# Multidimensional Dynamic Programming Algorithm for N -Level Batching with Hierarchical Clustering Structure 

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#### Abstract

This study focuses on the $N$-level batching problem with a hierarchical clustering structure. Clustering is the task of grouping a set of item types in such a way that item types in the same cluster are more similar (in some sense or another) to each other than to those in other clusters. In hierarchical clustering structure, more and more different item types are clustered together as the level of the hierarchy increases. $N$-level batching is the process by which items with different types are grouped into several batches passed from level 1 to level $N$ sequentially for given hierarchical clustering structure such that batches in each level should satisfy the maximum and minimum batch size requirements of the level. We consider two types of processing costs of the batches: unit processing cost and batch processing cost. We formulate the $N$-level batching problem with a hierarchical clustering structure as a nonlinear integer programming model with the objective of minimizing the total processing cost. To solve the problem optimally, we propose a multidimensional dynamic programming algorithm with an example.


## 1. Introduction

According to Wikipedia, clustering problem is the task of grouping a set of item types in such a way that item types in the same cluster are more similar (in some sense or another) to each other than to those in other clusters. It is the main task of exploratory data mining and a common technique for statistical data analysis, used in many fields, including machine learning, pattern recognition, image analysis, information retrieval, bioinformatics, data compression, and computer graphics. Hierarchical clustering (also called hierarchical cluster analysis or HCA) is a method of cluster analysis which seeks to build a hierarchy of clusters. Strategies for hierarchical clustering generally fall into two types: agglomerative clustering and divisive one. Agglomerative clustering is a bottom-up approach: that is, each observation starts in its own cluster, and pairs of clusters are merged as one moves up the hierarchy. To the contrary, divisive clustering is a top-down approach: that is, all observations start in one cluster, and splits are performed recursively as one moves
down the hierarchy. In general, the merges and splits are determined in a greedy manner. The results of hierarchical clustering are usually presented in a dendrogram.

The hierarchical clustering problem has been studied for several decades in a wide range of fields including manufacturing, biotechnology, information technology (IT), logistics and transportation, financial, and postal industries. In the manufacturing sector, hierarchical clustering has been used to form manufacturing cells and processing batches. Vakharia and Wemmerlöv [1] investigated the performance of seven hierarchical agglomerative clustering techniques and eight dissimilarity measures in the context of cell formation in the cellular manufacturing system. Chen et al. [2] proposed a constrained agglomerative clustering algorithm for the single batch processing machine scheduling problem often encountered in semiconductor manufacturing and metal heat treatment. Hierarchical clustering is one of the most commonly used methods in biotechnology for classification. Cheng et al. [3] suggested hierarchical model-based clustering of DNA sequences by upgrading Bayesian model-based
clustering. Cameron et al. [4] proposed hierarchical clustering of gene expression patterns consistent with the lineage of differentiating excitatory neurons during early neocortical development. Saunders et al. [5] used Markov clustering and hierarchical clustering to classify protein families of rust pathogens and rank them according to their likelihood of being effectors. Barzinpour et al. [6] proposed a spectral approach to community detection, where the multiplex is mapped onto Euclidean Space (using the top few eigenvectors) and applied $k$-mean clustering. See Andreopoulos et al. [7] for a review of the clustering algorithms applied in bioinformatics.

Clustering is one of the most important techniques for image segmentation and data analytics in the IT industry. Arifin and Asano [8] presented a histogram thresholding algorithm using hierarchical cluster analysis for image segmentation. Nunez-Iglesias et al. [9] proposed an active machine learning approach for performing hierarchical agglomerative clustering from superpixels to improve segmentation of 2D and 3D images. See Zaitoun and Aqel [10] for a survey of image segmentation techniques. In relation to data analytics, Bouguettaya et al. [11] proposed a set of agglomerative hierarchical clustering methods, and Costa et al. [12] proposed a hierarchical approach for clustering XML documents with multiple forms of structural components. Hierarchical clustering also has been successfully applied to the logistics and transportation sector. Özdamar and Demir [13] proposed a multilevel clustering algorithm that groups demand nodes into smaller clusters at each planning level for coordinating vehicle routing in large-scale postdisaster distribution and evacuation activities. Zhu and Guo [14] extended the traditional hierarchical clustering method by generalizing flows to different hierarchical levels to aggregate and map large taxi flow data in an urban area. The hierarchical clustering problem arises in the postal industry as well. Lim et al. [15] studied the three-level presorting loading problem which occurs in the commercial bulk mail service. They considered the problem as a three-level hierarchical clustering problem and proposed an optimal solution algorithm. For the financial sector application, Aghabozorgi and Teh [16] suggested a novel three-phase clustering model to categorize companies based on the similarity in the shape of their stock markets. See Murtagh and Contreras [17] for an extensive survey on the agglomerative hierarchical clustering algorithms.

In this study, we consider an $N$-level batching with agglomerative hierarchical clustering structure in which the highest possible level of the hierarchy is $N$. $N$-level batching is the process by which items with different types are grouped into several batches passed from level 1 to level $N$ sequentially for a given hierarchical clustering structure such that batches in each level of the hierarchy should satisfy the maximum and minimum batch size requirements of the level. We assume that types of items that can be clustered together are given in each level (i.e., hierarchical clustering structure). Also, we assume that there exist the maximum and minimum batch size requirements at each level of the hierarchy. We consider two kinds of costs for processing batched items: batch processing cost and unit processing cost. If items in
a batch are closely related, they can be processed as a batch simultaneously; hence, a batch processing cost occurs to the batch. On the other hand, if items in a batch are loosely related, they have to be processed separately; hence, a unit processing cost occurs to process each item in the batch. The objective of the problem is to minimize the total cost for processing all items.

## 2. $N$-Level Batching Problem with Hierarchical Clustering Structure

Now we describe the $N$-level batching problem (NLBP) with agglomerative hierarchical clustering structure considered in this study. The paper develops an integer nonlinear programming model for the NLBP using the notations (see Notations).

An integer nonlinear programming formulation for the NLBP is now presented.

$$
\begin{align*}
& \text { [NLBP] Minimize } \sum_{n=1}^{N} \sum_{l^{(n)} \in \Lambda_{U}^{(n)}} C_{l^{(n)}}^{U} \Omega_{l^{(n)}} \\
& +\sum_{n=1}^{N} \sum_{l^{(n)} \in \Lambda_{B}^{(n)}} C_{l^{(n)}}^{B}\left\lceil\frac{\Omega_{l^{(n)}}}{W_{l^{(n)}}}\right\rceil,  \tag{1}\\
& \text { subject to } \quad Q_{l^{(n)}}=R_{l^{(n)}}+\Omega_{l^{(n)}} \quad \forall n, l^{(n)} \text {, }  \tag{2}\\
& Q_{l^{(n)}}=\sum_{l^{(n-1)} \in \Delta_{l^{(n)}}} R_{l^{(n-1)}}  \tag{3}\\
& \forall n \geq 1, l^{(n)}, \\
& R_{l^{(N)}}=0 \quad \forall l^{(N)},  \tag{4}\\
& \left(\frac{\Omega_{l^{(n)}}}{\left\lceil\Omega_{l^{(n)}} / W_{l^{(n)}}\right\rceil}\right) \leq W_{l^{(n)}}  \tag{5}\\
& \text { if } \frac{\Omega_{l^{(n)}}}{\left\lceil\Omega_{l^{(n)}} / W_{l^{(n)}}\right\rceil} \geq D_{l^{(n)}} \forall n \geq 1, l^{(n)} \text {, } \\
& D_{l^{(n)}} \leq\left(\frac{\Omega_{l^{(n)}}}{\left\lfloor\Omega_{l^{(n)}} / W_{l^{(n)}}\right\rfloor}\right) \leq W_{l^{(n)}} \\
& \text { if } \frac{\Omega_{l^{(n)}}}{\left\lceil\Omega_{l^{(n)}} / W_{l^{(n)}}\right\rceil}<D_{l^{(n)}}, \Omega_{l^{(n)}} \geq D_{l^{(n)}} \forall n \geq 1, l^{(n)},  \tag{6}\\
& \Omega_{l^{(n)}}=0 \\
& \text { if } Q_{l^{(n)}}<D_{l^{(n)}} \forall n \geq 1, l^{(n)},  \tag{7}\\
& Q_{l^{(n)}} \geq 0 \text { and integer } \forall n, l^{(n)} \text {, }  \tag{8}\\
& \Omega_{l^{(n)}} \geq 0 \text { and integer } \forall n, l^{(n)},  \tag{9}\\
& R_{l^{(n)}} \geq 0 \text { and integer } \forall n, l^{(n)} . \tag{10}
\end{align*}
$$

The objective function (1) to be minimized denotes the total processing cost for all batched items. Both of unit


Figure 1: An example of a NLBP with $N=3$ and nine original item types.
processing cost and batch processing cost are involved in the total cost. Constraint (2) balances the number of items to be batched, the number of items batched, and number of items not batched for all hierarchical clusters. Constraint (3) ensures that the total number of items to be batched at any cluster should be equal to the number of items not batched in the clusters at the immediate preceding level. Constraint (4) ensures that there is no remained item not batched until level $N$. Constraints (5)-(7) indicate that items batched at any cluster should satisfy both the minimum and the maximum batch size requirements. Constraints (8)-(10) represent decision variables.

Figure 1 provides an example of a NLBP with $N=3$ and nine original item types: that is, $\Lambda^{(0)}=\left\{1^{(0)}, 2^{(0)}, \ldots, 9^{(0)}\right\}$. As shown in the figure, NLBP can be represented as a network flow problem. The network consists of nine source nodes with $Q_{l^{(0)}}$ items to be batched through 3-level batching. That is, there are nine level-1 clusters $\left(l^{(1)}\right.$ for $\left.l=1,2, \ldots, 9\right)$ where
the first level batches are formed with $\Omega_{l^{(1)}}$ satisfying both the minimum and the maximum batch size requirements of the clusters at level 1 , three level-2 clusters $\left(l^{(2)}\right.$ for $\left.l=1,2,3\right)$ where different types of items are batched (for example, four different types of items are batched at $1^{(2)}$ cluster), one level3 cluster where all nine item types can be batched together, and finally one destination node 0 . In the network, items are taken out to form batches passed from level-1 clusters to level-3 cluster sequentially with the objective of minimizing total processing costs of batched items. Here, processing costs of batched items, in general, increase as the level of cluster is deeper. Also, the minimum and maximum batch size requirements of clusters may be different. Item quantities to be batched at level $n(1 \leq n \leq 3)$ are the total number of items not batched at level $n-1$.

Lim et al. [15] developed an optimal solution algorithm for a special type of 3-level batching problem that has tapering discount structure in unit processing cost of batched item:
that is, $C_{l^{(2)}}^{U}-C_{l^{(1)}}^{U} \geq C_{l^{(3)}}^{U}-C_{l^{(2)}}^{U}$ for any $l$. In this study, we challenge a more general problem than that of Lim et al. [15] by extending the hierarchical level to $N$ and considering more general cost structure for batched items. This paper develops a dynamic programming solution algorithm for the NLBP to obtain an optimal $N$-level batching with hierarchical clustering structure.

