{(v+1)} y_{n+v} + \alpha_0^{(v+1)} y_n = h^2 \sum_{i=0}^{2v} \beta_i^{(v+1)} f_{n+i},$$

$$\vdots$$

$$y_{n+2v} + \alpha_v^{(2v)} y_{n+v} + \alpha_0^{(2v)} y_n = h^2 \sum_{i=0}^{2v} \beta_i^{(2v)} f_{n+i}$$

$$(15)$$

and also evaluate $U'(x)$ at x_i, $i = 0(1)(2v)$ to obtain

$$hy_n' + \alpha_v'^{(0)} y_{n+v} + \alpha_0'^{(0)} y_n = h^2 \sum_{i=0}^{2v} \beta_i'^{(0)} f_{n+i},$$

$$hy_{n+1}' + \alpha_v'^{(1)} y_{n+v} + \alpha_0'^{(1)} y_n = h^2 \sum_{i=0}^{2v} \beta_i'^{(1)} f_{n+i},$$

$$(16)$$

$$\vdots$$

$$hy_{n+2v}' + \alpha_v'^{(2v)} y_{n+v} + \alpha_0'^{(2v)} y_n = h^2 \sum_{i=0}^{2v} \beta_i'^{(2v)} f_{n+i}.$$

We note that the formulas in 3 are of $O(h^{k+4})$ while the derivative formulas are of $O(h^{k+3})$. Equations 3 and (16) can be compactly written in matrix form by introducing the following matrix notations. Let R be a $2N \times 2N$ matrix defined by

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}, \tag{17}$$

where R_{ij} are $N \times N$ matrices given as

R_{11}

$$= \begin{pmatrix} 1 & & & \alpha_v'^{(0)} & & & & & \\ & \ddots & & \alpha_v^{(1)} & & & & & \\ & & 1 & \alpha_v^{(v-1)} & & & & & \\ & & & \alpha_v^{(v+1)} & 1 & & & & \\ & & & & & \ddots & & & \\ & & & \alpha_v^{(2v)} & & 1 & & & \\ & & & & & & 1 & \alpha_v'^{(0)} & & \alpha_0'^{(0)} \\ & & & & & & & \alpha_v^{(1)} & & \alpha_0^{(1)} \\ & & & & & & & \ddots & & \\ & & & & & & 1 & \alpha_v^{(v-1)} & & \alpha_0^{(v-1)} \\ & & & & & & & \alpha_v^{(v+1)} & 1 & \alpha_0^{(v+1)} \\ & & & & & & & & \ddots & \\ & & & & & & & \alpha_v^{(2v)} & 1 & \alpha_0^{(2v)} \end{pmatrix}, \tag{18}$$

$$R_{21} = \begin{pmatrix} \alpha_v'^{(1)} & & & \\ \vdots & & & \\ \alpha_v'^{(2v)} & & & \\ & \alpha_v'^{(1)} & & \alpha_0'^{(1)} \\ & \vdots & & \vdots \\ & \alpha_v'^{(2v)} & & \alpha_0'^{(2v)} \end{pmatrix},$$

R_{12} is an $N \times N$ null matrix, and R_{22} is an $N \times N$ identity matrix.

Similarly, let S be a $2N \times 2N$ matrix defined by

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \tag{19}$$

where S_{ij} are $N \times N$ matrices given as

$$S_{11} = h^2 \begin{pmatrix} \begin{matrix} \beta_1'^{(0)} & \beta_2'^{(0)} & \cdots & \beta_{2v}'^{(0)} \\ \beta_1^{(1)} & \beta_2^{(1)} & \cdots & \beta_{2v}^{(1)} \\ \vdots & & & \\ \beta_1^{(v-1)} & \beta_2^{(v-11)} & \cdots & \beta_{2v}^{(v-11)} \\ \beta_1^{(v+1)} & \beta_2^{(v+1)} & \cdots & \beta_{2v}^{(v+1)} \\ \vdots & & & \\ \beta_1^{(2v)} & \beta_2^{(2v)} & \cdots & \beta_{2v}^{(2v)} \end{matrix} & & \\ & \begin{matrix} \beta_1'^{(0)} & \beta_2'^{(0)} & \cdots & \beta_{2v}'^{(0)} \\ \beta_1^{(1)} & \beta_2^{(1)} & \cdots & \beta_{2v}^{(1)} \\ \vdots & & & \\ \beta_1^{(v-1)} & \beta_2^{(v-1)} & \cdots & \beta_{2v}^{(v-1)} \\ \beta_1^{(v+1)} & \beta_2^{(v+1)} & \cdots & \beta_{2v}^{(v+1)} \\ \vdots & & & \\ \beta_1^{(2v)} & \beta_2^{(2v)} & \cdots & \beta_{2v}^{(2v)} \end{matrix} & \\ & & \begin{matrix} \ddots & \ddots & \ddots & \ddots \\ \beta_1^{(2v)} & \beta_2^{(2v)} & \cdots & \beta_{2v}^{(2v)} \end{matrix} \end{pmatrix},$$

$$S_{21} = h^2 \begin{pmatrix} \beta_1'^{(1)} & \beta_2'^{(1)} & \cdots & \beta_{2v}'^{(1)} \\ \vdots & & & \\ \beta_1'^{(2v)} & \beta_2'^{(2v)} & \cdots & \beta_{2v}'^{(2v)} \\ & & & & \beta_1'^{(1)} & \beta_2'^{(1)} & \cdots & \beta_{2v}'^{(1)} \\ & & & & \vdots & & & \\ & & & & \beta_1'^{(2v)} & \beta_2'^{(2v)} & \cdots & \beta_{2v}'^{(2v)} \\ & & & & & & & & \ddots & \ddots & \ddots & \ddots \\ & & & & & & & & \beta_1'^{(2v)} & \beta_2'^{(2v)} & \cdots & \beta_{2v}'^{(2v)} \end{pmatrix};$$

$$(20)$$

S_{12}, S_{22} are $N \times N$ null matrices.

We further define the following vectors:

$$\overline{Y} = \left(y_{n+1}, \ldots, y_{n+2v}, h y'_{n+1}, \ldots, h y'_{n+2v} \right)^T,$$

$$Y = \left(y(x_{n+1}), \ldots, y(x_{n+2v}), h y'(x_{n+1}), \ldots, \right.$$
$$\left. h y'(x_{n+2v}) \right)^T,$$

$$F = \left(f_{n+1}, \ldots, f_{n+2v}, h f'_{n+1}, \ldots, h f'_{n+2v} \right)^T,$$

$$L(h) = \left(l_1, \ldots, l_N, h l'_1, \ldots, h l'_N \right)^T,$$

$$(21)$$

$$C = \left(\beta_0'^{(0)} h^2 f_0 - h y'_0 - \alpha_0'^{(0)} y_0, \beta_0^{(1)} h^2 f_0 - \alpha_0^{(1)} y_0, \ldots, \right.$$

$$\beta_0^{(v-1)} h^2 f_0 - \alpha_0^{(v-1)} y_0, \beta_0^{(v+1)} h^2 f_0 - \alpha_0^{(v+1)} y_0, \ldots,$$

$$\left. \beta_0^{(2v)} h^2 f_0 - \alpha_0^{(2v)} y_0, 0, \ldots, 0 \right)^T.$$

The exact form of the system formed by 3 and (16) is given by

$$RY - SF(Y) + C + L(h) = 0, \qquad (22)$$

where $L(h)$ is the truncation error vector of the formulas in 3 and (16). The approximate form of the system is given by

$$R\overline{Y} - SF\left(\overline{Y}\right) + C = 0, \qquad (23)$$

where $\overline{Y}$ is the approximate solution of vector Y.

Subtracting (22) from (23) and letting $E = \overline{Y} - Y = (e_1, \ldots, e_N, e'_1, \ldots, e'_N)^T$ and using the mean value theorem, we have the error system

$$(R - SJ) E = L(h), \qquad (24)$$

where J is the Jacobian matrix and its entries J_{11}, J_{12}, J_{21}, and J_{22} are defined as

$$J_{11} = \begin{pmatrix} \dfrac{\partial f_1}{\partial y_1} & \cdots & \dfrac{\partial f_1}{\partial y_N} \\ \vdots & \vdots & \vdots \\ \dfrac{\partial f_N}{\partial y_1} & \cdots & \dfrac{\partial f_N}{\partial y_N} \end{pmatrix},$$

$$J_{12} = \begin{pmatrix} \dfrac{\partial f_1}{\partial y'_1} & \cdots & \dfrac{\partial f_1}{\partial y'_N} \\ \vdots & \vdots & \vdots \\ \dfrac{\partial f_N}{\partial y'_1} & \cdots & \dfrac{\partial f_N}{\partial y'_N} \end{pmatrix},$$

$$(25)$$

$$J_{21} = h \begin{pmatrix} \dfrac{\partial f'_1}{\partial y_1} & \cdots & \dfrac{\partial f'_1}{\partial y_N} \\ \vdots & \vdots & \vdots \\ \dfrac{\partial f'_N}{\partial y_1} & \cdots & \dfrac{\partial f'_N}{\partial y_N} \end{pmatrix},$$

$$J_{22} = h \begin{pmatrix} \dfrac{\partial f'_1}{\partial y'_1} & \cdots & \dfrac{\partial f'_1}{\partial y'_N} \\ \vdots & \vdots & \vdots \\ \dfrac{\partial f'_N}{\partial y'_1} & \cdots & \dfrac{\partial f'_N}{\partial y'_N} \end{pmatrix}.$$