## 3. Dynamic Programming Algorithm for the NLBP

In the dynamic programming algorithm for the NLBP, stage $n(n=1,2, \ldots, N)$ is represented by the level and the state at a stage $n$ is the numbers of items of cluster $l^{(n)}$ not batched yet until level $n$ : that is $R_{1^{(n)}}, R_{2^{(n)}}, \ldots, R_{l^{(n)}}$. Also, possible alternatives at stage $n$ are the numbers of items of cluster $l^{(n)}$ batched at level $n$ : that is, $\Omega_{1^{(n)}}, \Omega_{2^{(n)}}, \ldots, \Omega_{l^{(n)}}$, satisfying the balancing constraints (2)-(4) and the minimum and maximum batch size constraints (5)-(7). First, we give notation used in the DP recursive equations as follows:

$$
\begin{aligned}
& \Psi_{n}\left(R_{1^{(n)}}, R_{2^{(n)}}, \ldots, R_{\left.l^{(n)}\right)}\right. \text { : the minimum processing cost } \\
& \text { for batched items during level } 1 \text { through } n \text { when } \\
& \text { the numbers of items not batched at level } n \text { are } \\
& R_{1^{(n)}}, R_{2^{(n)}}, \ldots, R_{l^{(n)}} \\
& \Gamma_{n}\left(\Omega_{1^{(n)}}, \Omega_{2^{(n)}}, \ldots, \Omega_{l^{(n)}}\right) \text { : the processing cost for } \\
& \text { batched items at level } n \text { when the numbers } \\
& \text { of items batched at level } n \text { are } \Omega_{1^{(n)}}, \Omega_{2^{(n)}}, \ldots, \Omega_{l^{(n)}} \\
& \text { (here, } \Gamma_{n}\left(\Omega_{1^{(n)}}, \Omega_{2^{(n)}}, \ldots, \Omega_{l^{(n)}}\right)=\sum_{l^{(n)} \in \Lambda_{U}^{(n)}} C_{l^{(n)}} \Omega_{l^{(n)}}+ \\
& \sum_{l^{(n)} \in \Lambda_{B}^{(n)}} C_{l^{(n)}}^{B}\left\lceil\Omega_{l^{(n)}} / W_{\left.l^{(n)}\right]}\right)
\end{aligned}
$$

The forward DP recursive equations for the NLBP are

$$
\begin{align*}
& \Psi_{1}\left(R_{1^{(1)}}, R_{2^{(1)}}, \ldots, R_{l^{(1)}}\right) \\
& =\min _{\Omega_{l^{(1)}}=R_{l^{(0)}}-R_{l^{(1)}} \forall l^{(1)}} \quad \Gamma_{1}\left(\Omega_{1^{(1)}},\right. \\
& \text { Batches from } \Omega_{l(1)} \text { should satisfy the batch size constraints. } \\
& \left.\Omega_{2^{(1)}}, \ldots, \Omega_{l^{(1)}}\right), \\
& \Psi_{n}\left(R_{1^{(n)}}, R_{2^{(n)}}, \ldots, R_{l^{(n)}}\right) \\
& =\min _{\substack{\left.R_{l^{(n-1)}}-R_{l^{(n)}}\right) l^{(n)}}}^{\Omega_{l^{(n)}}=\sum_{l \in \Delta_{l^{(n)}}}} \Gamma_{n}\left(\Omega_{1^{(n)}},\right. \tag{11}
\end{align*}
$$

$$
\begin{aligned}
& \left.\quad \Omega_{2^{(n)}}, \ldots, \Omega_{l^{(n)}}\right)+\Psi_{n-1}\left(R_{1^{(n-1)}}, R_{2^{(n-1)}}, \ldots, R_{l^{(n-1)}}\right) \\
& \quad \text { for } n=2,3, \ldots, N-1, \\
& \Psi_{N}\left(R_{1^{(N)}}\right) \\
& =\min _{\Omega_{1^{(N)}}=\sum_{l \in \Delta_{l^{(N)}} R_{l}(N-1)}} \quad \Gamma_{N}\left(\Omega_{1^{(N)}}\right) \\
& \quad+\Psi_{N-1}\left(R_{1^{(N-1)}}, R_{2^{(N-1)}}, \ldots, R_{l^{(N-1)}}\right) .
\end{aligned}
$$

Optimal objective value for the NLBP is $\Psi_{N}\left(R_{1^{(N)}}\right)=\Psi_{N}(0)$.

It is necessary to reduce the number of states for computational efficiency. We find the range of $R_{l^{(n)}}$ needed to be considered in the DP recursive equations to find an optimal solution of the NLBP when unit processing cost, $C_{l^{(n)}}^{U}$, is charged for batched items.

Property 1. For a given cluster $l^{(n)}$ charged by unit processing cost, $R_{l^{(n)}} \leq\left\lfloor\left(\left(C_{l^{(N)}}^{U}-C_{l^{(n)}}^{U}\right) /\left(C_{l^{(n+1)}}^{U}-C_{l^{(n)}}^{U}\right)\right) \widehat{R}_{l^{(n)}}^{\prime}\right\rfloor$, where $\widehat{R}_{l^{(n)}}^{\prime}$ is as follows:
(a) If $Q_{l^{(n)}} / W_{l^{(n)}}$ is an integer, $\widehat{R}_{l^{(n)}}^{\prime}=\min \left\{\left(Q_{l^{(n)}} / W_{l^{(n)}}\right)\right.$. $\left.\left(W_{l^{(n)}}-D_{l^{(n)}}\right), D_{l^{(n)}}-1\right\}$.
(b) If $Q_{l^{(n)}} / W_{l^{(n)}}$ is not an integer and $Q_{l^{(n)}} /\left\lceil Q_{l^{(n)}} / W_{l^{(n)}}\right\rceil \geq$ $D_{l^{(n)}}, \widehat{R}_{l^{(n)}}^{\prime}=\min \left\{Q_{l^{(n)}}-\left(\left\lceil Q_{l^{(n)}} / W_{l^{(n)}}\right\rceil \cdot D_{l^{(n)}}\right), D_{l^{(n)}}-1\right\}$.
(c) If $Q_{l^{(n)}} / W_{l^{(n)}}$ is not an integer and $Q_{l^{(n)}} /\left\lceil Q_{l^{(n)}} / W_{l^{(n)}}\right\rceil<$ $D_{l^{(n)}}, \widehat{R}_{l^{(n)}}^{\prime}=\min \left\{Q_{l^{(n)}}-\left(\left\lfloor Q_{l^{(n)}} / W_{l^{(n)}}\right\rfloor \cdot D_{l^{(n)}}\right), D_{l^{(n)}}-1\right\}$.

Proof. Let $R_{l^{(n)}}^{\prime}$ be the number of items of cluster $l^{(n)}$ not batched at level $n$ and assume that $R_{l^{(n)}}^{\prime} \leq D_{l^{(n)}}-1$. In this case (Case 1), the maximum processing cost of cluster $l^{(n)}$ becomes $\Omega_{l^{(n)}}^{\prime} C_{l^{(n)}}^{U}+R_{l^{(n)}}^{\prime} C_{l^{(N)}}^{U}$ when $R_{l^{(n)}}^{\prime}$ is batched at the last level $N$. Here, $\Omega_{l^{(n)}}^{\prime}$ is the number of items of cluster $l^{(n)}$ batched at level $n$ with $R_{l^{(n)}}^{\prime}$ not batched items. Let $R_{l^{(n)}}^{\prime \prime}$ be the number of items of cluster $l^{(n)}$ not batched at level $n$ but $R_{l^{(n)}}^{\prime \prime} \geq D_{l^{(n)}}$. Also, let $\Omega_{l^{(n)}}^{\prime \prime}$ be the number of items of cluster $l^{(n)}$ batched at level $n$ with $R_{l^{(n)}}^{\prime \prime}$ not batched items. In this case (Case 2), the minimum processing cost of cluster $l^{(n)}$ becomes $\Omega_{l^{(n)}}^{\prime \prime} C_{l^{(n)}}^{U}+R_{l^{(n)}}^{\prime \prime} C_{l^{(n+1)}}^{U}$ when $R_{l^{(n)}}^{\prime \prime}$ is batched at the next level $n+1$. The difference between the maximum cost of Case 1 and the minimum cost of Case 2 is $\Omega_{l^{(n)}}^{\prime} C_{l^{(n)}}^{U}+R_{l^{(n)}}^{\prime} C_{l^{(N)}}^{U}-\left(\Omega_{l^{(n)}}^{\prime \prime} C_{l^{(n)}}^{U}+\right.$ $\left.R_{l^{(n)}}^{\prime \prime} C_{l^{(n+1)}}^{U}\right)=\left(\Omega_{l^{(n)}}^{\prime}-\Omega_{l^{(n)}}^{\prime \prime}\right) C_{l^{(n)}}^{U}+R_{l^{(n)}}^{\prime} C_{l^{(N)}}^{U}-R_{l^{(n)}}^{\prime \prime} C_{l^{(n+1)}}^{U}=$ $\left(R_{l^{(n)}}^{\prime \prime}-R_{l^{(n)}}^{\prime}\right) C_{l^{(n)}}^{U}+R_{l^{(n)}}^{\prime} C_{l^{(N)}}^{U}-R_{l^{(n)}}^{\prime \prime} C_{l^{(n+1)}}^{U}\left(\right.$ since $\Omega_{l^{(n)}}^{\prime}+R_{l^{(n)}}^{\prime}=$ $\left.\Omega_{l^{(n)}}^{\prime \prime}+R_{l^{(n)}}^{\prime \prime}\right)=\left(R_{l^{(n)}}^{\prime \prime}-R_{l^{(n)}}^{\prime}\right) C_{l^{(n)}}^{U}+R_{l^{(n)}}^{\prime} C_{l^{(N)}}^{U}-R_{l^{(n)}}^{\prime \prime} C_{l^{(n+1)}}^{U}=\left(C_{l^{(N)}}^{U}-\right.$ $\left.C_{l^{(n)}}^{U}\right) R_{l^{(n)}}^{\prime}-\left(C_{l^{(n+1)}}^{U}-C_{l^{(n)}}^{U}\right) R_{l^{(n)}}^{\prime \prime}$. As a result, it is better to keep $R_{l^{(n)}}^{\prime} \leq D_{l^{(n)}}-1$ items not batched at level $n$ than to keep $R_{l^{(n)}}^{\prime \prime} \geq$ $D_{l^{(n)}}$ if $\left(C_{l^{(N)}}^{U}-C_{l^{(n)}}^{U}\right) R_{l^{(n)}}^{\prime}-\left(C_{l^{(n+1)}}^{U}-C_{l^{(n)}}^{U}\right) R_{l^{(n)}}^{\prime \prime} \leq 0$. That is, keeping $R_{l^{(n)}}^{\prime}$ items at level $n$ gives less processing cost of batched items if $R_{l^{(n)}}^{\prime \prime} \geq\left(\left(C_{l^{(N)}}^{U}-C_{l^{(n)}}^{U}\right) /\left(C_{l^{(n+1)}}^{U}-C_{l^{(n)}}^{U}\right)\right) R_{l^{(n)}}^{\prime}$. In other words, it is sufficient to consider $R_{l^{(n)}}<\left\lfloor\left(\left(C_{l^{(N)}}^{U}-C_{l^{(n)}}^{U}\right) /\left(C_{l^{(n+1)}}^{U}-C_{l^{(n)}}^{U}\right)\right) R_{l^{(n)}}^{\prime}\right\rfloor$ to obtain an optimal solution of the NLBP. Here, note that $\left(C_{l^{(N)}}^{U}-C_{l^{(n)}}^{U}\right) /\left(C_{l^{(n+1)}}^{U}-C_{l^{(n)}}^{U}\right) \geq 1$ since $C_{l^{(n)}}^{U}<C_{l^{(n+1)}}^{U}$ for all $n$. There are three cases where $R_{l^{(n)}}^{\prime} \leq D_{l^{(n)}}-1$. The first case is when $Q_{l^{(n)}} / W_{l^{(n)}}$ is an integer. In this case, each batch can contain $W_{l^{(n)}}$ items and $\left(W_{l^{(n)}}-D_{l^{(n)}}\right)$ items in each batch can remain not batched. As a result, $R_{l^{(n)}}^{\prime}=\min \left\{\left(Q_{l^{(n)}} / W_{l^{(n)}}\right)\right.$. $\left.\left(W_{l^{(n)}}-D_{l^{(n)}}\right), D_{l^{(n)}}-1\right\}$ in this case. The second case is when $\mathrm{Q}_{l^{(n)}} / W_{l^{(n)}}$ is not an integer and $\mathrm{Q}_{l^{(n)}} /\left\lceil\mathrm{Q}_{l^{(n)}} / W_{l^{(n)}}\right\rceil \geq D_{l^{(n)}}$. We can form $\left\lceil Q_{l^{(n)}} / W_{l^{(n)}}\right\rceil$ batches and $Q_{l^{(n)}}-\left(\left\lceil Q_{l^{(n)}} / W_{l^{(n)}}\right\rceil\right.$. $D_{l^{(n)}}$ ) items can remain not batched. As a result, $R_{l^{(n)}}^{\prime}=$ $\min \left\{Q_{l^{(n)}}-\left(\left\lceil Q_{l^{(n)}} / W_{l^{(n)}}\right\rceil \cdot D_{l^{(n)}}\right), D_{l^{(n)}}-1\right\}$ in this case. The third case is when $Q_{l^{(n)}} / W_{l^{(n)}}$ is not an integer and $Q_{l^{(n)}} /\left\lceil Q_{l^{(n)}} / W_{l^{(n)}}\right\rceil<D_{l^{(n)}}$. We can form $\left\lfloor Q_{l^{(n)}} / W_{l^{(n)}}\right\rfloor$ batches and $Q_{l^{(n)}}-\left(\left\lfloor Q_{l^{(n)}} / W_{l^{(n)}}\right\rfloor \cdot D_{l^{(n)}}\right)$ items can remain not batched.