Let $M = -SJ$ be a matrix of dimension $2N$ so that (24) becomes

$$(R + M) E = L(h), \qquad (26)$$

and for sufficiently small h, $R + M$ is a monotone matrix and thus nonsingular (Jator and Li [14]). Therefore

$$(R + M)^{-1} = D = \left(d_{ij}\right) \geq 0,$$

$$\sum_{j=1}^{2N} d_{ij} = O\left(h^{-2}\right),$$

$$E = DL\left(h\right), \tag{27}$$

$$\|E\| = \|DL\left(h\right)\|,$$

$$= O\left(h^{-2}\right) O\left(h^{k+4}\right),$$

$$= O\left(h^{k+2}\right)$$

which shows that the methods are convergent and the global error is of order $O(h^{k+2})$.

3.1. Computational Complexities. In this section, we discuss the use of 3 and (16), which we henceforth call BVM (BVM), for $= 0(2\nu)(N-2\nu)$, where N is a multiple of 2ν. We emphasize that the methods in the BVM are all main methods since they are weighted the same and their use lead to a single matrix equation which can be solved for the unknowns. For example, for BVM6 (order 6), we make use of each of the methods above in steps of 4; that is, $n = 0, 4, \ldots, N - 4$. This results in a system of $2N$ equations in $2N$ unknowns which can be easily solved for the unknowns. Below is an algorithm for the use of the methods.

The methods are implemented as BVMs by efficiently using the following steps.

Step 1. Use the BVM for $n = 0$ to obtain $\mathbf{Y}_1$ in the interval $[y_0, y_{2\nu}]$ and for $n = 1$ $\mathbf{Y}_2$ is obtained in the interval $[y_{2\nu}, y_{4\nu}]$. Similarly, for $n = 2, 3, \ldots, (\Gamma - 1)$ we obtain $\mathbf{Y}_3, \ldots, \mathbf{Y}_\Gamma$, where $N = 2\nu \times \Gamma$ in the intervals, $[y_{4\nu}, y_{6\nu}], [y_{6\nu}, y_{8\nu}], \ldots, [y_{N-2\nu}, y_N]$, respectively.

Step 2. The unified block given by the system $\mathbf{Y}_1 \cup \mathbf{Y}_2 \cup \cdots \cup \mathbf{Y}_{\Gamma-1} \cup \mathbf{Y}_\Gamma$ obtained in Step 1 results in a system of $2N$ equations in $2N$ unknowns which can be easily solved.

Step 3. The values of the solution and the first derivatives of (1) are generated by the sequence $\{y_n\}, \{y'_n\}, n = 0, \ldots, N$, obtained as the solution in Step 2.

4. Lanczos-Chebyshev Reduction Technique

We consider the second-order problem of the form

$$\mathbf{L}u\left(x, y\right) = f \quad \text{in } \Upsilon \tag{28}$$

with the boundary condition

$$\mathbf{L}u\left(x, y\right) = g \quad \text{on } \Omega, \tag{29}$$

where Ω is the boundary of the region Υ. The Lanczos τ-method involves the replacement of one of the two functions in the trial solution by an approximate polynomial of the form

$$G_i = x^{i-1} \tag{30}$$

and $u(x, y)$ is approximated by

$$u\left(x, y\right) \approx u_N\left(x, y\right) = \sum_{i=1}^{N} p_i\left(y\right) x^{i-1} \tag{31}$$

and Ω must be bounded by the lines $x = \pm 1$ in the x-direction. The problem is slightly perturbed to obtain

$$\mathbf{L}u_N\left(x, y\right) = \tau_1\left(y\right) C_{N-2}^*\left(x\right) + \tau_2\left(y\right) C_{N-1}^*\left(x\right) + f, \tag{32}$$

$$\mathbf{L}u_N\left(x, y\right) = g, \quad x \in [a, b], \tag{33}$$

where $\tau_i(y)$ are arbitrary functions and $C_j^*(x)$ are the jth order shifted Chebyshev polynomials in the range $x \in [a, b]$. Equating the powers of x in (32) together with the boundary conditions (33) gives $(N + 2)$ equations in $(N + 2)$ unknowns $\tau_1(y), \tau_2(y), p_1(y), \ldots, p_N(y)$. The operator $\mathbf{L}$ may also contain polynomial functions. The inclusion of (33) is to ensure the exact satisfaction of the exact solutions at $x = a$ and $x = b$. The arbitrary τ functions are eliminated to give a set of N second-order ordinary differential equations. Hence, $p_i(y)$ can be uniquely determined.

5. Numerical Illustrations

In this section, we consider five numerical examples using the BVMs of order 6 to solve the ODE system arising from the semidiscretization of the PDEs. All computations were carried out using a written code in Mathematica 11.0. In all examples, a uniform stepsize was used and the maximum absolute errors were computed as

$$\max \left\|\left(u_{ij} - u\left(x_i, y_j\right)\right)\right\|, \quad 0 \leq i \leq N, \ 0 \leq j \leq M, \tag{34}$$

where u_{ij} is the numerical approximation of the exact solution $u(x_i, y_j)$ at the mesh point (x_i, y_j).

Example 1. We consider the following one-dimensional elliptic PDE:

$$u_{xx} + u_{yy} = 2\left(x^2 + y^2 - 2\right), \quad x, y \in [-1, 1],$$

$$u\left(x, \pm 1\right) = 0 \tag{35}$$

Exact: $u\left(x, y\right) = \left(x^2 - 1\right)\left(y^2 - 1\right)$.

We seek an approximation of the form

$$u_N\left(x, y\right) = \sum_{i=1}^{N} p_i\left(y\right) x^{i-1} \tag{36}$$

to obtain

$$\nabla^2 u = \sum_{i=1}^{N} \left\{p_i''\left(y\right) + i\left(i + 1\right) p_{i+2}\left(y\right)\right\} x^{i-1} \tag{37}$$

and the perturbed form

$$\nabla^2 u_N\left(x, y\right) = 2\left(x^2 + y^2 - 2\right) + \tau_1 C_{N-2}\left(x\right)$$

$$+ \tau_2\left(y\right) C_{N-1}\left(x\right) \tag{38}$$

TABLE 1: Maximum errors on the line $x = 0$ for the elliptic PDE in Example 1.

	N			
4	5	6	7	8
$4.441e - 16$	$2.776e - 16$	$2.220e - 16$	$2.43e - 16$	$2.220e - 16$

with

$$\nabla^2 u_N (x, y) = 2 (x^2 + y^2 - 2)\big|_{x=\pm 1}, \tag{39}$$

where

$$C_N (x) = \sum_{i=1}^{N} C_i^N x^i, \quad -1 \le x \le 1. \tag{40}$$

Equating the powers of x in (38), we obtain

$$p_i'' (y) + i (i + 1) p_{i+2} (y)$$
$$= \tau_1 (y) C_{i-1}^{N-2} + \tau_2 (y) C_{i-1}^{N-1}$$

$$i = 2, 4, 5, \ldots, N - 2,$$

$$p_1'' (y) + 2 p_3 (y)$$
$$= -4 + 2 y^2 + \tau_1 (y) C_0^{N-2} + \tau_2 (y) C_0^{N-1}, \tag{41}$$

$$p_3'' (y) + 12 p_5 (y) = 2 + \tau_1 (y) C_2^{N-2} + \tau_2 (y) C_2^{N-1},$$

$$p_{N-1}'' (y) = \tau_1 (y) C_{N-2}^{N-2} + \tau_2 (y) C_{N-2}^{N-1},$$

$$p_N'' (y) = \tau_2 (y) C_{N-1}^{N-1}.$$

From (39) and (41), we eliminate $\tau_i(y)$ to obtain N second-order differential equations in N unknowns $p_i(y)$, $i = 1(1)N$. To use the BVMs to solve the resulting second-order system, we require N boundary conditions which is obtained from $u(x, \pm 1) = 0$; that is, $p_0(\pm 1) = \cdots = p_N(\pm 1) = 0$. Table 1 shows the computational results; Figures 1–3 show the exact, approximate, and error function of the elliptic problem, respectively.

Example 2. We consider the one-dimensional wave equation with variable coefficients

$$\frac{x^2}{2} u_{xx} - u_{yy} = 0, \quad x, y \in [0, 1],$$

$$u (x, 0) = x, \tag{42}$$

$$u_y (x, 0) = x^2$$

Exact: $u (x, y) = x + x^2 \sinh (y)$.