As a result, $R_{l^{(n)}}^{\prime}=\min \left\{Q_{l^{(n)}}-\left(\left\lfloor Q_{l^{(n)}} / W_{l^{(n)}}\right\rfloor \cdot D_{l^{(n)}}\right), D_{l^{(n)}}-1\right\}$ in this case. This completes the proof.

Let $\Phi\left(l^{(n)}\right)$ be the set of $R_{l^{(n)}}$ needed to be considered in DP recursive equations to obtain an optimal solution of the NLBP when unit processing cost, $C_{l^{(n)}}^{U}$, is charged for batched items.

Property 2. For a given $l^{(n)}$ charged by unit processing cost satisfying $1 \leq n \leq N-1, \Phi\left(l^{(n)}\right)$ is given as follows.
(a) If $Q_{l^{(n)}} / W_{l^{(n)}}$ is an integer, $\Phi\left(l^{(n)}\right)=\left\{R_{l^{(n)}} \mid 0 \leq\right.$ $R_{l^{(n)}} \leq\left\lfloor\left(\left(C_{l^{(N)}}^{U}-C_{l^{(n)}}^{U}\right) /\left(C_{l^{(n+1)}}^{U}-C_{l^{(n)}}^{U}\right)\right) \hat{R}_{l^{(n)}}^{\prime}\right\rfloor$ and $\left(Q_{l^{(n)}}-\right.$ $\left.\left.R_{l^{(n)}}\right) /\left\lceil\left(Q_{l^{(n)}}-R_{l^{(n)}}\right) / W_{l^{(n)}}\right\rceil \geq D_{l^{(n)}}\right\}$ when $\left\lfloor\left(\left(C_{l^{(N)}}^{U}-\right.\right.\right.$ $\left.\left.\left.C_{l^{(n)}}^{U}\right) /\left(C_{l^{(n+1)}}^{U}-C_{l^{(n)}}^{U}\right)\right) \widehat{R}_{l^{(n)}}^{\prime}\right\rfloor \geq D_{l^{(n)}}$ whereas $\Phi\left(l^{(n)}\right)=$ $\left\{R_{l^{(n)}} \mid 0 \leq R_{l^{(n)}} \leq \widehat{R}_{l^{(n)}}^{\prime}\right.$ and $\left(Q_{l^{(n)}}-R_{l^{(n)}}\right) / \Gamma\left(Q_{l^{(n)}}-\right.$ $\left.\left.\left.R_{l^{(n)}}\right) / W_{l^{(n)}}\right\rceil \geq D_{l^{(n)}}\right\}$ when $\left\lfloor\left(\left(C_{l^{(N)}}^{U}-C_{l^{(n)}}^{U}\right) /\left(C_{l^{(n+1)}}^{U}-\right.\right.\right.$ $\left.\left.\left.C_{l^{(n)}}^{U}\right)\right) \widehat{R}_{l^{(n)}}^{\prime}\right\rfloor \leq D_{l^{(n)}}-1$ where $\widehat{R}_{l^{(n)}}^{\prime}=\min \left\{\left(Q_{l^{(n)}} / W_{l^{(n)}}\right)\right.$. $\left.\left(W_{l^{(n)}}-D_{l^{(n)}}\right), D_{l^{(n)}}-1\right\}$.
(b) If $Q_{l^{(n)}} / W_{l^{(n)}}$ is not an integer and $Q_{l^{(n)}} /\left\lceil Q_{l^{(n)}} / W_{l^{(n)}}\right\rceil \geq$ $D_{l^{(n)}}, \Phi\left(l^{(n)}\right)=\left\{R_{l^{(n)}} \mid 0 \leq R_{l^{(n)}} \leq\left\lfloor\left(\left(C_{l^{(N)}}^{U}-\right.\right.\right.\right.$ $\left.\left.\left.C_{l^{(n)}}^{U}\right) /\left(C_{l^{(n+1)}}^{U}-C_{l^{(n)}}^{U}\right)\right) \widehat{R}_{l^{(n)}}^{\prime}\right\rfloor$ and $\left(Q_{l^{(n)}}-R_{l^{(n)}}\right) / \Gamma\left(Q_{l^{(n)}}-\right.$ $\left.\left.\left.R_{l^{(n)}}\right) / W_{l^{(n)}}\right\rceil \geq D_{\left.l^{(n)}\right\}}\right\}$ when $\left\lfloor\left(\left(C_{l^{(N)}}^{U}-C_{l^{(n)}}^{U}\right) /\left(C_{l^{(n+1)}}^{U}-\right.\right.\right.$ $\left.\left.\left.C_{l^{(n)}}^{U}\right)\right) \hat{R}_{l^{(n)}}^{\prime}\right\rfloor \geq D_{l^{(n)}}$, whereas $\Phi\left(l^{(n)}\right)=\left\{R_{l^{(n)}} \mid 0 \leq\right.$ $R_{l^{(n)}} \leq \widehat{R}_{l^{(n)}}^{\prime}$ and $\left(Q_{l^{(n)}}-R_{l^{(n)}}\right) /\left\lceil\left(Q_{l^{(n)}}-R_{l^{(n)}}\right) / W_{l^{(n)}}\right\rceil \geq$ $\left.D_{l^{(n)}}\right\}$ when $\left\lfloor\left(\left(C_{l^{(N)}}^{U}-C_{l^{(n)}}^{U}\right) /\left(C_{l^{(n+1)}}^{U}-C_{l^{(n)}}^{U}\right)\right) \widehat{R}_{l^{(n)}}^{\prime}\right\rfloor \leq D_{l^{(n)}}-$ 1, where $\widehat{R}_{l^{(n)}}^{\prime}=\min \left\{Q_{l^{(n)}}-\left(\left\lceil Q_{l^{(n)}} / W_{l^{(n)}}\right\rceil \cdot D_{l^{(n)}}\right), D_{l^{(n)}}-\right.$ $1\}$.
(c) If $Q_{l^{(n)}} / W_{l^{(n)}}$ is not an integer and $Q_{l^{(n)}} /\left\lceil Q_{l^{(n)}} / W_{l^{(n)}}\right\rceil<$ $D_{l^{(n)}}$ and $Q_{l^{(n)}} \geq D_{l^{(n)}}, \Phi\left(l^{(n)}\right)=\left\{R_{l^{(n)}} \mid Q_{l^{(n)}}-\right.$ $\left(\left\lfloor Q_{l^{(n)}} / W_{l^{(n)}}\right\rfloor \cdot W_{l^{(n)}}\right) \leq R_{l^{(n)}} \leq\left\lfloor\left(\left(C_{l^{(N)}}^{U}-C_{l^{(n)}}^{U}\right) /\left(C_{l^{(n+1)}}^{U}-\right.\right.\right.$ $\left.\left.\left.C_{l^{(n)}}^{U}\right)\right) \widehat{R}_{l^{(n)}}^{\prime}\right\rfloor$ and $\left(Q_{l^{(n)}}-R_{l^{(n)}}\right) /\left\lceil\left(Q_{l^{(n)}}-R_{l^{(n)}}\right) / W_{l^{(n)}}\right\rceil \geq$ $\left.D_{l^{(n)}}\right\}$ when $\left\lfloor\left(\left(C_{l^{(N)}}^{U}-C_{l^{(n)}}^{U}\right) /\left(C_{l^{(n+1)}}^{U}-C_{l^{(n)}}^{U}\right)\right) \widehat{R}_{l^{(n)}}^{\prime}\right\rfloor \geq D_{l^{(n)}}$, whereas $\Phi\left(l^{(n)}\right)=\left\{R_{l^{(n)}} \mid Q_{l^{(n)}}-\left(\left\lfloor Q_{l^{(n)}} / W_{l^{(n)}}\right\rfloor \cdot W_{l^{(n)}}\right) \leq\right.$ $R_{l^{(n)}} \leq \widehat{R}_{l^{(n)}}^{\prime}$ and $\left(Q_{l^{(n)}}-R_{l^{(n)}}\right) /\left\lceil\left(Q_{l^{(n)}}-R_{l^{(n)}}\right) / W_{l^{(n)}}\right\rceil \geq$ $\left.D_{l^{(n)}}\right\}$ when $\left\lfloor\left(\left(C_{l^{(N)}}^{U}-C_{l^{(n)}}^{U}\right) /\left(C_{l^{(n+1)}}^{U}-C_{l^{(n)}}^{U}\right)\right) \widehat{R}_{l^{(n)}}^{\prime}\right\rfloor \leq D_{l^{(n)}}-$ 1, where $\widehat{R}_{l^{(n)}}^{\prime}=\min \left\{Q_{l^{(n)}}-\left(\left\lfloor Q_{l^{(n)}} / W_{l^{(n)}}\right\rfloor \cdot D_{l^{(n)}}\right), D_{l^{(n)}}-\right.$ $1\}$.
(d) If $Q_{l^{(n)}} / W_{l^{(n)}}$ is not an integer and $Q_{l^{(n)}} /\left\lceil Q_{l^{(n)}} / W_{l^{(n)}}\right\rceil<$ $D_{l^{(n)}}$ and $Q_{l^{(n)}}<D_{l^{(n)}}, \Phi\left(l^{(n)}\right)=\left\{Q_{l^{(n)}}\right\}$.

Proof. It is obvious from Property 1.
The next property gives the range of $R_{l^{(n)}}$ needed to be considered in the DP recursive equations to find an optimal solution of the NLBP when batch processing cost $C_{l^{(n)}}^{B}$ is charged instead of unit processing $\operatorname{cost} C_{l^{(n)}}^{U}$.

Property 3. For a given cluster $l^{(n)}$ charged by batch processing cost, it is sufficient to consider $R_{l^{(n)}} \leq\left\lfloor\left(R_{l^{(n)}}^{\prime} C_{l^{(n)}}^{B}+\right.\right.$ $\left.\left.W_{l^{(n)}} C_{l^{(N)}}^{B}\right) /\left(C_{l^{(n)}}^{B}+C_{l^{(n+1)}}^{B}\right)\right\rfloor$ in DP recursive equations.