The perturbed form of the problem is

$$\frac{x^2}{2} (u_N)_{xx} - (u_N)_{yy} = \tau_1 (y) C_{N-2}^* (x) \tag{43}$$
$$+ \tau_2 (y) C_{N-1}^* (x)$$
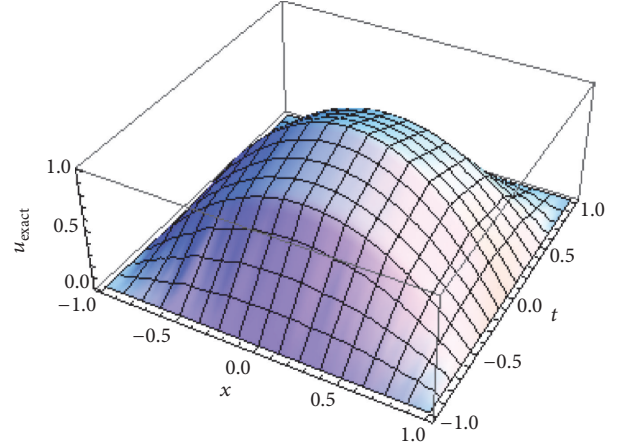
FIGURE 1: Graph of the exact solution for Example 1.
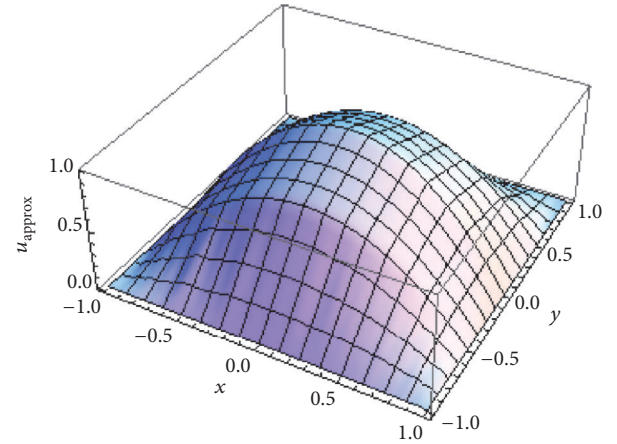
FIGURE 2: Graph of the approximate solution for Example 1.

with

$$\frac{x^2}{2} (u_N)_{xx} - (u_N)_{yy} = 0\big|_{x=0,1} \tag{44}$$

$$C_N^* (x) = \sum_{i=0}^{N} C_i^N x^i, \quad 0 \le x \le 1.$$

Equating powers of x, we obtain

$$\frac{1}{2} (i - 1) (i + 1) p_i (y) - p_i'' (y)$$
$$= \tau_1 (y) C_{i-1}^{N-2} + \tau_2 (y) C_{i-1}^{N-1} \quad i = 3, \ldots, N - 1,$$

$$p_1'' (y) = -\tau_1 (y) C_0^{N-2} - \tau_2 (y) C_0^{N-1}, \tag{45}$$

$$p_2'' (y) = -\tau_1 (y) C_1^{N-2} - \tau_2 (y) C_1^{N-1},$$

$$p_N'' (y) = -\tau_2 (y) C_{N-1}^{N-1}.$$

Eliminating $\tau_i(y)$ from (44) and (45), we obtain N second-order differential equations in N unknowns $p_i(y)$, $i = 1(1)N$.
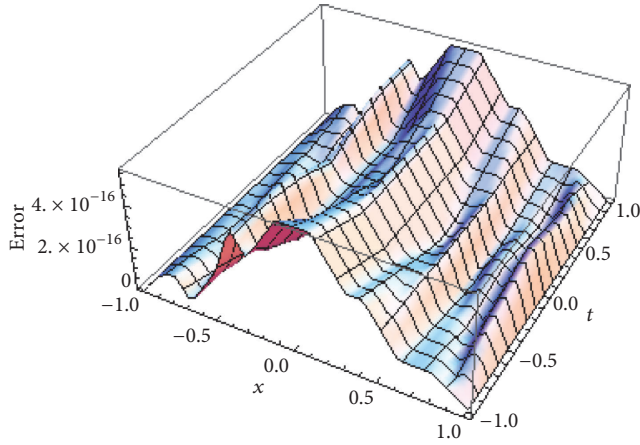
FIGURE 3: Graph of the error function for Example 1.
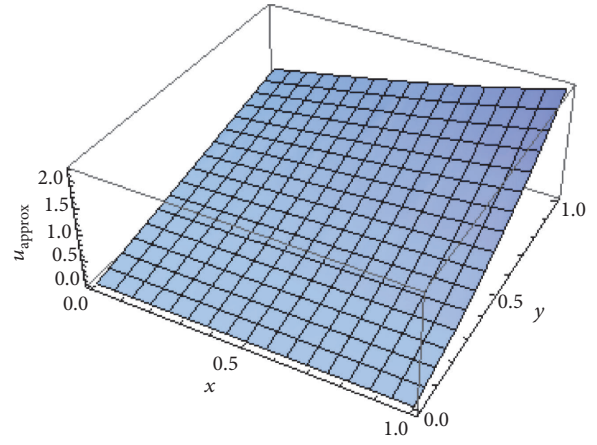
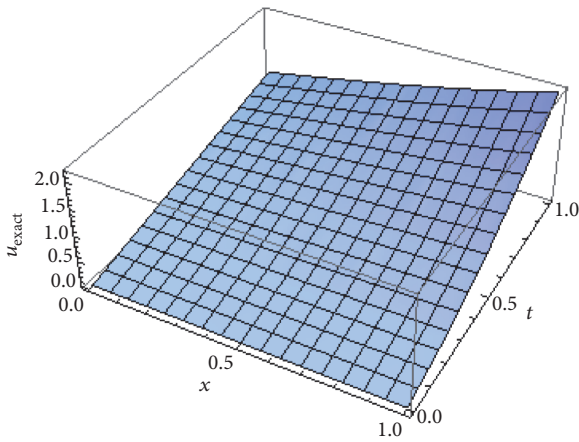FIGURE 5: Graph of the approximate solution for Example 2.

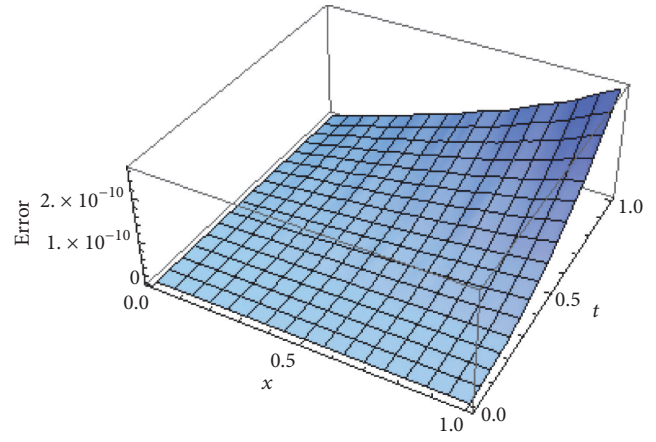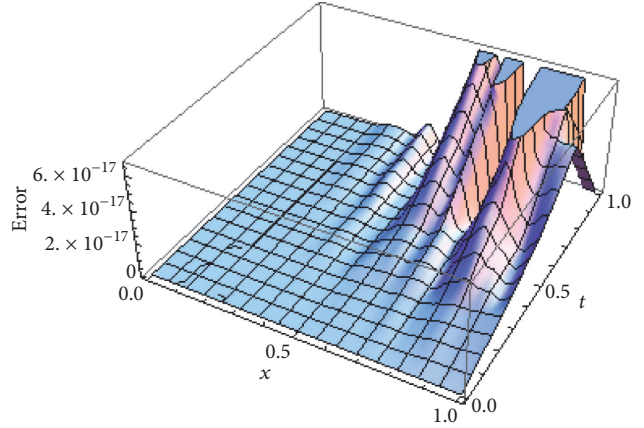FIGURE 4: Graph of the exact solution for Example 2.

FIGURE 6: Graph of the error function for Example 2.

The initial conditions to be used with this system of second-order ODE are

$$p_1(0) = 0,$$

$$p_2(0) = 1,$$

$$p_3(0) = \cdots = p_N(0) = 0,$$

$$p_1'(0) = 0, \tag{46}$$

$$p_2'(0) = 0,$$

$$p_3'(0) = 1,$$

$$p_4'(0) = \cdots = p_N'(0) = 0.$$

Table 2 shows the maximum errors on the line $x = 0.5$ for different N. Also Figures 4–6 depict the exact, approximate, and error function of the wave equation.

TABLE 2: Maximum errors on the line $x = 0.5$ for the wave equation in Example 2.

		N		
4	5	6	7	8
$6.579e-11$	$4.211e-11$	$2.924e-11$	$2.148e-11$	$1.645e-11$

Example 3. Consider the one-dimensional linear inhomogeneous Klein-Gordon equation

$$u_{tt} - u_{xx} + u = 6x^3 t + t^3 \left(x^3 - 6x\right), \quad x, t \in [0,1],$$

$$u(x,0) = 0,$$

$$u_t(x,0) = 0 \tag{47}$$

Exact: $u(x,t) = x^3 t^3$.