Proof. Let $R_{l^{(n)}}^{\prime}$ be the number of items of cluster $l^{(n)}$ not batched at level $n$ and assume that $R_{l^{(n)}}^{\prime} \leq D_{l^{(n)}}-1$. In this case (Case 1), the maximum processing cost of cluster $l^{(n)}$ becomes $\left\lceil\Omega_{l^{(n)}}^{\prime} / W_{l^{(n)}}\right] C_{l^{(n)}}^{B}+C_{l^{(N)}}^{B} \leq\left\{\left(\Omega_{l^{(n)}}^{\prime} / W_{l^{(n)}}\right)+1\right\} C_{l^{(n)}}^{B}+C_{l^{(N)}}^{B}$ (since $\left\lceil\Omega_{l^{(n)}}^{\prime} / W_{l^{(n)}}\right\rceil \leq\left(\Omega_{l^{(n)}}^{\prime} / W_{l^{(n)}}\right)+1$ ) when $R_{l^{(n)}}^{\prime}$ is batched at the last level $N$. Here, $\Omega_{l^{(n)}}^{\prime}$ is the number of items of cluster $l^{(n)}$ batched at level $n$ with $R_{l^{(n)}}^{\prime}$ not batched items. Let $R_{l^{(n)}}^{\prime \prime}$ be the number of items of cluster $l^{(n)}$ not batched at level $n$ but $R_{l^{(n)}}^{\prime \prime} \geq D_{l^{(n)}}$. Also, let $\Omega_{l^{(n)}}^{\prime \prime}$ be the number of items of cluster $l^{(n)}$ batched at level $n$ with $R_{l^{(n)}}^{\prime \prime}$ not batched items. In this case (Case 2), the minimum processing cost of cluster $l^{(n)}$ becomes $\left\lceil\Omega_{l^{(n)}}^{\prime \prime} / W_{l^{(n)}}\right\rceil C_{l^{(n)}}^{B}+\left\lceil R_{l^{(n)}}^{\prime \prime} / W_{l^{(n)}}\right\rceil C_{l^{(n+1)}}^{B} \geq\left(\Omega_{l^{(n)}}^{\prime \prime} / W_{l^{(n)}}\right) C_{l^{(n)}}^{B}+$ $\left(R_{l^{(n)}}^{\prime \prime} / W_{l^{(n)}}\right) C_{l^{(n+1)}}^{B}\left(\right.$ since $\left\lceil\Omega_{l^{(n)}}^{\prime \prime} / W_{l^{(n)}}\right\rceil \leq\left(\Omega_{l^{(n)}}^{\prime \prime} / W_{l^{(n)}}\right)+1$ and $\left.\left\lceil R_{l^{(n)}}^{\prime \prime} / W_{l^{(n)}}\right\rceil \leq\left(R_{l^{(n)}}^{\prime \prime} / W_{l^{(n)}}\right)+1\right)$ when $R_{l^{(n)}}^{\prime \prime}$ is batched at the next level $n+1$. The difference between the maximum cost of Case 1 and the minimum cost of Case 2 is $\left\{C_{l^{(n)}}^{B}\left(\Omega_{l^{(n)}}^{\prime}-\Omega_{l^{(n)}}^{\prime \prime}\right) / W_{l^{(n)}}\right\}+$ $C_{l^{(N)}}^{B}-\left\{\left(C_{l^{(n+1)}}^{B} R_{l^{(n)}}^{\prime \prime}\right) / W_{l^{(n)}}\right\}=\left\{C_{l^{(n)}}^{B}\left(R_{l^{(n)}}^{\prime}-R_{l^{(n)}}^{\prime \prime}\right) / W_{l^{(n)}}\right\}+C_{l^{(N)}}^{B}-$ $\left\{\left(C_{l^{(n+1)}}^{B} R_{l^{(n)}}^{\prime \prime}\right) / W_{l^{(n)}}\right\}$ since $\Omega_{l^{(n)}}^{\prime}+R_{l^{(n)}}^{\prime}=\Omega_{l^{(n)}}^{\prime \prime}+R_{l^{(n)}}^{\prime \prime}$. As a result, it is better to keep $R_{l^{(n)}}^{\prime} \leq D_{l^{(n)}}-1$ items not batched at level $n$ than to keep $R_{l^{(n)}}^{\prime \prime} \geq D_{l^{(n)}}$ if $\left\{C_{l^{(n)}}^{B}\left(R_{l^{(n)}}^{\prime}-R_{l^{(n)}}^{\prime \prime}\right) / W_{l^{(n)}}\right\}+C_{l^{(N)}}^{B}-$ $\left\{\left(C_{l^{(n+1)}}^{B} R_{l^{(n)}}^{\prime \prime}\right) / W_{l^{(n)}}\right\} \leq 0$. That is, keeping $R_{l^{(n)}}^{\prime}$ items at level $n$ gives less processing cost of batched items if $R_{l^{(n)}}^{\prime \prime} \geq\left(R_{l^{(n)}}^{\prime} C_{l^{(n)}}^{B}+\right.$ $\left.W_{l^{(n)}} C_{l^{(N)}}^{B}\right) /\left(C_{l^{(n)}}^{B}+C_{l^{(n+1)}}^{B}\right)$. In other words, it is sufficient to consider $R_{l^{(n)}}<\left(R_{l^{(n)}}^{\prime} C_{l^{(n)}}^{B}+W_{l^{(n)}} C_{l^{(N)}}^{B}\right) /\left(C_{l^{(n)}}^{B}+C_{l^{(n+1)}}^{B}\right)$ to obtain an optimal solution of the NLBP.

Let $\widetilde{\Phi}\left(l^{(n)}\right)$ be the set of $R_{l^{(n)}}$ needed to be considered in DP recursive equations to obtain an optimal solution of the NLBP when batch processing cost, $C_{l^{(n)}}^{B}$, is charged for batched items.

Property 4. For a given $l^{(n)}$ charged by batch processing cost satisfying $1 \leq n \leq N-1, \widetilde{\Phi}\left(l^{(n)}\right)$ is given as follows.
(a) If $Q_{l^{(n)}} / W_{l^{(n)}}$ is an integer, $\widetilde{\Phi}\left(l^{(n)}\right)=\left\{R_{l^{(n)}} \mid 0 \leq\right.$ $R_{l^{(n)}} \leq\left\lfloor\left(R_{l^{(n)}}^{\prime} C_{l^{(n)}}^{B}+W_{l^{(n)}} C_{l^{(N)}}^{B}\right) /\left(C_{l^{(n)}}^{B}+C_{l^{(n+1)}}^{B}\right)\right\rfloor$ and $\left.\left(Q_{l^{(n)}}-R_{l^{(n)}}\right) /\left\lceil\left(Q_{l^{(n)}}-R_{l^{(n)}}\right) / W_{l^{(n)}}\right\rceil \geq D_{l^{(n)}}\right\}$ when $\left\lfloor\left(R_{l^{(n)}}^{\prime} C_{l^{(n)}}^{B}+W_{l^{(n)}} C_{l^{(N)}}^{B}\right) /\left(C_{l^{(n)}}^{B}+C_{l^{(n+1)}}^{B}\right)\right\rfloor \geq D_{l^{(n)}}$, whereas $\widetilde{\Phi}\left(l^{(n)}\right)=\left\{R_{l^{(n)}} \mid 0 \leq R_{l^{(n)}} \leq \widehat{R}_{l^{(n)}}^{\prime}\right.$ and $\left(Q_{l^{(n)}}-\right.$ $\left.\left.R_{l^{(n)}}\right) /\left\lceil\left(Q_{l^{(n)}}-R_{l^{(n)}}\right) / W_{l^{(n)}}\right\rceil \geq D_{l^{(n)}}\right\}$ when $\left\lfloor\left(R_{l^{(n)}}^{\prime} C_{l^{(n)}}^{B}+\right.\right.$ $\left.\left.W_{l^{(n)}} C_{l^{(N)}}^{B}\right) /\left(C_{l^{(n)}}^{B}+C_{l^{(n+1)}}^{B}\right)\right\rfloor \leq D_{l^{(n)}}-1$, where $\widehat{R}_{l^{(n)}}^{\prime}=$ $\min \left\{\left(Q_{l^{(n)}} / W_{l^{(n)}}\right) \cdot\left(W_{l^{(n)}}-D_{l^{(n)}}\right), D_{l^{(n)}}-1\right\}$.
(b) If $Q_{l^{(n)}} / W_{l^{(n)}}$ is not an integer and $Q_{l^{(n)}} /\left\lceil Q_{l^{(n)}} / W_{l^{(n)}}\right\rceil \geq$ $D_{l^{(n)}}, \widetilde{\Phi}\left(l^{(n)}\right)=\left\{R_{l^{(n)}} \mid 0 \leq R_{l^{(n)}} \leq\left\lfloor\left(R_{l^{(n)}}^{\prime} C_{l^{(n)}}^{B}+\right.\right.\right.$ $\left.\left.W_{l^{(n)}} C_{l^{(N)}}^{B}\right) /\left(C_{l^{(n)}}^{B}+C_{l^{(n+1)}}^{B}\right)\right\rfloor$ and $\left(Q_{l^{(n)}}-R_{l^{(n)}}\right) /\left\lceil\left(Q_{l^{(n)}}-\right.\right.$ $\left.\left.\left.R_{l^{(n)}}\right) / W_{l^{(n)}}\right\rceil \geq D_{l^{(n)}}\right\}$ when $\left\lfloor\left(R_{l^{(n)}}^{\prime} C_{l^{(n)}}^{B}+W_{l^{(n)}} C_{l^{(N)}}^{B}\right) /\right.$ $\left.\left(C_{l^{(n)}}^{B}+C_{l^{(n+1)}}^{B}\right)\right\rfloor \geq D_{l^{(n)}}$, whereas $\widetilde{\Phi}\left(l^{(n)}\right)=\left\{R_{l^{(n)}} \mid 0 \leq\right.$ $R_{l^{(n)}} \leq \widehat{R}_{l^{(n)}}^{\prime}$ and $\left(Q_{l^{(n)}}-R_{l^{(n)}}\right) /\left\lceil\left(Q_{l^{(n)}}-R_{l^{(n)}}\right) / W_{l^{(n)}}\right\rceil \geq$ $\left.D_{l^{(n)}}\right\}$ when $\left\lfloor\left(R_{l^{(n)}}^{\prime} C_{l^{(n)}}^{B}+W_{l^{(n)}} C_{l^{(N)}}^{B}\right) /\left(C_{l^{(n)}}^{B}+C_{l^{(n+1)}}^{B}\right)\right\rfloor \leq$ $D_{l^{(n)}}-1$, where $\widehat{R}_{l^{(n)}}^{\prime}=\min \left\{Q_{l^{(n)}}-\left(\left\lceil Q_{l^{(n)}} / W_{l^{(n)}}\right\rceil\right.\right.$. $\left.\left.D_{l^{(n)}}\right), D_{l^{(n)}}-1\right\}$.
(c) If $Q_{l^{(n)}} / W_{l^{(n)}}$ is not an integer and $Q_{l^{(n)}} /\left\lceil Q_{l^{(n)}} / W_{l^{(n)}}\right\rceil<$ $D_{l^{(n)}}$ and $Q_{l^{(n)}} \geq D_{l^{(n)}}, \widetilde{\Phi}\left(l^{(n)}\right)=\left\{R_{l^{(n)}} \mid Q_{l^{(n)}}-\right.$ $\left(\left\lfloor Q_{l^{(n)}} / W_{l^{(n)}}\right\rfloor \cdot W_{l^{(n)}}\right) \leq R_{l^{(n)}} \leq\left\lfloor\left(R_{l^{(n)}}^{\prime} C_{l^{(n)}}^{B}+\right.\right.$ $\left.\left.W_{l^{(n)}} C_{l^{(N)}}^{B}\right) /\left(C_{l^{(n)}}^{B}+C_{l^{(n+1)}}^{B}\right)\right\rfloor$ and $\left(Q_{l^{(n)}}-R_{l^{(n)}}\right) / \Gamma\left(Q_{l^{(n)}}-\right.$ $\left.\left.\left.R_{l^{(n)}}\right) / W_{l^{(n)}}\right\rceil \geq D_{l^{(n)}}\right\}$ when $\left\lfloor\left(R_{l^{(n)}}^{\prime} C_{l^{(n)}}^{B}+W_{l^{(n)}} C_{l^{(N)}}^{B}\right) /\right.$ $\left.\left(C_{l^{(n)}}^{B}+C_{l^{(n+1)}}^{B}\right)\right\rfloor \geq D_{l^{(n)}}$, whereas $\widetilde{\Phi}\left(l^{(n)}\right)=\left\{R_{l^{(n)}} \mid\right.$ $Q_{l^{(n)}}-\left(\left\lfloor Q_{l^{(n)}} / W_{l^{(n)}}\right\rfloor \cdot W_{l^{(n)}}\right) \leq R_{l^{(n)}} \leq \widehat{R}_{l^{(n)}}^{\prime}$ and $\left(Q_{l^{(n)}}-\right.$ $\left.\left.R_{l^{(n)}}\right) /\left\lceil\left(Q_{l^{(n)}}-R_{l^{(n)}}\right) / W_{l^{(n)}}\right\rceil \geq D_{l^{(n)}}\right\}$ when $\left\lfloor\left(R_{l^{(n)}}^{\prime} C_{l^{(n)}}^{B}+\right.\right.$ $\left.\left.W_{l^{(n)}} C_{l^{(N)}}^{B}\right) /\left(C_{l^{(n)}}^{B}+C_{l^{(n+1)}}^{B}\right)\right\rfloor \leq D_{l^{(n)}}-1$, where $\widehat{R}_{l^{(n)}}^{\prime}=$ $\min \left\{Q_{l^{(n)}}-\left(\left\lfloor Q_{l^{(n)}} / W_{l^{(n)}}\right\rfloor \cdot D_{l^{(n)}}\right), D_{l^{(n)}}-1\right\}$.
(d) If $Q_{l^{(n)}} / W_{l^{(n)}}$ is not an integer and $Q_{l^{(n)}} /\left\lceil Q_{l^{(n)}} / W_{l^{(n)}}\right\rceil<$ $D_{l^{(n)}}$ and $Q_{l^{(n)}}<D_{l^{(n)}}, \widetilde{\Phi}\left(l^{(n)}\right)=\left\{Q_{l^{(n)}}\right\}$.