With the procedures of the previous examples, we obtain the maximum errors as shown in Table 3. Figures 7–9 also show the exact, approximate, and error function of the Klein-Gordon equation.
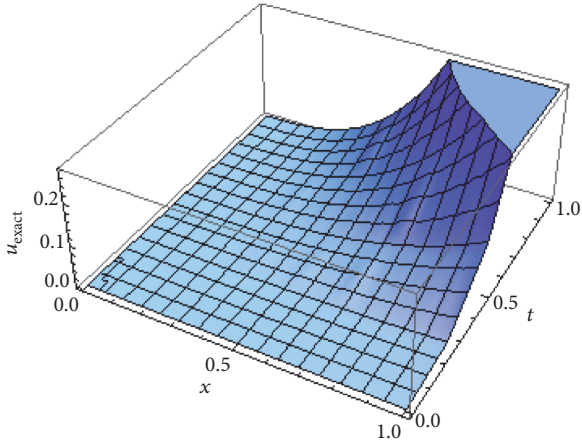
FIGURE 7: Graph of the exact solution for Example 3.

FIGURE 9: Graph of the error function for Example 3.
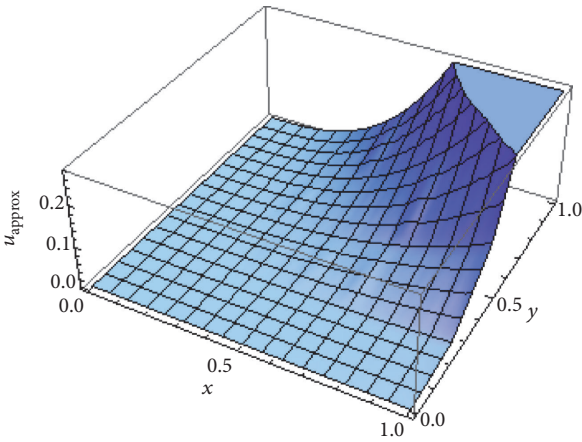
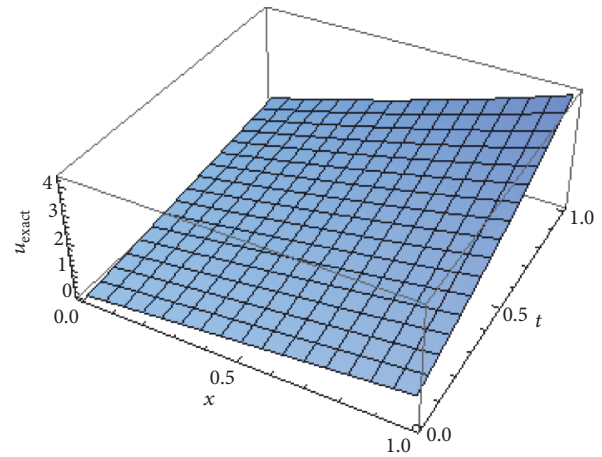FIGURE 8: Graph of the approximate solution for Example 3.

FIGURE 10: Graph of the exact solution for Example 4.

TABLE 3: Maximum errors on the line $x = 0.5$ for the Klein-Gordon equation in Example 3.

		N		
4	5	6	7	8
$4.163e - 17$	$1.388e - 17$	$6.939e - 18$	$3.469e - 18$	$3.469e - 18$

TABLE 4: Maximum errors on the line $x = 0.5$ for the Helmholtz equation in Example 4.

	N		
4	6	8	10
$9.554e - 03$	$.170e - 04$	$1.998e - 06$	$1.639e - 08$

Example 4. Consider the Helmholtz equation

$$u_{xx} + u_{yy} - u = 0, \quad x, y \in [0, 1],$$
$$u(x, 0) = 0,$$
$$u_y(x, 0) = e^x \tag{48}$$

Exact: $u(x, y) = ye^x + x\cosh(y)$.

Table 4 shows the numerical results for the Helmholtz equation. We observe that, for this example, as N increases the method gives more accurate results. This is due to the accurate matching of the derivative conditions as N increases. Figures 10–12 show the plot of the exact, approximate, and error function for this example.

Example 5. Lastly, we consider the nonlinear equation

$$u_{tt} = (uu_x)_x, \quad x, t \in [0, 1],$$
$$u(x, 0) = x^2,$$
$$u_t(x, 0) = -2x^2 \tag{49}$$

Exact: $u(x, t) = \left(\dfrac{x}{t + 1}\right)^2$.

Table 5 and Figures 13–15 show the numerical results for the nonlinear equation.
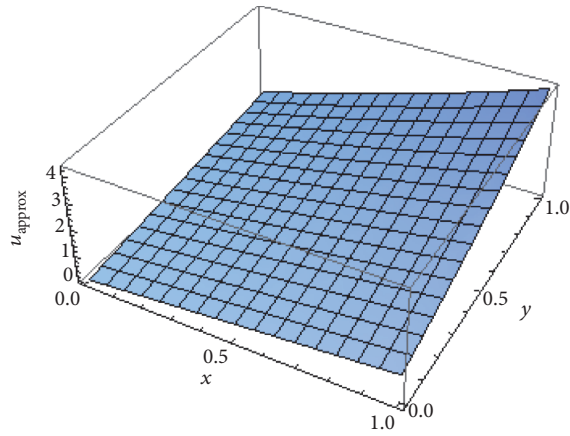
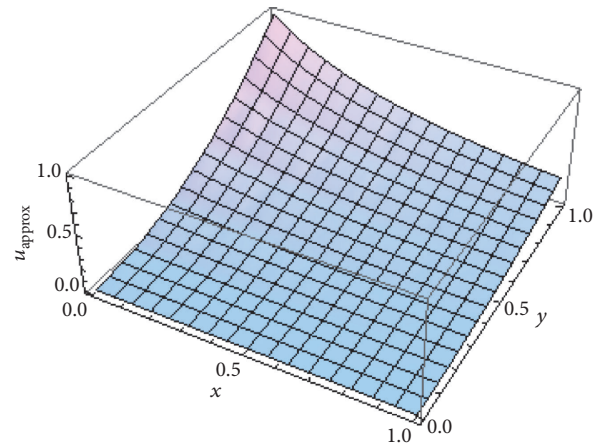FIGURE 11: Graph of the approximate solution for Example 4.

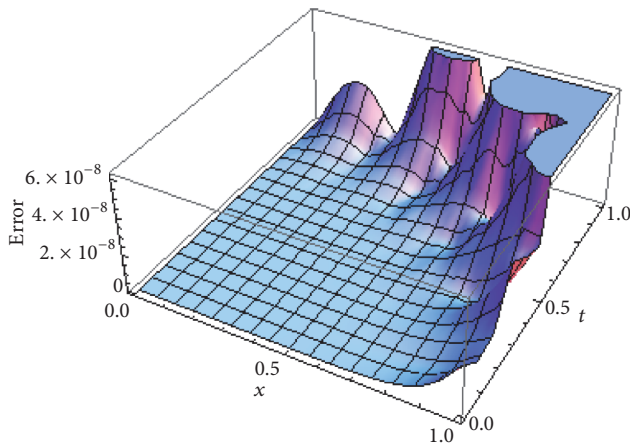FIGURE 14: Graph of the approximate solution for Example 5.

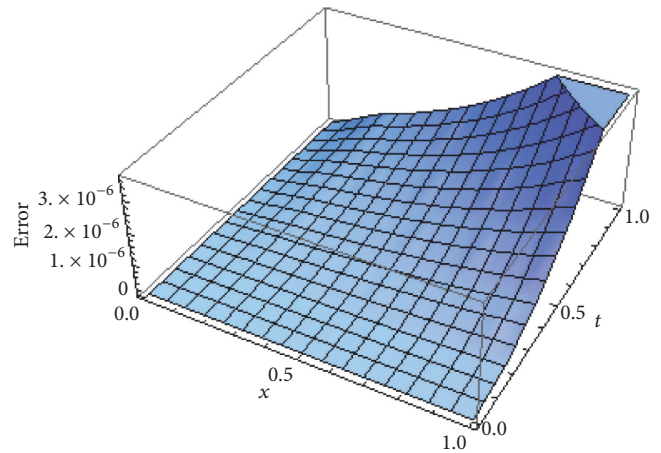FIGURE 12: Graph of the error function for Example 4.

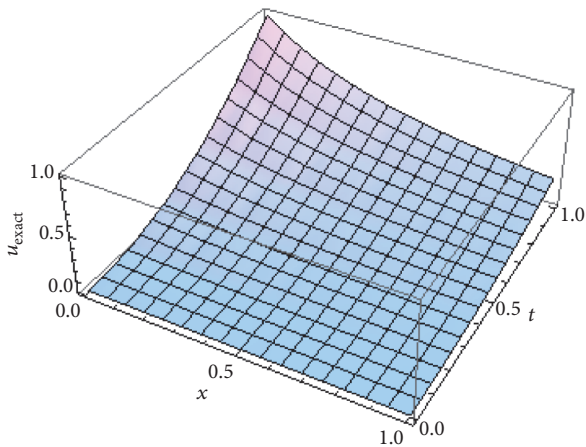FIGURE 15: Graph of the error function for Example 5.

TABLE 5: Maximum errors on the line $x = 0.5$ for the nonlinear equation in Example 5.

	N	
4		5
$1.454e - 06$		$9.304e - 07$

accuracy and simplicity of the approach is evidenced by the preceding examples.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

FIGURE 13: Graph of the exact solution for Example 5.

6. Conclusion

The Boundary Value Methods have been used to approximate second-order PDEs. This has been achieved by using the Lanczos-Chebyshev reduction technique to transform the PDEs into an equivalent second-order system. The high

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