Proof. It is obvious from Property 3.
Now, we can redefine the forward DP recursive equations as follows:

$$
\text { for } n=2,3, \ldots, N-1 \text {, }
$$

## 4. An Example for the Dynamic Programming Algorithm

In this section, we give an example to explain how to solve the NLBP with the DP recursive equations. Note that this example is the same as that given in Figure 1. Problem data is as follows. Here, we assume that all batched items are charged by unit processing cost.

$$
\begin{aligned}
& W_{l^{(n)}}=100 \quad \forall n, l^{(n)}, \\
& D_{l^{(1)}}=80 \quad \forall l^{(1)} \\
& D_{l^{(2)}}=75 \quad \forall l^{(2)},
\end{aligned}
$$

$$
\begin{aligned}
& \Psi_{N}\left(R_{1^{(N)}}\right) \\
& =\min _{\Omega_{1^{(N)}}=\sum_{l \in \Delta_{l(N)}} R_{l^{(N-1)}}} \Gamma_{N}\left(\Omega_{1^{(N)}}\right) \\
& R_{1}(N) \in \Phi\left(1^{(N)}\right) \forall l^{(N)} \in \Lambda_{U}^{(N)} \\
& R_{1}(N) \in \widetilde{\Phi}\left(1^{(N)}\right) \forall l^{(N)} \in \Lambda_{B}^{(N)} \\
& +\Psi_{N-1}\left(R_{1^{(N-1)}}, R_{2^{(N-1)}}, \ldots, R_{l^{(N-1)}}\right) .
\end{aligned}
$$

$$
\begin{aligned}
& \Psi_{1}\left(R_{1^{(1)}}, R_{2^{(1)}}, \ldots, R_{l^{(1)}}\right) \\
& =\min _{\Omega_{l^{(1)}}=R_{l^{(0)}}-R_{l^{(1)} \forall l^{(1)}}} \Gamma_{1}\left(\Omega_{1^{(1)}}, \Omega_{2^{(1)}}, \ldots, \Omega_{l^{(1)}}\right), \\
& R_{l^{(1)}} \in \Phi\left(l^{(1)}\right) \forall l^{(1)} \in \Lambda_{U}^{(1)} \\
& R_{l^{(1)}} \in \widetilde{\Phi}\left(l^{(1)}\right) \forall l^{(1)} \in \Lambda_{B}^{(1)} \\
& \Psi_{n}\left(R_{1^{(n)}}, R_{2^{(n)}}, \ldots, R_{l^{(n)}}\right) \\
& =\min _{\Omega_{l^{(n)}}=\sum_{l \in \Delta}^{l^{(n)}}} R_{l^{(n-1)}-R_{l^{(n)}} \forall l^{(n)}} \Gamma_{n}\left(\Omega_{1^{(n)}}, \Omega_{2^{(n)}}, \ldots, \Omega_{l^{(n)}}\right) \\
& R_{l^{(n)}} \in \Phi\left(l^{(n)}\right) \forall l^{(n)} \in \Lambda_{U}^{(n)} \\
& R_{l^{(n)}} \in \widetilde{\Phi}\left(l^{(n)}\right) \forall l^{(n)} \in \Lambda_{B}^{(n)} \\
& +\Psi_{n-1}\left(R_{1^{(n-1)}}, R_{2^{(n-1)}}, \ldots, R_{l^{(n-1)}}\right)
\end{aligned}
$$

$$
\begin{align*}
& D_{l^{(3)}}=0 \quad \forall l^{(3)}, \\
& C_{l^{(1)}}^{U}=100 \quad \forall l^{(1)}, \\
& C_{l^{(2)}}^{U}=125 \quad \forall l^{(2)}, \\
& C_{l^{(3)}}^{U}=140 \quad \forall l^{(3)}, \\
& Q_{1^{(0)}}=Q_{1^{(1)}}=128, \\
& Q_{2^{(0)}}=Q_{2^{(1)}}=258, \\
& Q_{3^{(0)}}=Q_{3^{(1)}}=78, \\
& Q_{4^{(0)}}=Q_{4^{(1)}}=480, \\
& Q_{5^{(0)}}=Q_{5^{(1)}}=298, \\
& Q_{6^{(0)}}=Q_{6^{(1)}}=400, \\
& Q_{7^{(0)}}=Q_{7^{(1)}}=99, \\
& Q_{8^{(0)}}=Q_{8^{(1)}}=555, \\
& Q_{9^{(0)}}=Q_{9^{(1)}}=171 . \tag{13}
\end{align*}
$$

Table 1 shows DP calculations at the first stage. Also, the processing cost for batched items at this stage (i.e., level 1) for given $\left(\Omega_{1^{(1)}}, \Omega_{2^{(1)}}, \ldots, \Omega_{9^{(1)}}\right)$ is computed as

$$
\begin{equation*}
\Gamma_{1}\left(\Omega_{1^{(1)}}, \Omega_{2^{(1)}}, \ldots, \Omega_{9^{(1)}}\right)=\sum_{l=1}^{9} \Omega_{l^{(1)}} C_{l^{(1)}}^{U} \tag{14}
\end{equation*}
$$

The minimum processing cost for batched items to level 1 for given $\left(R_{1^{(1)}}, R_{2^{(1)}}^{(1)}, \ldots, R_{9^{(1)}}\right)$ is computed as

$$
\begin{align*}
\Psi_{1} & \left(R_{1^{(1)}}, R_{2^{(1)}}, \ldots, R_{9^{(1)}}\right) \\
& =\min _{\substack{\Omega_{l^{(1)}}=R_{l^{(0)}}-R_{l^{(1)}} \forall l^{(1)} \\
R_{l^{(1)}} \in \Phi\left(l^{(1)}\right) \forall l^{(1)}}} \Gamma_{1}\left(\Omega_{1^{(1)}}, \Omega_{2^{(1)}}, \ldots, \Omega_{9^{(1)}}\right) . \tag{15}
\end{align*}
$$

Also, we can define the set of $R_{l^{(1)}}$ needed to be considered in DP recursive equations to obtain an optimal solution of the NLBP as follows:

$$
\begin{aligned}
& \widehat{R}_{1^{(1)}}^{\prime}=\min \left\{Q_{1^{(1)}}-D_{1^{(1)}} \cdot\left\lfloor\frac{Q_{1^{(1)}}}{W_{1^{(1)}}}\right\rfloor, D_{1^{(1)}}-1\right\} \\
& \quad=\min \left\{128-80 \cdot\left\lfloor\left.\frac{128}{100} \right\rvert\,, 79\right\}=48\right. \\
& \frac{\left(C_{1^{(3)}}^{U}-C_{1^{(1)}}^{U}\right)}{\left(C_{1^{(2)}}^{U}-C_{1^{(1)}}^{U}\right)} \widehat{R}_{1^{(1)}}^{\prime}=\frac{(140-100)}{(125-100)} 48=76.8 \\
& \Phi\left(1^{(1)}\right)=\left\{R_{1^{(1)}} \left\lvert\, Q_{1^{(1)}}-\left(\left\lfloor\frac{Q_{1^{(1)}}}{W_{1^{(1)}}}\right\rfloor \cdot W_{1^{(1)}}\right) \leq R_{1^{(1)}}\right.\right. \\
& \left.\quad \leq \widehat{R}_{1^{(1)}}^{\prime}, \frac{\left(Q_{1^{(1)}}-R_{1^{(1)}}\right)}{\left\lceil\left(Q_{1^{(1)}}-R_{1^{(1)}}\right) / W_{1^{(1)}}\right\rceil} \geq D_{1^{(1)}}\right\},
\end{aligned}
$$

$$
\begin{aligned}
& \Phi\left(1^{(1)}\right)=\left\{R_{1^{(1)}} \left\lvert\, 128-\left(\left\lfloor\frac{128}{100}\right\rfloor \cdot 100\right) \leq R_{1^{(1)}}\right.\right. \\
& \left.\leq 48, \frac{\left(128-R_{1^{(1)}}\right)}{\left\lceil\left(128-R_{1^{(1)}}\right) / 100\right\rceil} \geq 80\right\} \text {, } \\
& \Phi\left(1^{(1)}\right)=\left\{R_{1^{(1)}} \mid 28 \leq R_{1^{(1)}}\right. \\
& \left.\leq 48, \frac{\left(128-R_{1^{(1)}}\right)}{\left\lceil\left(128-R_{1^{(1)}}^{(1)}\right) / 100\right\rceil} \geq 80\right\} \text {, } \\
& \Phi\left(1^{(1)}\right)=\{28,29, \ldots, 48\}, \\
& \widehat{R}_{2^{(1)}}^{\prime}=\min \left\{Q_{2^{(1)}}-D_{2^{(1)}} \cdot\left\lceil\frac{Q_{2^{(1)}}}{W_{2^{(1)}}}\right\rceil, D_{2^{(1)}}-1\right\} \\
& =\min \left\{258-100 \cdot\left\lfloor\frac{258}{100}\right\rfloor, 79\right\}=58 \text {, } \\
& \frac{\left(C_{1^{(3)}}^{U}-C_{1^{(1)}}^{U}\right)}{\left(C_{1^{(2)}}^{U}-C_{1^{(1)}}^{U}\right)} \widehat{R}_{2^{(1)}}^{\prime}=\frac{(140-100)}{(125-100)} 58=92.8, \\
& \Phi\left(2^{(1)}\right)=\left\{R_{2^{(1)}} \mid 0 \leq R_{2^{(1)}}\right. \\
& \leq\left\lfloor\frac{\left(C_{1^{(3)}}^{U}-C_{2^{(1)}}^{U}\right)}{\left(C_{1^{(2)}}^{U}-C_{2^{(1)}}^{U}\right)} \widehat{R}_{2^{(1)}}^{\prime}\right\rfloor, \frac{\left(Q_{2^{(1)}}-R_{2^{(1)}}\right)}{\left\lceil\left(Q_{2^{(1)}}-R_{2^{(1)}}\right) / W_{2^{(1)}}\right\rceil} \\
& \left.\geq D_{2^{(1)}}\right\} \text {, } \\
& \Phi\left(2^{(1)}\right)=\left\{R_{2^{(1)}} \mid 0 \leq R_{2^{(1)}}\right. \\
& \left.\leq\left\lfloor\frac{(140-100)}{(125-100)} 58\right\rfloor, \frac{\left(258-R_{2^{(1)}}\right)}{\left\lceil\left(258-R_{2^{(1)}}\right) / 100\right\rceil} \geq 80\right\}, \\
& \Phi\left(2^{(1)}\right)=\left\{R_{2^{(1)}} \mid 0 \leq R_{2^{(1)}}\right. \\
& \left.\leq 92, \frac{\left(258-R_{2^{(1)}}\right)}{\left\lceil\left(258-R_{2^{(1)}}\right) / 100\right\rceil} \geq 80\right\}, \\
& \Phi\left(2^{(1)}\right)=\{0,1,2, \ldots, 18,58,59, \ldots, 92\} \text {, } \\
& \Phi\left(3^{(1)}\right)=\{78\}, \\
& \widehat{R}_{4^{(1)}}^{\prime}=\min \left\{Q_{4^{(1)}}-D_{4^{(1)}} \cdot\left\lceil\frac{Q_{4^{(1)}}}{W_{4^{(1)}}}\right\rceil, D_{4^{(1)}}-1\right\} \\
& =\{480-80 \cdot 5,79\}=79, \\
& \frac{\left(C_{1^{(3)}}^{U}-C_{4^{(1)}}^{U}\right)}{\left(C_{1^{(2)}}^{U}-C_{4^{(1)}}^{U}\right)} \widehat{R}_{4^{(1)}}^{\prime}=\frac{(140-100)}{(125-100)} 79=126.4, \\
& \Phi\left(4^{(1)}\right)=\left\{R_{4^{(1)}} \mid 0 \leq R_{4^{(1)}}\right.
\end{aligned}
$$

$$
\begin{aligned}
& \leq\left\lfloor\frac{\left(C_{1^{(3)}}^{U}-C_{4^{(1)}}^{U}\right)}{\left(C_{1^{(2)}}^{U}-C_{4^{(1)}}^{U}\right)} \widehat{R}_{4^{(1)}}^{\prime}\right\rfloor, \frac{\left(Q_{4^{(1)}}-R_{4^{(1)}}\right)}{\left\lceil\left(Q_{4^{(1)}}-R_{4^{(1)}}\right) / W_{4^{(1)}}\right\rceil} \\
& \left.\geq D_{4^{(1)}}\right\} \text {, } \\
& \Phi\left(4^{(1)}\right)=\left\{R_{4^{(1)}} \mid 0 \leq R_{4^{(1)}}\right. \\
& \left.\leq\left\lfloor\frac{(140-100)}{(125-100)} 79\right\rfloor, \frac{\left(480-R_{4^{(1)}}\right)}{\left\lceil\left(480-R_{4^{(1)}}\right) / 100\right\rceil} \geq 80\right\}, \\
& \Phi\left(4^{(1)}\right)=\left\{R_{4^{(1)}} \mid 0 \leq R_{4^{(1)}}\right. \\
& \left.\leq 126, \frac{\left(480-R_{4^{(1)}}\right)}{\left\lceil\left(480-R_{4^{(1)}}\right) / 100\right\rceil} \geq 80\right\}, \\
& \Phi\left(4^{(1)}\right)=\{0,1,2, \ldots, 126\}, \\
& \widehat{R}_{5^{(1)}}^{\prime}=\min \left\{Q_{5^{(1)}}-D_{5^{(1)}} \cdot\left\lceil\frac{Q_{5^{(1)}}}{W_{5^{(1)}}}\right\rceil, D_{5^{(1)}}-1\right\} \\
& =\{298-80 \cdot 3,79\}=58, \\
& \frac{\left(C_{1^{(3)}}^{U}-C_{5^{(1)}}^{U}\right)}{\left(C_{2^{(2)}}^{U}-C_{5^{(1)}}^{U}\right)} \widehat{R}_{5^{(1)}}^{\prime}=\frac{(140-100)}{(125-100)} 58=92.8, \\
& \Phi\left(5^{(1)}\right)=\left\{R_{5^{(1)}} \mid 0 \leq R_{5^{(1)}}\right. \\
& \leq\left\lfloor\frac{\left(C_{1^{(3)}}^{U}-C_{5^{(1)}}^{U}\right)}{\left(C_{2^{(2)}}^{U}-C_{5^{(1)}}^{U}\right)} \widehat{R}_{5^{(1)}}^{\prime}\right\rfloor, \frac{\left(Q_{5^{(1)}}-R_{5^{(1)}}\right)}{\left\lceil\left(Q_{5^{(1)}}-R_{5^{(1)}}\right) / W_{5^{(1)}}\right\rceil} \\
& \left.\geq D_{5^{(1)}}\right\} \text {, } \\
& \Phi\left(5^{(1)}\right)=\left\{R_{5^{(1)}} \mid 0 \leq R_{5^{(1)}}\right. \\
& \left.\leq\left\lfloor\frac{(140-100)}{(125-100)} 58\right\rfloor, \frac{\left(298-R_{5^{(1)}}\right)}{\left\lceil\left(298-R_{5^{(1)}}\right) / 100\right\rceil} \geq 80\right\} \text {, } \\
& \Phi\left(5^{(1)}\right)=\left\{R_{5^{(1)}} \mid 0 \leq R_{5^{(1)}}\right. \\
& \left.\leq 92, \frac{\left(298-R_{5^{(1)}}\right)}{\left\lceil\left(298-R_{5^{(1)}}\right) / 100\right\rceil} \geq 80\right\}, \\
& \Phi\left(5^{(1)}\right)=\{0,1,2, \ldots, 58\}, \\
& \widehat{R}_{6^{(1)}}^{\prime}=\min \left\{\left(\frac{Q_{6^{(1)}}}{W_{6^{(1)}}}\right) \cdot\left(W_{6^{(1)}}-D_{6^{(1)}}\right), D_{6^{(1)}}-1\right\} \\
& =\min \left\{\left(\frac{400}{100}\right) \cdot(100-80), 79\right\}=79,
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\left(C_{1^{(3)}}^{U}-C_{1^{(1)}}^{U}\right)}{\left(C_{1^{(2)}}^{U}-C_{1^{(1)}}^{U}\right)} \widehat{R}_{6^{(1)}}^{\prime}=\frac{(140-100)}{(125-100)} 79=126.4, \\
& \Phi\left(6^{(1)}\right)=\left\{R_{6^{(1)}} \mid 0 \leq R_{6^{(1)}}\right. \\
& \leq\left\lfloor\frac{\left(C_{1^{(3)}}^{U}-C_{6^{(1)}}^{U}\right)}{\left(C_{2^{(2)}}^{U}-C_{6^{(1)}}^{U}\right)} \widehat{R}_{6^{(1)}}^{\prime}\right\rfloor, \frac{\left(Q_{6^{(1)}}-R_{6^{(1)}}\right)}{\left\lceil\left(Q_{6^{(1)}}-R_{6^{(1)}}\right) / W_{6^{(1)}}\right\rceil} \\
& \left.\geq D_{6^{(1)}}\right\} \text {, } \\
& \Phi\left(6^{(1)}\right)=\left\{R_{6^{(1)}} \mid 0 \leq R_{6^{(1)}}\right. \\
& \left.\leq 126, \frac{\left(400-R_{6^{(1)}}\right)}{\left\lceil\left(400-R_{6^{(1)}}\right) / 100\right\rceil} \geq 80\right\}, \\
& \Phi\left(6^{(1)}\right)=\{0,1,2, \ldots, 80,100,101, \ldots, 126\}, \\
& \widehat{R}_{7^{(1)}}^{\prime}=\min \left\{Q_{7^{(1)}}-D_{7^{(1)}} \cdot\left\lceil\frac{Q_{7^{(1)}}}{W_{7^{(1)}}}\right\rceil, D_{7^{(1)}}-1\right\} \\
& =\min \{99-80 \cdot 1,79\}=19 \text {, } \\
& \frac{\left(C_{1^{(3)}}^{U}-C_{7^{(1)}}^{U}\right)}{\left(C_{3^{(2)}}^{U}-C_{7^{(1)}}^{U}\right)} \widehat{R}_{7^{(1)}}^{\prime}=\frac{(140-100)}{(125-100)} 19=30.4, \\
& \Phi\left(7^{(1)}\right)=\left\{R_{7^{(1)}} \mid 0 \leq R_{7^{(1)}}\right. \\
& \left.\leq \widehat{R}_{7^{(1)}}^{\prime}, \frac{\left(Q_{7^{(1)}}-R_{7^{(1)}}\right)}{\left\lceil\left(Q_{7^{(1)}}-R_{7^{(1)}}\right) / W_{7^{(1)}}\right\rceil} \geq D_{7^{(1)}}\right\}, \\
& \Phi\left(7^{(1)}\right)=\left\{R_{7^{(1)}} \mid 0 \leq R_{7^{(1)}} \leq 19, \frac{\left(99-R_{7^{(1)}}\right)}{\left\lceil\left(99-R_{7^{(1)}}\right) / 100\right\rceil}\right. \\
& \geq 80\}, \\
& \Phi\left(7^{(1)}\right)=\{0,1,2, \ldots, 19\}, \\
& \widehat{R}_{8^{(1)}}^{\prime}=\min \left\{Q_{8^{(1)}}-D_{8^{(1)}} \cdot\left\lceil\frac{Q_{8^{(1)}}}{W_{8^{(1)}}}\right\rceil, D_{8^{(1)}}-1\right\} \\
& =\min \{555-80 \cdot 6,79\}=75, \\
& \frac{\left(C_{1^{(3)}}^{U}-C_{8^{(1)}}^{U}\right)}{\left(C_{3^{(2)}}^{U}-C_{8^{(1)}}^{U}\right)} \widehat{R}_{8^{(1)}}^{\prime}=\frac{(140-100)}{(125-100)} 75=120, \\
& \Phi\left(8^{(1)}\right)=\left\{R_{8^{(1)}} \mid 0 \leq R_{8^{(1)}}\right. \\
& \leq\left\lfloor\frac{\left(C_{1^{(3)}}^{U}-C_{8^{(1)}}^{U}\right)}{\left(C_{3^{(2)}}^{U}-C_{8^{(1)}}^{U}\right)} \widehat{R}_{8^{(1)}}^{\prime}\right\rfloor, \frac{\left(Q_{8^{(1)}}-R_{8^{(1)}}\right)}{\left\lceil\left(Q_{8^{(1)}}-R_{8^{(1)}}\right) / W_{8^{(1)}}\right\rceil}
\end{aligned}
$$

$$
\begin{align*}
& \left.\geq D_{8^{(1)}}\right\}, \\
& \Phi\left(8^{(1)}\right)=\left\{R_{8^{(1)}} \mid 0 \leq R_{8^{(1)}}\right. \\
& \left.\leq 120, \frac{\left(555-R_{8^{(1)}}\right)}{\left\lceil\left(555-R_{8^{(1)}}\right) / 100\right\rceil} \geq 80\right\} \text {, } \\
& \Phi\left(8^{(1)}\right)=\{0,1,2, \ldots, 120\}, \\
& \widehat{R}_{9^{(1)}}^{\prime}=\min \left\{Q_{9^{(1)}}-D_{9^{(1)}} \cdot\left\lceil\frac{Q_{9^{(1)}}}{W_{9^{(1)}}}\right\rceil, D_{9^{(1)}}-1\right\} \\
& =\min \{171-80 \cdot 2,79\}=11, \\
& \frac{\left(C_{1^{(3)}}^{U}-C_{9^{(1)}}^{U}\right)}{\left(C_{3^{(2)}}^{U}-C_{9^{(1)}}^{U}\right)} \widehat{R}_{9^{(1)}}^{\prime}=\frac{(140-100)}{(125-100)} 11=17.6, \\
& \Phi\left(9^{(1)}\right)=\left\{R_{9^{(1)}} \mid 0 \leq R_{9^{(1)}}\right. \\
& \left.\leq \widehat{R}_{9^{(1)}}^{\prime}, \frac{\left(Q_{9^{(1)}}-R_{9^{(1)}}\right)}{\left\lceil\left(Q_{9^{(1)}}-R_{9^{(1)}}\right) / W_{9^{(1)}}\right\rceil} \geq D_{9^{(1)}}\right\}, \\
& \Phi\left(9^{(1)}\right)=\left\{R_{8^{(1)}} \mid 0 \leq R_{8^{(1)}}\right. \\
& \left.\leq 11, \frac{\left(171-R_{9^{(1)}}\right)}{\left\lceil\left(171-R_{9^{(1)}}\right) / 100\right\rceil} \geq 80\right\}, \\
& \Phi\left(9^{(1)}\right)=\{0,1,2, \ldots, 11\} . \tag{16}
\end{align*}
$$

All possible states at stage 1 are defined with all combinations of $R_{1^{(1)}}, R_{2^{(1)}}, \ldots, R_{9^{(1)}}$ of the sets $\Phi\left(1^{(1)}\right), \Phi\left(2^{(1)}\right), \ldots, \Phi\left(9^{(1)}\right)$ and given in the first column of Table 1. That is, the number of possible states is $21 \times 93 \times 1 \times 127 \times 59 \times 127 \times 20 \times 121 \times 12$. Since the time complexity of the DP algorithm depends on both the number of stages and the number of possible states at each stage, the DP algorithm has exponential time complexity because the number of states increases exponentially due to the multidimensionality from the agglomerative hierarchical clustering structure. However, we expect that the DP algorithm works well for moderate-sized problem instances because the size of the solution space can be dramatically reduced in many cases by reducing the number of states using Properties 1 and 2.

Table 2 shows DP calculations at stage 2. The processing cost for batched items at this stage (i.e., level 2) for given $\left(\Omega_{1^{(2)}}, \Omega_{2^{(2)}}, \Omega_{3^{(2)}}\right)$ is computed as

$$
\begin{equation*}
\Gamma_{2}\left(\Omega_{1^{(2)}}, \Omega_{2^{(2)}}, \Omega_{3^{(2)}}\right)=\sum_{l=1}^{3} \Omega_{l^{(2)}} C_{l^{(2)}}^{U} \tag{17}
\end{equation*}
$$

Table 1: DP calculations at stage 1.

| $\left(R_{1^{(1)}}, R_{2^{(1)}}, \ldots, R_{9^{(1)}}\right)$ | $\left(\Omega_{1^{(1)}}, \Omega_{2^{(1)}}, \ldots, \Omega_{g^{(1)}}\right)$ | $\Gamma_{1}\left(\Omega_{1^{(1)}}, \Omega_{2^{(1)}}, \ldots, \Omega_{g^{(1)}}\right)$ | $\Psi_{1}\left(R_{1^{(1)}}, R_{2^{(1)}}, \ldots, R_{g^{(1)}}\right)$ |
| :--- | :---: | :---: | :---: |
| $(28,0,78,0,0,0,0,0,0)$ | $(100,258,0,480,298,400,99,555,171)$ | $(100+\cdots+171) \times 100=236100$ | 236100 |
| $(28,0,78,0,0,0,0,0,1)$ | $(100,258,0,480,298,400,99,555,170)$ | $(100+\cdots+170) \times 100=236000$ | 236000 |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $(48,92,78,126,58,126,19,120,11)$ | $(80,166,0,354,240,274,80,435,160)$ | $(80+\cdots+160) \times 100=178900$ | 178900 |

TAble 2: DP calculations at stage 2.

| $\left(R_{1^{(1)}}, R_{2^{(1)}}, \ldots, R_{9^{(1)}}\right)$ | $\left(R_{1^{(2)}}, R_{2^{(2)}}, R_{3^{(2)}}\right)$ | $\left(\Omega_{1^{(2)}}, \Omega_{2^{(2)}}, \Omega_{3^{(2)}}\right)$ | $\begin{gathered} \Gamma_{2}\left(\Omega_{1^{(2)}}, \Omega_{2^{(2)}}, \Omega_{3^{(2)}}\right)+ \\ \Psi_{1}\left(R_{1^{(1)}}, R_{2^{(1)}}, \ldots, R_{9^{(1)}}\right) \\ \hline \end{gathered}$ | $\Psi_{2}\left(R_{1^{(2)}}, R_{2^{(2)}}, R_{3^{(2)}}\right)$ |
| :---: | :---: | :---: | :---: | :---: |
| $(28,0,78,0,0,0,0,0,0)$ | $(6,18,0)$ | (100, 0, 0) | $\begin{gathered} (100+0+0) \times 125+ \\ 236100=248600 \end{gathered}$ |  |
|  | (7, 18, 0) | $(99,0,0)$ | $\begin{gathered} (99+0+0) \times 125+ \\ 236100=248475 \end{gathered}$ |  |
|  |  | $\ldots$ | $\ldots$ |  |
|  | $(31,18,0)$ | ( $75,0,0$ ) | $\begin{gathered} (75+0+0) \times 125+ \\ 236100=245475 \\ \hline \end{gathered}$ |  |
| $(28,0,78,0,0,0,0,0,1)$ | $(6,18,1)$ | $(100,0,0)$ | $\begin{gathered} (100+0+0) \times 125+ \\ 236000=248500 \end{gathered}$ |  |
|  | $(7,18,1)$ | $(99,0,0)$ | $\begin{gathered} (99+0+0) \times 125+ \\ 236000=248375 \end{gathered}$ |  |
|  | $\cdots$ |  | $\cdots$ |  |
|  | $(31,18,1)$ | $(75,0,0)$ | $\begin{gathered} (75+0+0) \times 125+ \\ 236000=245375 \end{gathered}$ |  |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |  |
| $(28,1,78,0,0,0,0,0,0)$ | $(7,18,0)$ | (100, 0, 0) | $\begin{gathered} (100+0+0) \times 125+ \\ 236000=248500 \end{gathered}$ |  |
|  | $(8,18,0)$ | $(99,0,0)$ | $\begin{gathered} (99+0+0) \times 125+ \\ 236000=248375 \end{gathered}$ |  |
|  | $\ldots$ | $\ldots$ | ... |  |
|  | $(32,18,0)$ | $(75,0,0)$ | $\begin{gathered} (75+0+0) \times 125+ \\ 236000=245375 \end{gathered}$ |  |
| $\cdots$ | $\ldots$ | $\cdots$ | ... |  |
| $(48,92,78,126,58,126,19,120,11)$ | $(0,0,0)$ | $(344,184,150)$ |  |  |
|  | $(0,1,0)$ | $(344,183,150)$ |  |  |
|  | $(44,34,0)$ | $(300,150,150)$ |  |  |

The minimum processing cost for batched items until level 2 for given $\left(R_{1^{(2)}}, R_{2^{(2)}}, R_{3^{(2)}}\right)$ is computed as

$$
\begin{align*}
\Psi_{2} & \left(R_{1^{(2)}}, R_{2^{(2)}}, R_{3^{(2)}}\right) \\
= & \min _{\substack{\Omega_{l^{(2)}}=\sum_{l \in \Delta l^{(2)}} \\
R_{l^{(1)}}-R_{l^{(2)}} \in \Phi\left(l^{(2)}\right) \forall l^{(2)}}} \Gamma_{2}\left(\Omega_{1^{(2)}}, \Omega_{2^{(2)},}, \Omega_{3^{(2)}}\right)  \tag{18}\\
& +\Psi_{1}\left(R_{1^{(1)}}, R_{2^{(1)}}, \ldots, R_{9^{(1)}}\right) .
\end{align*}
$$

Next, we define the set of $R_{l^{(2)}}$ needed to be considered in DP recursive equations to obtain an optimal solution of the NLBP as follows. As shown in Table 2, possible states at stage 2 are defined with the states at stage 1 . For example, 26 states $(26 \times 1 \times 1)$ at stage 2 are defined with a state
$\left(R_{1^{(1)}}, R_{2^{(1)}}, R_{3^{(1)}}, R_{4^{(1)}}, R_{5^{(1)}}, R_{6^{(1)}}, R_{7^{(1)}}, R_{8^{(1)}}, R_{9^{(1)}}\right)=(28,0,78$, $0,0,0,0,0,0)$ as follows. Here, note that a state at stage 2 can be defined by several different states at stage 1 . See the state $\left(R_{1^{(2)}}, R_{2^{(2)}}, R_{3^{(2)}}\right)=(7,18,0)$ of Table 2. This state can be defined by the state $\left(R_{1^{(1)}}, R_{2^{(1)}}, R_{3^{(1)}}\right.$, $\left.R_{4^{(1)}}, R_{5^{(1)}}, R_{6^{(1)}}, R_{7^{(1)}}, R_{8^{(1)}}, R_{g^{(1)}}\right)=(28,0,78,0,0,0,0,0,0)$ and the state $\left(R_{1^{(1)}}, R_{2^{(1)}}, R_{3^{(1)}}, R_{4^{(1)}}, R_{5^{(1)}}, R_{6^{(1)}}, R_{7^{(1)}}, R_{8^{(1)}}\right.$, $\left.R_{9^{(1)}}\right)=(28,1,78,0,0,0,0,0,0)$. As a result, $\Psi_{2}\left(R_{1^{(2)}}, R_{2^{(2)}}\right.$, $R_{3^{(2)}}$ ) should be determined among several different states $\left(R_{1^{(1)}}, R_{2^{(1)}}, R_{3^{(1)}}, R_{4^{(1)}}, R_{5^{(1)}}, R_{6^{(1)}}, R_{7^{(1)}}, R_{8^{(1)}}, R_{9^{(1)}}\right)$ at the first stage making the same state $\left(R_{1^{(2)}}, R_{2^{(2)}}, R_{3^{(2)}}\right)$ at the second stage.

$$
\begin{aligned}
& Q_{1^{(2)}}=R_{1^{(1)}}+R_{2^{(1)}}+R_{3^{(1)}}+R_{4^{(1)}}=28+0+78+0 \\
& \quad=106
\end{aligned}
$$

$$
\begin{align*}
& Q_{2^{(2)}}=R_{5^{(1)}}+R_{6^{(1)}}=0+18=18, \\
& Q_{3^{(2)}}=R_{7^{(1)}}+R_{8^{(1)}}+R_{9^{(1)}}=0+0+0=0, \\
& \widehat{R}_{1^{(2)}}^{\prime}=\min \left\{Q_{1^{(2)}}-D_{1^{(2)}} \cdot\left\lfloor\frac{Q_{1^{(2)}}}{W_{1^{(2)}}}\right\rfloor, D_{1^{(2)}}-1\right\} \\
& =\min \left\{106-75 \cdot\left\lfloor\frac{106}{100}\right\rfloor, 74\right\}=31, \\
& \frac{\left(C_{1^{(3)}}^{U}-C_{1^{(2)}}^{U}\right)}{\left(C_{1^{(3)}}^{U}-C_{1^{(2)}}^{U}\right)} \widehat{R}_{1^{(2)}}^{\prime}=\frac{(140-125)}{(140-125)} 31=31, \\
& \Phi\left(1^{(2)}\right)=\left\{R_{1^{(2)}} \left\lvert\, Q_{1^{(2)}}-\left(\left\lfloor\frac{Q_{1^{(2)}}}{W_{1^{(2)}}}\right\rfloor \cdot W_{1^{(2)}}\right) \leq R_{1^{(2)}}\right.\right. \\
& \left.\leq \widehat{R}_{1^{(2)}}^{\prime}, \frac{\left(Q_{1^{(2)}}-R_{1^{(2)}}\right)}{\left\lceil\left(Q_{1^{(2)}}-R_{1^{(2)}}\right) / W_{1^{(2)}}\right\rceil} \geq D_{1^{(2)}}\right\} \text {, } \\
& \Phi\left(1^{(2)}\right)=\left\{R_{1^{(2)}} \left\lvert\, 106-\left(\left\lfloor\frac{106}{100}\right\rfloor \cdot 100\right) \leq R_{1^{(2)}}\right.\right. \\
& \left.\leq 31, \frac{\left(106-R_{1^{(2)}}\right)}{\left\lceil\left(106-R_{1^{(2)}}\right) / 100\right\rceil} \geq 75\right\} \text {, } \\
& \Phi\left(1^{(2)}\right)=\left\{R_{1^{(2)}} \mid 6 \leq R_{1^{(2)}}\right. \\
& \left.\leq 31, \frac{\left(106-R_{1^{(2)}}\right)}{\left\lceil\left(106-R_{1^{(2)}}\right) / 100\right\rceil} \geq 75\right\} \text {, } \\
& \Phi\left(1^{(2)}\right)=\{6,7, \ldots, 31\}, \\
& \Phi\left(2^{(2)}\right)=\{18\}, \\
& \Phi\left(3^{(2)}\right)=\{0\} . \tag{19}
\end{align*}
$$

We can define 1575 states ( $45 \times 35 \times 1$ ) with another state of stage $1,\left(R_{1^{(1)}}, R_{2^{(1)}}, R_{3^{(1)}}, R_{4^{(1)}}, R_{5^{(1)}}, R_{6^{(1)}}, R_{7^{(1)}}, R_{8^{(1)}}, R_{9^{(1)}}\right)=$ $(48,92,78,126,58,126,19,120,11)$ as follows:

$$
\begin{aligned}
& Q_{1^{(2)}}=R_{1^{(1)}}+R_{2^{(1)}}+R_{3^{(1)}}+R_{4^{(1)}}=48+92+78 \\
& \quad+126=344, \\
& Q_{2^{(2)}}=R_{5^{(1)}}+R_{6^{(1)}}=58+126=184, \\
& Q_{3^{(2)}}=R_{7^{(1)}}+R_{8^{(1)}}+R_{9^{(1)}}=19+120+11=150, \\
& \widehat{R}_{1^{(2)}}^{\prime}=\min \left\{Q_{1^{(2)}}-D_{1^{(2)}} \cdot\left[\frac{Q_{1^{(2)}}}{W_{1^{(2)}}}\right], D_{1^{(2)}}-1\right\} \\
& \quad=\{344-75 \cdot 4,74\}=44, \\
& \left(C_{1^{(3)}}^{U}-C_{1^{(2)}}^{U}\right) \\
& \left(C_{1^{(3)}}^{U}-C_{1^{(2)}}^{U}\right) \\
& \widehat{R}_{1^{(2)}}^{\prime}=\frac{(140-125)}{(140-125)} 44=44, \\
& \Phi\left(1^{(2)}\right)=\left\{R_{1^{(2)}} \mid 0 \leq R_{1^{(2)}}\right.
\end{aligned}
$$

$$
\frac{\left(C_{1^{(3)}}^{U}-C_{2^{(2)}}^{U}\right)}{\left(C_{1^{(3)}}^{U}-C_{2^{(2)}}^{U}\right)} \widehat{R}_{2^{(2)}}^{\prime}=\frac{(140-125)}{(140-125)} 34=34
$$

$$
\Phi\left(2^{(2)}\right)=\left\{R_{2^{(2)}} \mid 0 \leq R_{2^{(2)}}\right.
$$

$$
\left.\leq \widehat{R}_{2^{(2)}}^{\prime}, \frac{\left(Q_{2^{(2)}}-R_{2^{(2)}}\right)}{\left\lceil\left(Q_{2^{(2)}}-R_{2^{(2)}}\right) / W_{2^{(2)}}\right\rceil} \geq D_{2^{(2)}}\right\}
$$

$$
\Phi\left(2^{(2)}\right)=\left\{R_{2^{(2)}} \mid 0 \leq R_{2^{(2)}}\right.
$$

$$
\left.\leq 34, \frac{\left(184-R_{2^{(2)}}\right)}{\left\lceil\left(184-R_{2^{(2)}}\right) / 100\right\rceil} \geq 75\right\}
$$

$$
\Phi\left(2^{(2)}\right)=\{0,1,2, \ldots, 34\}
$$

$$
\widehat{R}_{3^{(2)}}^{\prime}=\min \left\{N_{3^{(2)}}-D^{(2)} \cdot\left\lceil\frac{N_{3^{(2)}}}{W}\right\rceil, D^{(2)}-1\right\}=\{150
$$

$$
-75 \cdot 2,74\}=0
$$

$$
\begin{align*}
& \frac{\left(C_{1^{(3)}}^{U}-C_{3^{(2)}}^{U}\right)}{\left(C_{1^{(3)}}^{U}-C_{3^{(2)}}^{U}\right)} \widehat{R}_{3^{(2)}}^{\prime}=\frac{(140-125)}{(140-125)} 0=0, \\
& \Phi\left(3^{(2)}\right)=\left\{R_{3^{(2)}} \mid 0 \leq R_{3^{(2)}}\right. \\
& \left.\quad \leq \widehat{R}_{3^{(2)}}^{\prime}, \frac{\left(Q_{3^{(2)}}-R_{3^{(2)}}\right)}{\left\lceil\left(Q_{3^{(2)}}-R_{3^{(2)}}\right) / W_{3^{(2)}}\right\rceil} \geq D_{3^{(2)}}\right\}, \\
& \Phi\left(3^{(2)}\right)=\{0\} . \tag{20}
\end{align*}
$$

Table 3 shows DP calculations at stage 3. The processing cost for batched items at this stage (i.e., level 3) for given $\Omega_{1^{(3)}}$ is computed as

$$
\begin{equation*}
\Gamma_{3}\left(\Omega_{1^{(3)}}\right)=\Omega_{1^{(3)}} C_{1^{(3)}}^{U} . \tag{21}
\end{equation*}
$$

$$
\begin{aligned}
& \left.\leq \widehat{R}_{1^{(2)}}^{\prime}, \frac{\left(Q_{1^{(2)}}-R_{1^{(2)}}\right)}{\left\lceil\left(Q_{1^{(2)}}-R_{1^{(2)}}\right) / W_{1^{(2)}}\right\rceil} \geq D_{1^{(2)}}\right\}, \\
& \Phi\left(1^{(2)}\right)=\left\{R_{1^{(2)}} \mid 0 \leq R_{1^{(2)}}\right. \\
& \left.\leq 44, \frac{\left(344-R_{1^{(2)}}\right)}{\left\lceil\left(344-R_{1^{(2)}}\right) / 100\right\rceil} \geq 75\right\} \text {, } \\
& \Phi\left(1^{(2)}\right)=\{0,1,2, \ldots, 44\}, \\
& \widehat{R}_{2^{(2)}}^{\prime}=\min \left\{Q_{2^{(2)}}-D_{2^{(2)}} \cdot\left\lceil\frac{Q_{2^{(2)}}}{W_{2^{(2)}}}\right\rceil, D_{2^{(2)}}-1\right\} \\
& =\{184-75 \cdot 2,74\}=34 \text {, }
\end{aligned}
$$

TAble 3: DP calculations at stage 3.

| $\left(R_{1^{(2)}}, R_{2^{(2)}}, R_{3^{(2)}}\right)$ | $R_{1^{(3)}}$ | $\Omega_{1^{(3)}}$ | $\Gamma_{3}\left(\Omega_{1^{(3)}}\right)+\Psi_{2}\left(R_{1^{(2)}}, R_{2^{(2)}}, R_{3^{(2)}}\right)$ | $\Psi_{3}\left(R_{1^{(3)}}\right)$ |
| :--- | :---: | :---: | :---: | :---: |
| $(6,18,0)$ | 0 | 24 | $(6+18+0) \times 140+\Psi_{2}(6,18,0)$ |  |
| $(7,18,0)$ | 0 | 25 | $(7+18+0) \times 140+\Psi_{2}(7,18,0)$ |  |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |  |
| $(44,34,0)$ | 0 | 78 | $(44+34+0) \times 140+\Psi_{2}(38,61,30)$ |  |

The minimum processing cost for batched items until level 3 for given $R_{1^{(3)}}$ is computed as

$$
\begin{align*}
\Psi_{3}\left(R_{1^{(3)}}\right)= & \Psi_{3}(0) \\
= & \min _{\substack{\Omega_{1^{(3)}}=\sum_{l \in \Lambda_{l^{(3)}}}^{R_{1^{(3)}} \in \Phi\left(1^{(3)}\right)}}} \Gamma_{3}\left(\Omega_{1^{(3)}}\right)  \tag{22}\\
& +\Psi_{2}\left(R_{1^{(2)}}, R_{2^{(2)}}, R_{3^{(2)}}\right) .
\end{align*}
$$

As shown in Table 3, there exists only one state at stage 3: that is, $R_{1^{(3)}}=0$. This means that all items should be batched until the last level.

## 5. Concluding Remarks

In this study, we consider the $N$-level batching problem (NLBP) with a hierarchical clustering structure for minimizing the total cost for processing all items. In the N level batching problem, given items with different types can be grouped into several batches at each level and this batching process is performed from level 1 to level $N$ (from the shallower to the deeper level) sequentially in the given hierarchical clustering structure until all of given items are grouped. In this problem, we assume that the less processing cost is incurred for the batches in the shallower level, since more similar items are required to be grouped in a batch of the shallower level. Both of unit processing cost and batch processing cost are considered for batches at each level for real-world applications.

We formulate the NLBP as a nonlinear integer programming model, propose a multidimensional dynamic programming algorithm for the NLBP, and develop several optimal properties by which the number of states is efficiently reduced in the proposed DP algorithm. For the clear understanding of the proposed DP algorithm and the properties, we provide the tangible example of NLBP and its solution. In the further research, we will apply the proposed algorithm to real world such as the batching processes of the semiconductor wafer fabrications to reduce the manufacturing cost. In addition, it is necessary to develop more efficient heuristic algorithms for the NLBP since the time and space complexity of the proposed DP algorithm is too high to solve large-sized problem instances.

## Notations

## Parameters

$n$ : $\quad$ Index of levels $(n=0,1,2, \ldots, N)$
$l^{(n)}$ : Index of clusters at level $n$ (here, note that any cluster includes item types and items of these item types should be batched. Each cluster at level $n$ is composed of clusters at level $n-1$ (i.e., agglomerative hierarchical clustering structure). Also, $l^{(0)}$ is the index of original item types not clustered)
$\Lambda^{(n)}$ : Set of all clusters at level $n$ (here, $\Lambda^{(0)}$ is the set of all original item types not clustered)
$\Lambda_{U}^{(n)}$ : Set of all clusters at level $n$ and charged by unit processing cost (here, we assume that clusters charged by unit processing cost are known in advance)
$\Lambda_{B}^{(n)}:$ Set of all clusters at level $n$ and charged by batch processing cost (here, we assume that clusters charged by batch processing cost are known in advance)
$\Delta_{l^{(n)}}$ : Set of clusters at level $n-1$ that consists of cluster $l^{(n)}$ (here, $\Delta_{l_{1}^{(n)}} \cap \Delta_{l_{2}^{(n)}}=\phi$ if $l_{1}^{(n)} \neq l_{2}^{(n)}$ and $\bigcup_{l^{(n)}} \Delta_{l^{(n)}}=\Lambda^{(n-1)}$ for all $n$. Also, we assume that any cluster at level $n-1$ can be included in only one cluster at level $n$ )
$W_{l^{(n)}}$ : The maximum batch size requirement of cluster $l^{(n)}$
$D_{l^{(n)}}$ : The minimum batch size requirement of cluster $l^{(n)}$
$C_{l^{(n)}}^{U}$ : Unit processing cost of item batched in the cluster $l^{(n)}$ (here, we assume that $C_{l^{(n)}}^{U}<C_{l^{(n+1)}}^{U}$ for all $n$ )
$C_{l^{(n)}}^{B}$ : Batch processing cost of item batched in the cluster $l^{(n)}$ (here, we assume that $C_{l^{(n)}}^{B}<C_{l^{(n+1)}}^{B}$ for all $n$ ).

## Decision Variables

$Q_{l^{(n)}}$ : Total number of items to be batched in the cluster $l^{(n)}$
$\Omega_{l^{(n)}}$ : Total number of items batched in the cluster $l^{(n)}$
(here, $\Omega_{l^{(0)}}=0$ for all $l^{(0)}$ )
$R_{l^{(n)}}$ : Total number of items not batched in the cluster $l^{(n)}$
(here, $R_{l^{(n)}}=Q_{l^{(n)}}-\Omega_{l^{(n)}}$ and $Q_{l^{(n)}}=\sum_{l^{(n-1)} \in \Delta_{l^{(n)}}} R_{l^{(n-1)}}$.
Also, $R_{l^{(0)}}$ is the number of items in the cluster $l^{(0)}$ to be batched).

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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